# On the Group Level Swiss Solvency Test[*]

Damir Filipović[†]        Michael Kupper[‡]

7 March 2007

### Abstract

In this paper we elaborate on Swiss Solvency Test (SST) consistent group diversification effects via optimizing the web of capital and risk transfer (CRT) instruments between the legal entities. A group level SST principle states that subsidiaries can be sold by the parent company at their economic value minus some minimum capital requirement. In a numerical example we examine the dependence of the optimal CRT on this minimum capital requirement. Our findings raise the question of how to actually implement this group level SST principle and how to define the respective level of minimum capital requirements, in particular.

**Key words**: Convex Optimization, Group Diversification, Minimum Capital Requirement, Swiss Solvency Test

## 1 Introduction

The Swiss Solvency Test (SST) provides a consistent framework both for legal entity and group solvency capital requirements. The underlying reference methodology has recently been outlined in a working paper [2]. This methodology relies on a set of principles which are summarized in stylized form below:

(i) An insurance group is composed of different legal entities (parent company and subsidiaries) which are potentially supervised by different regulators or unregulated, and a web of legally binding capital and risk transfer (CRT) instruments between these legal entities.

(ii) The calculation of available and required capital of a regulated entity has to include the CRT instruments and interdependencies with all other group entities.

---

(iii) Group diversification effects only exist due to the web of CRT instruments.

(iv) Subsidiaries can be sold by the parent company at their economic value (available capital) minus some minimum capital requirement[1]. Owning a subsidiary is in this sense a fungible[2] value.

Examples of risk transfer instruments are intra-group retrocession, securitization of future cash flows, guarantees and other contingent capital solutions while capital transfer instruments are for example cash-bonds or dividends.

The above principles go well with the bottom-up[3] framework for group diversification via optimal legally enforceable CRTs that we developed in [4]. The aim of this paper is to elaborate on the common methods in [4] and the group level SST. For this purpose, we formalize the above group level SST principles.

In a first step, a common set of legally binding CRT instruments is identified. The risk management's objective is then to minimize the group capital requirements by an optimal choice of the CRT. This leads to a well-posed convex optimization problem. The first order conditions induce a consistent valuation principle (compatible with any prior valuation principle) for CRT instruments.

We then distinguish a particular optimal ("equilibrium") CRT which does not affect the entities' individual available capitals, and which is fair in the sense that no lower than the group level of diversification can overturn the diversification benefit of the entire group. Due to the bottom-up approach, an extra capital allocation step is not necessary. In fact, in the context of the optimized capital and risk structure, the allocated capital is just given by the individual entity's required capital.

It turns out that the definition of the minimum capital requirement in Principle (iv) has a strong impact on the optimal CRT and the respective group diversification effects. Indeed, in a numerical example we show that if the minimum capital requirement is defined as market value margin (or cost of capital), as proposed in [2], then Principle (iv) dominates the effect of any other CRT. In fact, almost the fully consolidated diversification effect is obtained. This raises the question of how to implement, or modify, Principle (iv). We propose that, in any case, the minimum capital requirement in Principle (iv) be distinguished from the market value margin.

The remainder of the paper is as follows. Section 2 contains the formal probabilistic setup for the group capital structure (available capital). In Section 3 we discuss the solvency capital requirement (required capital) from different points of view: stand alone and non diversified, fully consolidated, and SST compatible via CRTs, which formalizes the above group level SST Principles (i)–(iv). Section 4 contains the main results. We characterize optimal CRTs and show how to find them by solving a well-posed convex optimization problem. In Section 5 we illustrate our findings by a concrete example and elaborate on the

---

[1] In the SST [2] framework, this is the "market value margin" or "cost of capital"

[2] Fungibility in this context refers to the ability to convert assets into cash or other forms of capital which can be transferred. In general, lack of fungibility has to be taken into account in the SST calculations.

[3] "Bottom-up" here means based on the risk assessments on a legal entity level.

impact of the minimum capital requirement on the optimal CRT. We conclude by Section 6. The Appendix (Sections A–C) contains some notation and facts from convex analysis and the proofs of our theorems.

## 2  Group Capital Structure

We consider an insurance group consisting of $m + 1$ legal entities: the parent company (entity 0) and $m$ subsidiaries (entities $1, \ldots, m$). Values at the beginning of the accounting year are deterministic and denoted by small letters. Values at the end of the accounting year are random and denoted by capital letters. We model this randomness, or risk, with the space of integrable random variables $L^1$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that all values are already discounted by the prevailing risk free rate.

The current available capital (value of asset-liability portfolio) of entity $i$ is defined as

$$c_i = a_i - \ell_i$$

where $a_i$ and $\ell_i$ denote the value of assets and best estimate of liabilities, respectively. The terminal value of the asset-liability portfolio of entity $i$ is given as

$$V_i = A_i - F_i - L_i \tag{1}$$

where $A_i$ and $L_i$ denote the terminal value of assets and best estimate of liabilities, respectively, and $F_i$ denotes the claims payments during the accounting year. As in [4] we assume a linear valuation principle $\mathcal{V} : L^1 \to \mathbb{R}$ such that

$$c_i = \mathcal{V}(V_i) \tag{2}$$

for all entities $i$.

**Remark 2.1.** In view of Principle (iv), owning the subsidiaries is an asset for the parent company. But to avoid double counting, we assume that $a_0$ and $A_0$ are net of the value of owning the subsidiaries. We do, however, take Principle (iv) into account for the realizable distribution of terminal available capital in (7) and (8) below. See also Remark 3.2

## 3  Required Capital

The SST risk measure is the expected shortfall ES on the confidence level of 99%. The stand alone solvency requirement for entity $i$ is

$$\mathrm{ES}(V_i - mvm_i) \leq 0 \tag{3}$$

where $mvm_i$ denotes the market value margin (or cost of capital, see [1]) that is needed at the end of the accounting year to assure the run-off of the asset-liability portfolio. Put (3) in words: the risk of missing the market value margin ($V_i < mvm_i$) is acceptably low.

The calculation of $mvm_i$ is part of the solo SST. Hence we can assume that $mvm_i$ is a given deterministic parameter. The stand alone required capital (or target capital) for entity $i$ accordingly is

$$k_{\text{stal},i} = \text{ES}(V_i - mvm_i - c_i) = \text{ES}(V_i) + mvm_i + c_i. \tag{4}$$

This results in a non diversified group required capital on a stand alone basis of

$$k_{\text{stal}} = \sum_i k_{\text{stal},i}. \tag{5}$$

**Remark 3.1.** The required capital as an indicator for the risk profile has to be considered with respect to the available capital. Indeed, suppose the available capital is increased by adding assets to its portfolio. In absolute terms, this certainly improves the financial strength for backing the liabilities. And yet, due to the riskiness of the additional assets, the required capital increases too. Hence optimizing the risk profile subject to regulatory requirements amounts to minimize the difference between required and available capital. This approach is taken up below.

## 3.1 Consolidated View

Under a fully consolidated view (one group balance sheet, assuming full fungibility of capital) the group solvency requirement would be $\text{ES}\left(\sum_i V_i - mvm_i\right) \leq 0$, and the fully diversified group required capital would amount to

$$k_{\text{cons}} = \text{ES}\left(\sum_i V_i - mvm_i - c_i\right) = \text{ES}\left(\sum_i V_i\right) + \sum_i (mvm_i + c_i). \tag{6}$$

In particular, the consolidated group market value margin is given as sum of the respective stand alone margins $mvm_i$.

Combining (4) and (6), we obtain a (hypothetical) consolidated relative diversification effect of

$$b_{\text{cons}} = 1 - \frac{k_{\text{cons}}}{k_{\text{stal}}}.$$

However, this approach is not in line with regulatory practice! According to Principle (iii) of the group level SST, group diversification effects can only be realized via legally binding CRTs.

## 3.2 CRT View

In view of Principle (iv), the surplus $(V_i - mcr_i)^+$ of subsidiary $i$ exceeding the minimum capital requirement $mcr_i$ is the maximal amount of capital which is fungible and can be transferred to the parent company. The gross[4] available

---

[4]That is, ingnoring the contingent CRT cash flows to be defined below in (10).

capitals of the parent company and its subsidiaries, respectively, become

$$C_0 = V_0 + \sum_{i=1}^{m} (V_i - mcr_i)^+ \tag{7}$$

$$C_i = \min\{V_i, mcr_i\}, \quad i = 1, \ldots, m. \tag{8}$$

We assume in the sequel that $mcr_i$ is a deterministic parameter which can be determined according to some SST guidelines.

**Remark 3.2.** In accordance to (7), the value of owning subsidiary $i$ becomes $\mathcal{V}((V_i - mcr_i)^+)$. In view of Principle (iv), this value could be counted towards the assets of the parent company, $a_0$. But then, to avoid double counting of capital, $a_i$ would have to be reduced by the same amount, lowering the rating of subsidiary $i$ substantially! This would not be in line with a going concern.

Anyhow, due to the cash-invariance of ES, the group capital requirement does not depend on the initial allocation of available capital. Indeed, the group management may redistribute current available capital across the group in order to increase the rating of its subsidiaries. For example, transferring 100 m euro risk free cash from the parent company to subsidiary 1, increases the subsidiary's available capital by 100 m euro (and reduces the parent company's by 100 m euro) without changing its required capital.

In view of Principle (i), we can assume that a well defined set of CRT instruments exists, with future contingent values modelled by some linearly independent random variables $Z_0, Z_1, \ldots, Z_n$ in $L^1$. We also assume that cash is fungible between the entities as long as the payments at the end are determined at the beginning of the accounting year. This is expressed by letting $Z_0 \equiv 1$ denote the payoff of a cash-bond.

Formally, a CRT is a matrix $x = (x_i^j)$ in $\mathbb{R}^{(m+1) \times (n+1)}$ satisfying the clearing condition

$$\sum_i x_i^j = 0, \quad j = 0, \ldots, n, \tag{9}$$

which yields the following realizable distribution of available capital across the entities

$$C_{\text{CRT},i} = C_i + \sum_j x_i^j Z_j, \quad i = 0, \ldots, m. \tag{10}$$

The objective of the group (see Remark 3.1) is to minimize the difference between required and available capital, hence to find an optimal CRT $\hat{x}$ which solves the optimization problem

$$\min_{(x_0, \ldots, x_m)} \sum_i \text{ES} \left( C_i + \sum_j x_i^j Z_j \right) \tag{11}$$

subject to the clearing condition (9). The resulting group capital requirement becomes

$$k_{\text{CRT}} = \sum_i k_{\text{CRT},i}$$

with

$$k_{\mathrm{CRT},i} = \mathrm{ES}(C_{\mathrm{CRT},i} - mvm_i - c_i) = \mathrm{ES}(C_{\mathrm{CRT},i}) + mvm_i + c_i. \qquad (12)$$

Notice that $k_{\mathrm{CRT},i}$ obtained in (12) can be interpreted as the capital allocated to entity $i$. Hence, in our framework, we do not need an exogenous capital allocation method. In fact, we will show in Theorem 4.4 below that this capital allocation is fair in some specific sense.

The realizable relative diversification effect becomes

$$b_{\mathrm{CRT}} = 1 - \frac{k_{\mathrm{CRT}}}{k_{\mathrm{stal}}}.$$

Benchmark is the consolidated diversification effect $b_{\mathrm{cons}}$. From theory (subadditivity of ES) we already know that $b_{\mathrm{CRT}} \leq b_{\mathrm{cons}}$. The goodness of our approach below will be measured by how small we can make the difference $b_{\mathrm{cons}} - b_{\mathrm{CRT}}$.

## 4   Optimal CRTs

We now formalize the proposed framework and introduce the functions

$$u_i(x) := \mathrm{ES}\left(C_i + \sum_j x^j Z_j\right), \qquad (13)$$

for $x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1}$ and $i = 0, \dots, m$. Note that $u_i$ is finite-valued since, by assumption, all random variables considered are in $L^1$.

As a consequence, we can express the constrained $(m+1) \times (n+1)$-dimensional optimization problem (11), subject to (9), as follows

$$\min_{\sum_i x_i = 0} \sum_i u_i(x_i). \qquad (14)$$

Coherence of ES implies that $u_i$ is convex and "cash-invariant"

$$u_i(x + re_0) = u_i(x) - r, \quad \forall r \in \mathbb{R}, \qquad (15)$$

where $e_0 = (1, 0, \dots, 0), e_1, \dots, e_n$ denotes the standard basis on $\mathbb{R}^{n+1}$. Hence $\partial u_i / \partial x^0 = -1$. To simplify the subsequent discussion[5], we assume that every

$$u_i \text{ is differentiable on } \mathbb{R}^{n+1}. \qquad (16)$$

Adding long (assets) or short (liabilities) positions in the CRT instruments to the portfolio (10) also changes its current value (available capital). To determine the available and required capital therefore one needs to know the value of adding positions in $Z_0, \dots, Z_n$. We assume that such value is given by a linear indifference valuation principle as follows. Let $x_i \in \mathbb{R}^{n+1}$ represent the portfolio

---

[5]The following results also hold without this technical assumption, see [4].

(10) of entity $i$. We call the linear functional $\mathcal{V} : L^1 \to \mathbb{R}$ an *indifference valuation principle for entity $i$ with respect to $x_i$* if adding positions $z \in \mathbb{R}^{n+1}$ to $x_i$ is less optimal (that is, requires more capital) than adding the value equivalent cash amount of

$$\sum_j z^j \mathcal{V}(Z_j) = p \cdot z,$$

where the value vector $p = p(\mathcal{V}) \in \mathbb{R}^{n+1}$ is defined as $p^j := \mathcal{V}(Z_j)$, and $\cdot$ denotes the scalar product. Formally, this means

$$u_i(x_i + z) \geq u_i(x_i + (p \cdot z)e_0) \quad \forall z \in \mathbb{R}^{n+1}. \tag{17}$$

From (17) and the cash-invariance property (15) we derive:

**Lemma 4.1.** $\mathcal{V}$ *is an indifference valuation principle for entity $i$ with respect to $x_i$ if and only if*

$$p = -\nabla u_i(x_i). \tag{18}$$

*In particular, we then have $p \cdot e_0 = p^0 = 1$. Hence the value of a unit of cash is one.*

*Proof.* Cash-invariance (15) and (17) imply

$$u_i(x_i + z) \geq u_i(x_i) - p \cdot z \quad \forall z \in \mathbb{R}^{n+1}.$$

Hence $-p \in \partial u_i(x_i)$ is a subgradient of $u_i$ at $x_i$. Since, by assumption (16), $u_i$ is differentiable at $x_i$, it thus follows that $-p = \nabla u_i(x_i)$, see Section A. The last statement follows again from the cash-invariance (15). $\square$

Consistent valuation across the entities therefore can only take place at CRTs $(x_0, \ldots, x_m)$ where $\nabla u_0(x_0) = \cdots = \nabla u_m(x_m)$. It turns out that this is just the first order condition for the optimization problem (14).

**Theorem 4.2.** *Let $(\hat{x}_0, \ldots, \hat{x}_m)$ satisfy the clearing condition (9). The following are equivalent:*

(i) $(\hat{x}_0, \ldots, \hat{x}_m)$ *is a minimizer for (14);*

(ii) $\nabla u_0(\hat{x}_0) = \cdots = \nabla u_m(\hat{x}_m)$;

(iii) $(\hat{x}_i^j, i = 1, \ldots, m, j = 1, \ldots, n) \in \mathbb{R}^{m \times n}$ *is a minimizer for the unconstrained $m \times n$-dimensional convex optimization problem*

$$\min_{(x_i^j) \in \mathbb{R}^{m \times n}} \left( u_0 \left( 0, -\sum_{i=1}^{m} x_i^1, \ldots, -\sum_{i=1}^{m} x_i^n \right) + \sum_{i=1}^{m} u_i(0, x_i^1, \ldots, x_i^n) \right), \tag{19}$$

*and $\hat{x}_0^j = -\sum_{i=1}^{m} \hat{x}_i^j$.*

**Remark 4.3.** Note that a minimizer for (14) is never unique: due to cash-invariance (15), rebalancing the cash $x_i^0 \mapsto x_i^0 + r_i$, for any transfer with $\sum_i r_i = 0$, does not alter the aggregated capital requirements, as

$$\sum_i u_i(x_i) = \sum_i u_i(x_i + r_i e_0).$$

Among the optimal CRTs there is a distinguished one:

**Theorem 4.4.** *Let $(\hat{x}_0, \dots, \hat{x}_m)$ be a minimizer for (14), and unambiguously denote $p = -\nabla u_i(\hat{x}_i)$. Then $\overline{x}_i := \hat{x}_i - (p \cdot \hat{x}_i)e_0$ defines an optimal CRT which is also individually optimal in the sense that*

$$p \cdot \overline{x}_i = 0 \quad and \quad \min_{p \cdot z = 0} u_i(z) = u_i(\overline{x}_i), \tag{20}$$

*for all entities $i = 0, \dots, m$. Moreover, it is fair in the sense that*

$$\sum_{i \in I} u_i(\overline{x}_i) \leq \min_{\sum_{i \in I} x_i = 0} \sum_{i \in I} u_i(x_i) \tag{21}$$

*for every level of diversification $I \subset \{0, \dots, m\}$.*

In view of (21) we may interpret

$$k_{\mathrm{CRT},i} = u_i(\overline{x}_i) + mvm_i + c_i, \quad i = 0, \dots, m \tag{22}$$

as a fair capital allocation, as announced in Section 3.2 above.

**Remark 4.5.** In economic theory, the allocation $(\overline{x}_i)$ satisfying (20) is called an *equilibrium*, see [4].

**Remark 4.6.** Property (20) says in particular that the net value of the equilibrium CRT $\overline{x}_i$ is zero under the valuation principle $p$ for every entity $i$. Hence it does not affect the current available capital, and is thus distinguished.

Strictly speaking, in order that $p$ be consistent with any prior linear valuation principle (2), we have to assume that $V_i$ does not lie in the linear span of $Z_0, \dots, Z_n$, for all entities $i$. Since in this case, we are indeed free to specify $\mathcal{V}(Z_j)$ to be equal to $p^j$, for all $j = 0, \dots, n$. This assumption is realistic, as in general the initial asset-liability portfolio of entity $i$ is more diverse than any portfolio consisting solely of the CRT instruments.

**Remark 4.7.** There is empirical evidence that insurance companies price CRT instruments based on (risk measure) equilibrium valuation principles, such as the present one. Indeed, using date from the U.S. property-liability industry, Cummins et al. [3] provide empirical tests which strongly support the theoretical prediction that prices of illiquid, imperfectly hedgeable intermediated risk products should depend upon firm capital structure, the covariability of the risks with the firms's other projects, and their marginal effects on the firm's insolvency risk.

As for the existence of an optimal CRT, we quote Corollary 7.2 in [4]:

**Theorem 4.8.** *A minimizer for (14) always exists.*

# 5 Example

For illustration we consider an insurance group consisting of the parent company and $m = 1$ subsidiary[6]. The current capital structure is

| $i$ | $a_i$ | $\ell_i$ | $c_i$ |
|-----|-------|----------|-------|
| 0 | 8 | 6 | 2 |
| 1 | 4 | 3 | 1. |

Hence the parent company is twice the subsidiary in size. For simplicity, we summarize $F_i$ and $L_i$ in one variable, denoted $L_i$, so that (1) reads $V_i = A_i - L_i$. For $i = 0, 1$, we model $A_i$ normal and $L_i$ log-normal as

$$A_i = a_i \left(1 + \mu + \sigma_A W_A\right)$$
$$L_i = \ell_i \exp\left(\sigma_L W_{L_i} - \sigma_L^2/2\right)$$

with asset return[7] $\mu = 0.01$ and volatility $\sigma_A = 0.02$, log-liability standard deviation $\sigma_L = 0.08$, and $W = (W_A, W_{L_0}, W_{L_1})$ a three dimensional standard[8] normal distributed vector. Hence the asset returns for parent company and subsidiary are perfectly correlated, while their liabilities are independent.

The SST field tests [1] have shown that the market value margin $mvm_i$ ranges between 10% and 60% of the one year risk capital $c_i + \mathrm{ES}(V_i)$. Consistently with these empirical facts, we set

$$mvm_i = 0.4 \times (c_i + \mathrm{ES}(V_i)), \quad i = 0, 1.$$

The minimum capital requirement $mcr_i$ will vary as a multiple $q_{\mathrm{mcr}} \geq 0$ of the stand alone one year required capital:

$$mcr_i = q_{\mathrm{mcr}} \times (c_i + \mathrm{ES}(V_i)), \quad i = 0, 1. \tag{23}$$

An interesting indicator is the probability

$$p_{\mathrm{default}} = \mathbb{P}[V_1 < mcr_1]$$

that the subsidiary defaults on the minimum capital requirement, see (7)–(8).

As CRT instrument we use quota share retrocession. The subsidiary can cede a proportion of its liabilities to the parent company, that is, we set

$$Z_0 = 1 \quad \text{and} \quad Z_1 = L_1.$$

The optimal quota follows by minimizing the group required capital (14).

Using Matlab with $10^6$ sample points for $W$, we obtain the following stand alone and consolidated numbers, respectively:

$$k_{\mathrm{stal},0} = 1.4 \times 1.3807 = 1.933, \quad k_{\mathrm{stal},1} = 1.4 \times 0.693 = 0.970,$$
$$k_{\mathrm{stal}} = 2.903, \quad k_{\mathrm{cons}} = 2.372, \quad b_{\mathrm{cons}} = 0.183. \tag{24}$$

---

[6]This can also be interpreted as all subsidiaries summarized by a representative one.

[7]All values are already discounted by the prevailing risk free rate.

[8]That is, its coordinates are mutually independent.

The CRT figures are determined for varying minimum capital requirement (23), by numerically solving the unconstrained 1-dimensional convex optimization problem (19). The results are as follows:

Figure 1 shows the non monotonic dependence of the group required capital $k_{cons}$ and the equilibrium value $p^1$ of the CRT instrument $Z_1 = L_1$ on the minimum capital requirement factor $q_{mcr}$. The maximal group required capital of 2.594 is attained at $q_{mcr} = 1.2$. On the other hand, for $q_{mcr} \leq 0.4$ we obtain almost the fully consolidated diversification effect, see also Figure 3. The minimal equilibrium value $p^1$ of 3.19 is attained at $q_{mcr} = 1.5$, which is still greater than the best estimate $\ell_1 = \mathbb{E}[L_1] = 3$ of $L_1$. The positive difference $p^1 - \ell_1$ equals the risk premium that the subsidiary is willing to pay for ceding a part of its liability risk to the parent company.

[Insert Figure 1 about here]

Figure 2 shows the dependence of the equilibrium capital allocation $k_{CRT,0} + k_{CRT,1} = k_{CRT}$, see (22), on the minimum capital requirement factor $q_{mcr}$. The required capital $k_{CRT,0}$ of the parent company attains its maximum of 1.85 at $q_{mcr} = 1.6$.

[Insert Figure 2 about here]

Figure 3 shows the non monotonic dependence of the relative diversification effect $b_{CRT}$ on the minimum capital requirement factor $q_{mcr}$. The worst relative diversification effect of 0.106 is attained at $q_{mcr} = 1.2$. For $q_{mcr} \leq 0.4$, we have $b_{CRT} \geq 0.180$, hence almost the fully consolidated relative diversification effect (24), which is due to the very small default probability $p_{default}$, see Figure 4.

[Insert Figure 3 about here]

Figure 4 shows the dependence of the optimal CRT $\hat{x}_1^1$ (in the figure denoted by "$x$") and the default probability $p_{default}$ on the minimum capital requirement factor $q_{mcr}$. Obviously, both variables are increasing. As for the optimal CRT, we obtain that $\hat{x}_1^1 \to 0.878$ for $q_{mcr} \to \infty$. On the other hand, we observe that $\hat{x}_1^1 \approx 0$ (up to 5 digits) for $q_{mcr} \leq 0.4$. This is associated with very small default probabilities of $p_{default} \leq 0.003$ for $q_{mcr} \leq 0.4$. Hence the effect of Principle (iv) is essentially equivalent to a fully consolidated view if the minimum capital requirement is defined as (small as) the market value margin. Is this reasonable? We propose that, in any case, the minimum capital requirement in Principle (iv) be distinguished from the market value margin.

[Insert Figure 4 about here]

Note that omitting Principle (iv) is equivalent to setting $q_{mcr} = \infty$. From Figure 3 we see that a minimum capital requirement of $q_{mcr} = 0.8$ yields approximately the same diversification effect via CRTs as if Principle (iv) were omitted.

# 6 Conclusion

We have formally implemented the stylized principles of the group level SST. We assumed that the risk management's objective is to minimize the group capital requirements by optimizing the web of CRT instruments between the entities. This led to a well-posed convex optimization problem. As byproducts we obtained a consistent valuation principle for the CRT instruments and a fair (equilibrium) capital allocation.

In a numerical example we have elaborated on how the optimal CRT and the respective group diversification effect depend on the minimum capital requirement in SST Principle (iv). It turned out that the effect of Principle (iv) is essentially equivalent to a fully consolidated view if the minimum capital requirement is defined as (small as) the market value margin. This raises the question of how to actually implement, or modify, Principle (iv). In any case, the minimum capital requirement in Principle (iv) should be distinguished from the market value margin. A systematic study is beyond the scope of this paper. We recommend that this aspect be further discussed in the Solvency 2 process.

# A  Some Facts from Convex Analysis

The following proofs rely on general principles in convex analysis, which can be found e.g. in [5]. Let $f : \mathbb{R}^d \to (-\infty, +\infty]$ be a lower semi-continuous convex function. Its *conjugate*,

$$f^*(q) := \sup_{x \in \mathbb{R}^d} (q \cdot x - f(x)),$$

is again a lower semi-continuous convex function $f^* : \mathbb{R}^d \to (-\infty, +\infty]$, and $f^{**} = f$ (see Theorem 12.2 in [5]). The *effective domain* of $f$ is defined as

$$\mathrm{dom}(f) = \{q \mid f(q) < \infty\}.$$

The *subgradients* of $f$ form a (possibly empty) convex set

$$\partial f(x) = \left\{ q \in \mathbb{R}^d \mid f(x+z) \geq f(x) + q \cdot z \; \forall z \in \mathbb{R}^d \right\},$$

and are characterized by

$$q \in \partial f(x) \quad \Leftrightarrow \quad f(x) + f^*(q) = q \cdot x, \tag{25}$$

see Theorem 23.5 in [5]. Furthermore, $\partial f(x)$ consists of a single element if and only if $f$ is differentiable at $x$. In this case $\partial f(x) = \{\nabla f(x)\}$, where $\nabla f$ denotes the gradient of $f$, see Theorem 25.1 in [5].

# B  Proof of Theorem 4.2

It follows by the finiteness of $u_i$ and Theorem 5.4 in [5] that

$$u(y) := \inf_{y = \sum_{i=0}^m x_i} \sum_{i=0}^m u_i(x_i) \tag{26}$$

defines a convex function $u : \mathbb{R}^{n+1} \to [-\infty, +\infty)$. Theorem 16.4 in [5] states that its conjugate satisfies

$$u^*(q) = \sum_{i=0}^{m} u_i^*(q) \quad \forall q \in \mathbb{R}^{n+1}. \tag{27}$$

Obviously, the constrained optimization problem (14) is equivalent to (26) for $y = 0$.

(i)$\Rightarrow$(ii). By assumption, all random variables considered are in $L^1$ and thus $u(0) = \sum_{i=0}^{m} u_i(\hat{x}_i) > -\infty$, see (13). It follows from Theorem 7.2 in [5] that $u$ is finite-valued on $\mathbb{R}^{n+1}$. In view of Theorem 23.4 in [5], there exists a subgradient $q \in \partial u(0)$. Using (25) and (27) and the clearing condition, we conclude that

$$u(0) = -u^*(q) = \sum_{i=0}^{m} -u_i^*(q) = \sum_{i=0}^{m}(q \cdot \hat{x}_i - u_i^*(q)) \leq \sum_{i=0}^{m} u_i(\hat{x}_i) = u(0).$$

Therefore $q \cdot \hat{x}_i - u_i^*(q) = u_i(\hat{x}_i)$ and thus $q = \nabla u_i(\hat{x}_i)$, see (25), for all $i$.

(ii)$\Rightarrow$(i). Write $q = \nabla u_i(\hat{x}_i)$. In view of (25) and (27), it follows that

$$u(0) \geq -u^*(q) = \sum_{i=0}^{m}(q \cdot \hat{x}_i - u_i^*(q)) = \sum_{i=0}^{m} u_i(\hat{x}_i).$$

Hence $(\hat{x}_0, \ldots, \hat{x}_m)$ is a minimizer for (14).

(i)$\Leftrightarrow$(iii). This follows from the cash-invariance (15).

## C   Proof of Theorem 4.4

We have, by the properties of $u_i$ and $p$,

$$\inf_{p \cdot y = 0} u_i(y) = \inf_{y} u_i(y - (p \cdot y)e_0) = \inf_{y}(u_i(y) + p \cdot y)$$
$$= -\sup_{y}(-p \cdot y - u_i(y)) = -u_i^*(-p),$$

and on the other hand,

$$u_i(\overline{x}_i) = u_i(\hat{x}_i) + p \cdot \hat{x}_i = -p \cdot \hat{x}_i - u_i^*(-p) + p \cdot \hat{x}_i = -u_i^*(-p). \tag{28}$$

This proves (20). Using the clearing condition and (28) again, we obtain

$$\sum_{i \in I} u_i(\overline{x}_i) = \sum_{i \in I} -u_i^*(-p)$$
$$= \inf_{\sum_{i \in I} x_i = 0} \sum_{i \in I}(-p \cdot x_i - u_i^*(-p)) \leq \inf_{\sum_{i \in I} x_i = 0} \sum_{i \in I} u_i(x_i),$$

and the theorem is proved.

# References

[1] Bundesamt für Privatversicherungen (2006), "The Swiss Experience with Market Consistent Technical Provisions - the Cost of Capital Approach," URL: www.bpv.admin.ch/themen/00506/00552/00727

[2] Bundesamt für Privatversicherungen (2006), "Draft: Modelling of Groups and Group Effects," URL: www.bpv.admin.ch/themen/00506/00530

[3] Cummins, J.D., Lin, Y. and Phillips, R.D. (2006), "Capital Allocation and the Pricing of Financially Intermediated Risks: An Empricial Investigation," Preprint.

[4] Filipović, D. and Kupper, M. (2007), "Optimal Capital and Risk Transfers for Group Diversification," *Mathematical Finance*, forthcoming.

[5] Rockafellar, R.T. (1997), *Convex Analysis*, Princeton Landmarks in Mathematics and Physics, Princeton University Press.
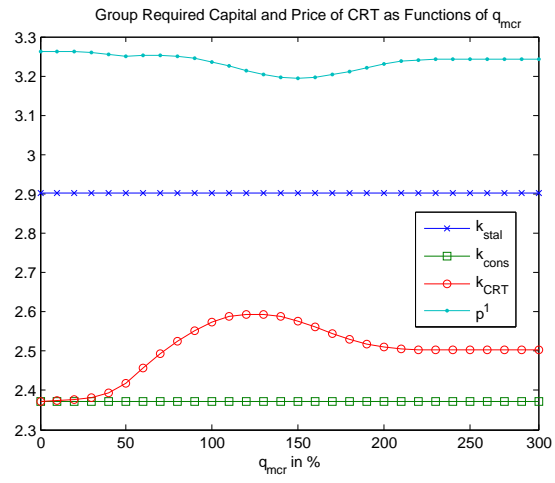
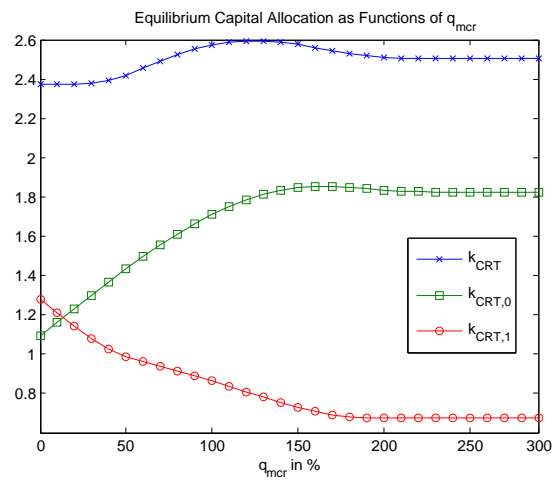Figure 1: Group required capital $k_{\mathrm{CRT}}$ and price $p^1$ of $Z_1$ as functions of $q_{\mathrm{mcr}}$.

Figure 2: Equilibrium capital allocation $k_{\mathrm{CRT},0} + k_{\mathrm{CRT},1} = k_{\mathrm{CRT}}$ as function of $q_{\mathrm{mcr}}$.
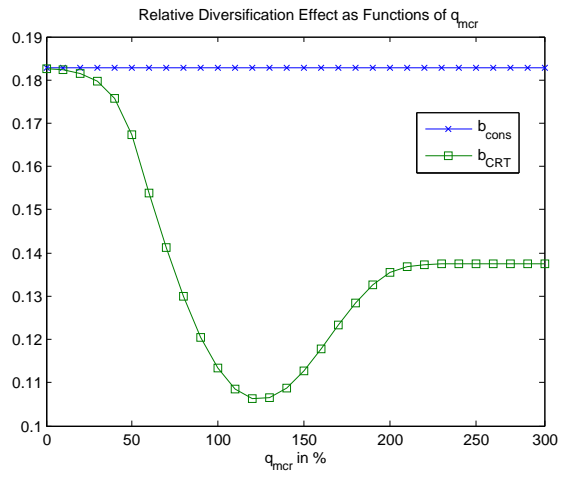
Figure 3: Relative diversification effect $b_{\mathrm{CRT}}$ as function of $q_{\mathrm{mcr}}$.
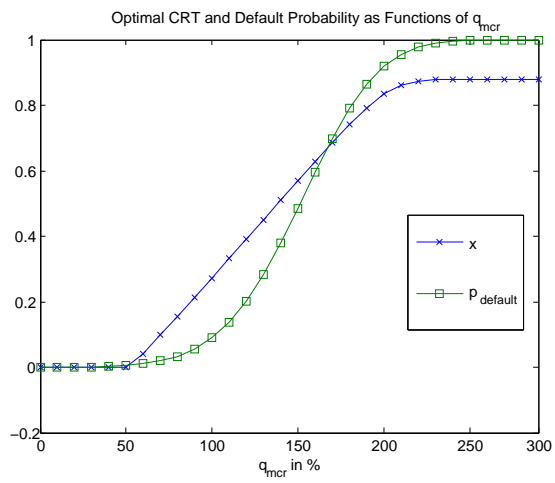
Figure 4: Optimal CRT $\hat{x}_1^1$ (in the figure denoted as "$x$") and default probability $p_{\text{default}}$ as functions of $q_{\text{mcr}}$.