

The Canonical Model Space for Law-invariant Convex Risk Measures is L^1 *

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Abstract

In this paper we establish a one-to-one correspondence between law-invariant convex risk measures on L^∞ and L^1 . This proves that the canonical model space for the predominant class of law-invariant convex risk measures is L^1 .

1 Introduction

Convex risk measures are best known on L^∞ . Indeed, Artzner et al. [1] introduced the seminal axioms of coherence, which then were further generalised to the convex case by Föllmer and Schied [6] and Frittelli and Rosazza-Gianin [8], on L^∞ . However, there is a growing mathematical finance literature dealing with convex risk measures beyond L^∞ , see e.g. [2, 3, 4, 10, 11, 13]. This extended approach is vital since important risk models, such as normal distributed random variables, are not contained in L^∞ .

In most of the above mentioned articles, the model space is chosen such that some preselected risk measure remains finite valued. In contrast, we believe that the model space should be maximal possible to capture the universe of financial risks from the outset. On the other hand, from a computational point of view, the model space should be standard and endowed with a topological structure supporting convex duality. In sum, we propose the model space L^p for $p \geq 1$.

Risk measures on L^p have been studied by several authors, e.g. [2, 3, 10, 11, 13]. But the interplay between convex risk measures on L^p and L^∞ has not been highlighted yet. In this paper, we establish a one-to-one correspondence between law-invariant convex risk measures on L^∞ and L^1 . This proves that

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the canonical model space for the predominant class of law-invariant convex risk measures is L^1 .

2 Extensions of Law-invariant Functions on L^∞

Throughout, we fix a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All equalities and inequalities between random variables are understood in the \mathbb{P} -almost sure (a.s.) sense. The topological dual space of $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$, for $p \in [1, \infty]$, is denoted by L^{p*} . It is well known that $(L^p)^* = L^q$ with $q = \frac{p}{p-1}$ for $p < \infty$, and that $L^{\infty*} \supset L^1$ can be identified with ba , the space of all bounded finitely additive measures μ on (Ω, \mathcal{F}) such that $\mathbb{P}(A) = 0$ implies $\mu(A) = 0$. With some facilitating abuse of notation, we shall write $(X, Z) \mapsto E[XZ]$ for the dual pairing on (L^p, L^{p*}) also for the case $p = \infty$.

We suppose the reader is familiar with standard terminology and basic duality theory for convex functions as outlined in [5] or [12]. We recall that a function $f : L^p \rightarrow [-\infty, \infty]$ is

- (i) *convex* if its *epigraph* $\text{epi } f := \{(X, y) \mid f(X) \leq y\}$ is a convex subset of $L^p \times \mathbb{R}$,
- (ii) *proper* if $f > -\infty$ and its *domain* $\text{dom } f := \{f < \infty\} \neq \emptyset$,
- (iii) *closed* if either $f \equiv -\infty$, $f \equiv \infty$ or f is proper and *lower semi-continuous (l.s.c.)*,
- (iv) *law-invariant* if $f(X) = f(Y)$ for all identically distributed $X \sim Y$.

The *dual* $f^*(Z) = \sup_{X \in L^p} (E[XZ] - f(X))$ of f is a closed convex function on L^{p*} . The Fenchel–Moreau theorem (proposition 4.1 in [5]) states that $f^{**} = f$ if and only if f is closed convex.

Definition 2.1. A convex function $\rho : L^p \rightarrow (-\infty, \infty]$ is called convex risk measure if it is

- (i) *cash-invariant*: $\rho(0) \in \mathbb{R}$ and $\rho(X + m) = \rho(X) - m$ for all $m \in \mathbb{R}$,
- (ii) *monotone*: $\rho(X) \leq \rho(Y)$ for $X \geq Y$.

A positively homogeneous convex risk measure ρ is called coherent.

Note that convex risk measures on L^p for $p \in [1, \infty)$ are not closed in general. Consider e.g. the law-invariant coherent risk measure

$$\rho : L^p \rightarrow (-\infty, \infty], \quad \rho(X) = E[-X] + \delta(X^- \mid L^\infty)$$

where $\delta(X \mid L^\infty)$ is defined as 0 if $X \in L^\infty$ and ∞ elsewhere. Clearly, the acceptance set $\mathcal{A}_\rho := \{X \in L^p \mid \rho(X) \leq 0\}$ is not closed, so ρ is not closed.

The following theorem is our main result, which is valid not only for convex risk measures.

Theorem 2.2. *Let $f : L^\infty \rightarrow [-\infty, \infty]$ be a law-invariant closed convex function. Then, for each $p \in [1, \infty]$, there exists a unique law-invariant closed convex function \bar{f}^p on L^p such that $\bar{f}^p|_{L^\infty} = f$. The function \bar{f}^p is given by*

$$\bar{f}^p(X) = \sup_{Z \in L^{p^*}} E[ZX] - f^*(Z), \quad X \in L^p,$$

where $f^* : L^{\infty^*} \rightarrow [-\infty, \infty]$ is the dual of f . Moreover, for all $p \in [1, \infty]$, we have that $\bar{f}^p = \bar{f}^1|_{L^p}$. In particular, the function f is $\sigma(L^\infty, L^\infty)$ -closed.

Hence there is a one-to-one correspondence between law-invariant closed convex functions, and thus between law-invariant closed convex risk measures, on L^∞ and L^1 . In this sense, we may conclude that the canonical model space for law-invariant convex risk measures is L^1 .

Now let $p \in [1, \infty]$, and $f : L^\infty \rightarrow [-\infty, \infty]$ be some convex function. Then f is trivially extended to a convex function \tilde{f} on L^p by letting $\tilde{f} = f$ on L^∞ and $\tilde{f} = \infty$ elsewhere. It is easily verified that $\tilde{f}^* = f^*$ on L^{p^*} . Hence, the function \bar{f}^p in theorem 2.2 is simply the well-known *closure* or *l.s.c. regularisation* of \tilde{f} in L^p (see [5] section 3.2 or [12] section 3). We chose the notation \bar{f}^p in order to emphasise the dependence on p . Clearly, in general \bar{f}^p will differ for differing $p \in [1, \infty]$ (for an example see [14]). But if f is law-invariant, then theorem 2.2 states that we can omit the p , which is a strong property of law-invariant closed convex functions.

The core of the proof of theorem 2.2 is lemma 2.4 below. It is based on results by Jouini, Schachermayer, and Touzi in [9].

Let $\mathcal{G} \subset \mathcal{F}$ be a sub- σ -algebra. As in [9] we define the conditional expectation on L^{∞^*} as a function $E[\cdot | \mathcal{G}] : L^{\infty^*} \rightarrow L^{\infty^*}$ where $E[\mu | \mathcal{G}]$ is given by

$$E[E[\mu | \mathcal{G}]X] := E[\mu E[X | \mathcal{G}]] \quad \forall X \in L^\infty.$$

Clearly, this definition is consistent with the ordinary conditional expectation in case $\mu \in L^1 \subset L^{\infty^*}$.

Remark 2.3. *If $\mathcal{G} = \sigma(A_1, \dots, A_n)$ is finite, then $E[\mu | \mathcal{G}] \in L^\infty$. In order to verify this, note that for all $X \in L^\infty$ we have*

$$E[E[\mu | \mathcal{G}]X] = E[\mu E[X | \mathcal{G}]] = \sum_{i=1}^n E[X 1_{A_i}] \frac{\mu(A_i)}{\mathbb{P}(A_i)}.$$

Hence, $E[\mu | \mathcal{G}] = \sum_{i=1}^n \frac{\mu(A_i)}{\mathbb{P}(A_i)} 1_{A_i} \in L^\infty$.

Note that (L^∞, L^r) is a dual pair for every $r \in [1, \infty]$.

Lemma 2.4. (i) *Let $D \subset L^\infty$ be a $\|\cdot\|_\infty$ -closed convex law-invariant set. Then D is $\sigma(L^\infty, L^r)$ -closed for every $r \in [1, \infty]$.*

(ii) *A law-invariant convex function $f : L^\infty \rightarrow [-\infty, \infty]$ is closed if and only if it is closed w.r.t. any $\sigma(L^\infty, L^r)$ -topology for every $r \in [1, \infty]$.*

Proof. (i): if $D = \emptyset$, the assertion is obvious. For the remainder of this proof, we assume thus that $D \neq \emptyset$.

According to lemma 4.2 in [9], for all $Y \in D$ and all sub- σ -algebras $\mathcal{G} \subset \mathcal{F}$ we have

$$E[Y \mid \mathcal{G}] \in D. \quad (2.1)$$

Now let $(X_i)_{i \in I}$ be a net in D converging to some $X \in L^\infty$ in the $\sigma(L^\infty, L^r)$ -topological sense, i.e. $E[ZX_i] \rightarrow E[ZX]$ for all $Z \in L^r$. Then, in view of remark 2.3, if \mathcal{G} is finite, we have $E[E[\mu \mid \mathcal{G}]X_i] \rightarrow E[E[\mu \mid \mathcal{G}]X]$ for all $\mu \in L^{\infty*}$. But by definition this equals $E[\mu E[X_i \mid \mathcal{G}]] \rightarrow E[\mu E[X \mid \mathcal{G}]]$ for all $\mu \in L^{\infty*}$. In other words, the net $(E[X_i \mid \mathcal{G}])_{i \in I}$ converges to $E[X \mid \mathcal{G}]$ in the $\sigma(L^\infty, L^{\infty*})$ -topology. Since, according to (2.1), $E[X_i \mid \mathcal{G}] \in D$ for all $i \in I$, we conclude that $E[X \mid \mathcal{G}] \in D$, because D is closed and convex and thus $\sigma(L^\infty, L^{\infty*})$ -closed. Hence, $E[X \mid \mathcal{G}] \in D$ for all finite sub- σ -algebras $\mathcal{G} \subset \mathcal{F}$. Recalling that we can approximate X in $(L^\infty, \|\cdot\|_\infty)$ by a sequence of conditional expectations $(E[X \mid \mathcal{G}_n])_{n \in \mathbb{N}}$ in which the \mathcal{G}_n s are all finite, we conclude by means of the norm-closedness of D that $X \in D$. Thus D is $\sigma(L^\infty, L^r)$ -closed, and (i) is proved.

(ii): suppose f is closed. Then, for every $k \in \mathbb{R}$ the level sets $\{X \in L^\infty \mid f(X) \leq k\}$ are $\|\cdot\|_\infty$ -closed, convex, and law-invariant. Hence, (i) yields the $\sigma(L^\infty, L^r)$ -closedness of the level sets, i.e. f is closed with respect to the $\sigma(L^\infty, L^r)$ -topology. The converse implication is trivial. \square

In the following we denote the (left continuous) quantile function of a random variable X by

$$q_X : (0, 1) \rightarrow \mathbb{R}, \quad q_X(s) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq s\}. \quad (2.2)$$

In the proof of theorem 2.2 we will need the following fact:

(F1) For $X \in L^p$ and $Z \in L^{p*} \cap L^1$ we have that

$$\int_0^1 q_X(s)q_Z(s) ds = \sup_{\hat{X} \sim X} E[\hat{X}Z] = \sup_{\hat{Z} \sim Z} E[X\hat{Z}].$$

The relation (F1) is stated and proved in [7] lemma 4.55 for the case $X \in L^1$ and $Z \in L^\infty$. This result in turn can be easily extended to $X \in L^p$ and $Z \in L^{p*} \cap L^1$ by suitable approximation, so we omit a proof here.

Proof of theorem 2.2. Since the case $p = \infty$ is trivial, we assume henceforth that $p \in [1, \infty)$. According to lemma 2.4 any law-invariant closed convex function $f : L^\infty \rightarrow [-\infty, \infty]$ is $\sigma(L^\infty, L^\infty)$ -closed. Hence, \bar{f}^p equals f on L^∞ . Applying (F1) we observe that $f^*|_{L^1}$ is law-invariant, because for all $Z \in L^1$:

$$\begin{aligned} f^*(Z) &= \sup_{X \in L^\infty} E[XZ] - f(X) = \sup_{X \in L^\infty} \left(\sup_{\hat{X} \sim X} E[\hat{X}Z] \right) - f(X) \\ &= \sup_{X \in L^\infty} \int_0^1 q_X(s)q_Z(s) ds - f(X) \end{aligned}$$

in which the latter expression depends on the law of Z only. Thus $(\bar{f}^p)^* = f^*|_{L^{p^*}}$ is law-invariant. Another application of (F1) similar to the one above yields the law-invariance of \bar{f}^p . In order to prove that \bar{f}^p is the *unique* law-invariant closed convex extension of f to L^p , let g be any such extension. For every $X \in L^p$ and all $m \in \mathbb{N}$ there exists a finite partition A_1^m, \dots, A_n^m of Ω such that the L^p -distance between X and the simple random variable $X_m := E[X | \sigma(A_1^m, \dots, A_n^m)] \in L^\infty$ is less than $1/m$. On the one hand, corollary 4.59 in [7] (in combination with lemma 2.4) states that $g(X_m) \leq g(X)$ for all $m \in \mathbb{N}$. On the other hand, by l.s.c. of g , we know that $g(X) \leq \liminf_{m \rightarrow \infty} g(X_m)$. Hence, $g(X) = \lim_{m \rightarrow \infty} g(X_m)$. Since the latter observation in particular holds for \bar{f}^p , we obtain

$$g(X) = \lim_{m \rightarrow \infty} g(X_m) = \lim_{m \rightarrow \infty} f(X_m) = \lim_{m \rightarrow \infty} \bar{f}^p(X_m) = \bar{f}^p(X)$$

and uniqueness is proved. Finally, by letting $g = \bar{f}^1|_{L^p}$, it immediately follows that $\bar{f}^1|_{L^p} = \bar{f}^p$. \square

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