

# Optimal Capital and Risk Transfers for Group Diversification\*

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## Abstract

Diversification is at the core of insurance and other financial business. It constitutes an important issue in the preparation of the new Solvency II framework for the regulation of European insurance undertakings. In this paper, we propose a conceptual framework for a legally enforceable capital and risk transfer which optimally accounts for the designated group diversification benefits. We also provide a consistent valuation principle which is compatible with any prior valuation method. This makes our framework fully flexible and universally applicable. A first simple numerical example illustrates the practicability of our proposal.

**Key words:** Diversification under legal constraints, convex risk measures, optimal capital and risk transfer, existence of equilibrium

## 1 Introduction

Diversification is at the core of insurance and other financial business. It constitutes an important issue in the preparation of the new Solvency II framework for the regulation of European insurance undertakings. For the industry it is vital to shift the focus from the capital requirements for individual business units to a group level.

But for diversification to work at a business group level, capital needs to flow freely between business units (fungibility). Regulators and local companies' management may constrain this fungibility unless there is some standardization for *capital and risk transfers (C&R-transfers)*. Standard methods, such as the covariance method, fail to take these vital aspects into account. This fact is

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acknowledged by regulators and the industry, as a recent document ([7]) of the European Federation of the National Insurance Associations (CEA) underlines: “Diversification is an area where further research and analysis will be required”.

In this paper, we propose a bottom-up<sup>1</sup> framework for a *legally enforceable C&R transfer* which optimally accounts for the designated top-down<sup>2</sup> group diversification benefits. We also provide a consistent valuation principle which is compatible with any prior valuation principle. This makes our framework fully flexible and universally applicable.

We consider an insurance group structured in business units. In a first step, a common set of legally enforceable C&R transfer instruments is identified. The risk management’s task is then to find an optimal C&R transfer across the business units which minimizes the regulatory group capital requirement. Unlike in many economic optimization problems, we have here a well defined and objectively known target function: the sum of regulatory specified local risk measures.

We provide sufficient conditions which guarantee the existence of such an optimal C&R transfer. It is shown that an optimal C&R transfer minimizes the group’s required capital while leaving the group’s available capital invariant. This invokes a valuation principle which is compatible with any prior valuation method. We then distinguish a particular optimal (“equilibrium”) C&R transfer which does not affect the business units’ individual available capitals, and which is fair in the sense that no lower than the group level of diversification can overturn the diversification benefit of the entire group. Due to the bottom-up approach, an extra capital allocation step is not necessary. In fact, in the context of the optimized capital and risk structure, the allocated capital is just given by the individual business unit’s required capital.

We also provide a simple numerical example to illustrate the practicability of our framework. The Solvency II framework envisages two levels of capitalization: the simple rules-based Minimum Capital Requirement (MCR) and the risk-sensitive principles-based Solvency Capital Requirement (SCR). The SCR should be interpreted as target capital level and reflects the company’s risk profile, while the MCR is a strict minimum level, below which the company is considered as insolvent. It is understood that, if the SCR is nevertheless calculated at the business unit level, the group’s SCR should be the aggregation of the units’ SCR taking into account the group diversification effects. We thus propose a bottom-up approach, where surplus capital exceeding MCR can be shared among the business units, while the group’s SCR is given as sum of the individual SCRs. It turns out that, under an optimal C&R transfer, the diversification benefits are essentially the same as designated by the usual (unrealizable) top-down approach.

For more background on the Solvency II framework we refer to the official web page of the European Commission, Section: Internal Market – Financial

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<sup>1</sup>“Bottom-up” means based on the risk assessments on a business unit level.

<sup>2</sup>“Top-down” means based on a (simplified) aggregated risk assessment on the group entity level.

Services – Insurance – Solvency [13], or the benchmarking study of internal models of the CRO Forum [17].

Our proposal is linked to the theory of equilibrium for financial markets and optimal risk exchange based on convex risk measures. In our framework, an optimal C&R transfer is Pareto optimal and can always be brought into an equilibrium by rebalancing the cash. There is a vast literature on equilibrium theory for financial markets. We just refer to the textbooks [9, 12] and the references therein. Convex risk measures have been the topic in [1, 10, 16, 18, 19, 20, 22]. Optimal risk exchange in an expected utility framework is explored in [4, 5], and based on convex risk measures in [2, 3, 6, 11, 15, 19, 21, 23]

The remainder of the paper is as follows. In Section 2 we introduce the formal framework and the diversification problem. In Section 3 we sketch the usual top-down methods and discuss some fallacy and shortcoming. This includes the covariance method in particular. In Section 4 we propose our modified bottom-up approach, which is then formalized in Section 5. The optimal  $C&R$  transfers are characterized, and we provide a consistent valuation principle. In Section 6 we distinguish a particular optimal  $C&R$  transfer which is fair in a game-theoretic sense and leaves the business units' prior available capitals invariant. In Section 7 we provide sufficient conditions for the existence of optimal  $C&R$  transfers. It turns out that, in practice, these conditions are usually satisfied (Corollary 7.2). On the other hand, we give an example for non-existence in Section 7.1. Section 8 contains a first example which illustrates the practicability of our framework. We conclude in Section 9. Section A contains some notation, definitions and facts from convex analysis, which are used throughout the text. To facilitate the reading, the proofs of all theorems are postponed to Sections B–E in the appendix.

## 2 Capital Requirements under Diversification

We consider an insurance group with  $m$  business units. Values at the beginning of the accounting year are deterministic and denoted by small letters. Values at the end of the accounting year are random and denoted by capital letters. We model this randomness, or risk, with some linear space  $E$  of random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . For example,  $E = L^p(\Omega, \mathcal{F}, \mathbb{P})$  for some  $p \in [0, \infty]$ . By convention, we identify  $X$  and  $Y$  in  $E$  if  $X = Y$  a.s. The riskiness of a portfolio  $X \in E$  is assessed by means of a *convex risk measure*, which is a map  $\rho : E \rightarrow (-\infty, +\infty]$  satisfying

- (i) monotonicity:  $\rho(X) \leq \rho(Y)$  if  $X \geq Y$  a.s.
- (ii) convexity:  $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$  for  $\lambda \in [0, 1]$ ;
- (iii) cash invariance:  $\rho(X + r) = \rho(X) - r$  for  $r \in \mathbb{R}$ .

Property (iii) implies the standing assumption that all current and future values are expressed in terms of some fixed numeraire (“cash”), and that the constant 1 is an element of  $E$ . If, in addition,  $\rho$  is positively homogeneous then it is called a

*coherent risk measure.* A coherent risk measure thus satisfies the sub-additivity property

$$\rho(X + Y) \leq \rho(X) + \rho(Y). \quad (1)$$

We define the *available capital* of business unit  $i$  as the value of its asset-liability portfolio. The available capital of business unit  $i$  at the beginning and the end of the accounting year is denoted by  $c_i \in \mathbb{R}$  and  $C_i \in E$ , respectively. This definition implies the assumption of a valuation principle  $V : E \rightarrow \mathbb{R}$  such that

$$c_i = V(C_i) \quad (2)$$

(see (PF5) and Remark 4.3 below). The available capital depends on the selection of liabilities to be covered and the assets backing these liabilities. That is, there may be off-balance sheet positions that are not considered for this assessment, see [17] for further details. We assume here that for each business unit the relevant portfolio is determined.

Business unit  $i$  uses a convex risk measure  $\rho_i$  to quantify its *required capital*

$$k_i = c_i + \rho_i(C_i). \quad (3)$$

This risk measure may be exogenously specified by the local regulator. The asset-liability portfolio is considered as acceptable if  $c_i \geq k_i$ .

**Remark 2.1.** The required capital as an indicator for the risk profile has to be considered with respect to the available capital. Indeed, suppose the available capital of the company is increased by adding assets to its portfolio. In absolute terms, this certainly improves the financial strength for backing the liabilities. And yet, due to the riskiness of the additional assets, the required capital increases too. Hence optimizing the risk profile subject to regulatory requirements amounts to minimize the difference between required and available capital. This approach is taken up below.

The objective of the group is to account for *diversification* effects across the business unites, which results in an aggregate group required capital,  $k_{\text{group}}$ , being less than the sum of the stand-alone business units' required capitals

$$k_{\text{group}} \leq k_1 + \dots + k_m.$$

It is understood that the group required capital  $k_{\text{group}}$  is allocated to the business units according to their (marginal) risk contributions:

$$k_{\text{group}} = \hat{k}_1 + \dots + \hat{k}_m, \quad (4)$$

where  $\hat{k}_i$  denotes the capital allocated to business unit  $i$ , see (7) for an example. We remark that there exists no distinguished allocation method. For more background and references on capital allocation methods we refer to Section 6.3 in [24]. See also Theorem 6.3 below.

### 3 Current Industry Practice

The current usual approach to the above diversification problem is based on the following assumptions.

#### Usual Framework

- (UF1) There is one coherent risk measure  $\rho_i \equiv \rho$  for all business units.
- (UF2) The aggregate group risk profile is given by the sum of stand alone risk profiles  $C := C_1 + \dots + C_m$ .
- (UF3) The group required capital accordingly is

$$k_{\text{group}} = \sum_{i=1}^m c_i + \rho(C_1 + \dots + C_m). \quad (5)$$

- (UF4) Diversification is then expressed through coherence of  $\rho$ , see (1),

$$k_{\text{group}} = \sum_{i=1}^m c_i + \rho(C_1 + \dots + C_m) \leq \sum_{i=1}^m (c_i + \rho(C_i)) = \sum_{i=1}^m k_i.$$

- (UF5) An exogenous allocation method is applied to determine  $\hat{k}_i$  in (4).

**Remark 3.1.** Observe that

$$\inf_{X_i \in E: \sum_{i=1}^m X_i = C} \sum_{i=1}^m \rho(X_i) = \sum_{i=1}^m \rho(\lambda_i C) = \rho(C) \quad (6)$$

for all  $\lambda_i \in [0, 1]$  with  $\sum_{i=1}^m \lambda_i = 1$ . In other words, (5) amounts to pooling the entire risk,  $C = C_1 + \dots + C_m$ , and redistribute  $C$  optimally across the business units,  $\sum_{i=1}^m X_i = C$ . Due to the coherence of  $\rho$ , every convex risk sharing of the form  $(\lambda_1 C, \dots, \lambda_m C)$  is optimal in that sense, and  $\rho(\lambda_i C)$  add up to  $\rho(C)$ .

A common industry practice is the *covariance method*, which assumes, in addition to (UF1)–(UF5), that  $C_1, \dots, C_m$  are jointly normal distributed and  $\rho$  is law-invariant. That is,  $\rho(X) = \rho(Y)$  whenever  $X$  and  $Y$  have the same law. Since any linear combination  $X$  of  $C_1, \dots, C_m$  is normal distributed, we conclude that  $\rho$  is linear in the standard deviation of  $X$ :

$$\rho(X) = \rho(\sqrt{\text{Var}[X]}Z + \mathbb{E}[X]) = \kappa\sqrt{\text{Var}[X]} - \mathbb{E}[X],$$

where  $Z$  is a standard normal distributed random variable and  $\kappa = \rho(Z)$ . This holds for the expected shortfall, *ES*, as well as for the value-at-risk, *VaR*, if  $E$  is the linear span of the constant 1 and  $C_1, \dots, C_m$ . The covariance capital allocation method is given by

$$\hat{k}_i = c_i + \frac{d}{d\epsilon} \rho(C + \epsilon C_i)|_{\epsilon=0} = c_i + \kappa \frac{\text{Cov}[C_i, C]}{\sqrt{\text{Var}[C]}} - \mathbb{E}[C_i]. \quad (7)$$

For more background on the covariance method we refer to Section 6.3 in [24].

The CRO Forum [8] distinguishes between 4 different levels of diversification. Levels 1 and 2 are within and across risk types within business units, respectively. Level 3 is across business units which are grouped within the same geographical zone, and Level 4 is across the entire group. However, it is a *fallacy* to expect a higher benefit for a single business unit when the level of diversification is increased, as the following example shows.

**Example 3.2.** Consider a group with 3 business units where the first two business units are located in country *A* and the third is located in country *B*. Let  $\kappa = 1$ ,  $c_i = E[C_i]$  and  $\text{Var}[C_i] = 100$ , such that the stand-alone capital requirement per business unit is  $k_i = 10$ , for all  $i$ . Let the correlation matrix for  $C_1, C_2, C_3$  be

$$\begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

Level 3 diversification within country *A* yields an allocated capital for business unit 1 of

$$\frac{100}{\sqrt{100 + 100}} = \sqrt{50} \approx 7.07,$$

whereas level 4 diversification across all three business units yields a higher capital

$$\frac{100 + 100}{\sqrt{100 + 100 + 100 + 2 \times 100}} = \sqrt{80} \approx 8.94.$$

That is, risk aggregation with business unit 3 penalizes business unit 1, which is due to their high correlation.

We will provide a game theoretic view of the “fairness” of a capital allocation method below in Section 6.

Our main concern, though, about this usual framework is the following: the extent to which the compensatory effects on the summed aggregate risk,  $C = C_1 + \dots + C_m$ , can be realized depends on the capital mobility between the business units. Consider, for instance, a group with 3 business units where the first two business units are located in a country *A* and the third is located in a separate country *B*. There may be states  $\omega \in \Omega$  where  $C_3(\omega) < 0$  while  $C_1(\omega) + C_2(\omega) > 0$  and  $C_1(\omega) + C_2(\omega) + C_3(\omega) > 0$ . That is, the part of the insurance group located within country *A* is solvent, whereas business unit 3 stand alone would be insolvent. Only if capital can flow freely from country *A* to *B*, the positive net value of the group can be realized. Regulators and local companies’ management may constrain this fungibility unless there is some **standardization for C&R transfers**. In other words, the risk sharing described in Remark 3.1 cannot be realized in practice.

## 4 Standardized C&R Transfers

In this section and the following we propose a modified framework which is capable to account for the above mentioned problem.

Standardized C&R transfers must consist of legally enforceable contingent capital instruments. A variety of such instruments exist to facilitate capital mobility: intra-group reinsurance, intra-group lending, securitization of future cash flows and earnings, issuance of surplus notes, etc. Responding to the preceding observations we propose the following bottom-up approach.

### Proposed Framework

- (PF1) Every  $\rho_i : E \rightarrow (-\infty, +\infty]$  is a convex risk measure, with  $\rho_i(C_i) < \infty$ .
- (PF2) There exists a well-specified finite set of **legally enforceable C&R transfer instruments** with future contingent values modelled by some linearly independent random variables  $Z_0, Z_1, \dots, Z_n$  in  $E$ .
- (PF3) **Cash** (the numeraire) **is fungible** between business units as long as the payments at the end are determined at the beginning of the accounting year. This is expressed by letting  $Z_0 \equiv 1$ .
- (PF4) The modified risk profile of business unit  $i$  becomes

$$C_i + \sum_{j=0}^n x_i^j Z_j, \quad i = 1, \dots, m, \quad (8)$$

for some *feasible C&R transfer*

$$x_i = (x_i^0, \dots, x_i^n) \in W_i, \quad i = 1, \dots, m, \quad (9)$$

such that

$$\sum_{i=1}^m \sum_{j=0}^n x_i^j Z_j \leq 0 \quad \text{a.s.} \quad (10)$$

where  $W_i$  is some closed convex subset in  $\mathbb{R}^{n+1}$  with

$$0 \in W_i \quad \text{and} \quad W_i + (r, 0, \dots, 0) = W_i \quad \forall r \in \mathbb{R}, \quad i = 1, \dots, m. \quad (11)$$

That is, there is no exogenous value added. **The modified risk profile is solely due to a feasible redistribution of capital and risk by means of the instruments  $Z_0, \dots, Z_n$ .** Compare this to Remark 3.1.

- (PF5) The aggregate group required capital is then

$$k_{\text{group}} = c_{\text{group}} + \sum_{i=1}^m \rho_i \left( C_i + \sum_{j=0}^n x_i^j Z_j \right), \quad (12)$$

where the aggregate group available capital

$$c_{\text{group}} := \sum_{i=1}^m V(C_i) + \sum_{i=1}^m \sum_{j=0}^n x_i^j V(Z_j)$$

is given by some appropriate linear valuation principle  $V : E \rightarrow \mathbb{R}$ .

**(PF6)** The objective of the group (see Remark 2.1) is to minimize the difference between required and available capital, hence to find an optimal C&R transfer which solves the optimization problem

$$\min_{(x_1, \dots, x_m)} \sum_{i=1}^m \rho_i \left( C_i + \sum_{j=0}^n x_i^j Z_j \right) \quad (13)$$

subject to the feasibility and clearing conditions (9) and (10), respectively. Compare this to the unconstrained version (6).

**Remark 4.1.** In view of (8), the linear independence assumption in (PF2) is no essential loss of generality, since otherwise one could simply reduce the number of instruments.

**Remark 4.2.** The unlimited cash fungibility assumption (PF3) and (11) may be subject to criticism. However, in all examples that we have encountered, the resulting net cash flow was small, see (29) and (43) below. Moreover, the present framework is far more realistic than the usual one where full fungibility of all contingent capital is assumed.

**Remark 4.3.** (PF5) implies the assumption that current asset-liability portfolio values (available capitals) are fully fungible and add up to the group available capital. Again, this assumption is debatable, but in line with the usual framework (UF3).

**Remark 4.4.** We do not assume that  $W_i$  contains an open neighborhood of 0. Hence there may be instruments  $Z_j$  that are not feasible for business unit  $i$ . See, however, assumption (PF7) below.

## 5 Optimal C&R Transfers

We now formalize the proposed framework and introduce the functions  $v_i(x) := \rho_i \left( C_i + \sum_{j=0}^n x^j Z_j \right)$  and

$$u_i(x) := v_i(x) + \delta(x | W_i) = \begin{cases} \rho_i \left( C_i + \sum_{j=0}^n x^j Z_j \right), & \text{if } x \in W_i, \\ +\infty, & \text{else.} \end{cases} \quad (14)$$

for  $x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1}$  and  $i = 1, \dots, m$ , where the indicator function  $\delta(\cdot | W_i)$  is defined in (46). For handling the clearing condition (10), we define



the following partial order on  $\mathbb{R}^{n+1}$

$$x \succeq y \quad :\Leftrightarrow \quad \sum_{j=0}^n x^j Z_j \geq \sum_{j=0}^n y^j Z_j \quad \text{a.s.} \quad (15)$$

Notice that  $\succeq$  is reflexive ( $x \succeq x$ ), transitive ( $x \succeq y$  and  $y \succeq z$  imply  $x \succeq z$ ) and antisymmetric ( $x \succeq y$  and  $y \succeq x$  imply  $x = y$ ), due to the linear independence of  $Z_0, \dots, Z_n$ .

**Definition 5.1.** We call a feasible C&R transfer  $(x_1, \dots, x_m) \in \prod_{i=1}^m W_i$  attainable if (10) is satisfied, which is equivalent to

$$0 \succeq \sum_{i=1}^m x_i.$$

As a consequence, we can express the constrained optimization problem (13), subject to (9) and (10), as follows

$$\inf_{0 \succeq \sum_{i=1}^m x_i} \sum_{i=1}^m u_i(x_i). \quad (16)$$

We next derive some basic properties of the functions  $u_i$ . We write  $e_0, \dots, e_n$  for the standard basis in  $\mathbb{R}^{n+1}$ ; that is,  $e_0 = (1, 0, \dots, 0)$ ,  $e_1 = (0, 1, 0, \dots, 0)$ , etc. The scalar product is denoted by  $x \cdot y := x^0 y^0 + \dots + x^n y^n$ . It follows by inspection that

$$\mathcal{P} := \{q \in \mathbb{R}^{n+1} \mid q \cdot x \leq 0 \ \forall x \succeq 0\} \quad (17)$$

is a closed convex cone. So is its *polar cone*

$$\mathcal{P}^\circ := \{x \in \mathbb{R}^{n+1} \mid q \cdot x \leq 0 \ \forall q \in \mathcal{P}\}.$$

The Bipolar Theorem (Theorem 14.1 in [25]) states that

$$x \succeq 0 \quad \Leftrightarrow \quad x \in \mathcal{P}^\circ.$$

We will make the following assumptions:

**(PF7)**  $W_1 + \dots + W_m$  contains an open neighborhood of 0, and

**(PF8)**  $W_i \subset \text{ri}(\text{dom } v_i)$ , for  $i = 1, \dots, m$ .

These assumptions are technical and facilitate the subsequent analysis as the following lemma indicates ((PF7) is used in the proof of (i) $\Rightarrow$ (iii) in Theorem 5.4). From the applied point of view they are no great loss of generality. Indeed, (PF7) means that every instrument  $Z_j$  must be feasible for at least one business unit (otherwise omit  $Z_j$ ) and must be provided from within the group (short and long positions in  $Z_j$  are feasible, but not necessarily for the same business unit). Assumption (PF8) asserts that  $\text{dom}(v_i)$  is large enough to allow for (PF7). If (PF8) were violated then we regard the instruments  $Z_j$  as not being compatible with the given risk measures  $\rho_i$  and profiles  $C_i$ . In that case, other C&R transfer instruments must be chosen.

**Lemma 5.2.** *Every  $u_i : \mathbb{R}^{n+1} \rightarrow (-\infty, +\infty]$  is a convex function that is continuous on  $W_i$  and satisfies*

$$\text{dom}(u_i) = W_i, \quad (18)$$

$$u_i(x) \leq u_i(y) \quad \forall x \succeq y, \quad x, y \in W_i \quad (\text{local monotonicity}) \quad (19)$$

$$u_i(x + re_0) = u_i(x) - r \quad \forall r \in \mathbb{R} \quad (\text{cash-invariance}). \quad (20)$$

Moreover, we have

$$\emptyset \neq \partial u_i(x) \subset \{q \in \mathbb{R}^{n+1} \mid q^0 = -1\} \quad \forall x \in W_i. \quad (21)$$

If, in addition,  $\mathcal{P}^\circ \subseteq W_i$  then also

$$\partial u_i(x) \subset \mathcal{P} \quad \forall x \in W_i. \quad (22)$$

*Proof.* Convexity of  $u_i > -\infty$  and properties (18)–(20) follow from the respective properties of  $\rho_i$  and  $W_i$ . Since  $v_i$  is continuous on the relative interior of its effective domain (see Theorem 10.1 in [25]) and in view of (PF8), we conclude that  $u_i$  is continuous on  $W_i$ . Moreover,  $\partial u_i(x) = \partial v_i(x) + \partial \delta(x \mid W_i) \supset \partial v_i(x) + 0 \neq \emptyset$  for all  $x \in W_i$  (see Theorems 23.4 and 23.8 in [25]).

Now let  $q \in \partial u_i(x)$ . Then

$$rq \cdot e_0 \leq u_i(x + re_0) - u_i(x) = -r \quad \forall r \in \mathbb{R},$$

hence  $q \cdot e_0 = -1$ , which is (21). If, in addition,  $\mathcal{P}^\circ \subseteq W_i$  then  $x \in W_i$  implies  $x + z = \lim_{k \rightarrow \infty} \left( \frac{k-1}{k}x + \frac{1}{k}kz \right) \in W_i$ , for all  $z \succeq 0$ . Therefore

$$q \cdot z \leq u_i(x + z) - u_i(x) \leq 0 \quad \forall z \succeq 0,$$

hence  $q \in \mathcal{P}$ , and the lemma is proved.  $\square$

By definition, the available capital,  $c_i$ , is the value of the asset-liability portfolio. Hence adding long (assets) or short (liabilities) positions in the instruments  $Z_0, \dots, Z_n$  to the portfolio (8) also changes the available capital. To determine the available and required capital therefore one needs to know the value of adding positions in  $Z_0, \dots, Z_n$ .

According to (PF5), we assume that such value is given by a linear indifference valuation principle as follows. Let  $x_i \in \mathbb{R}^{n+1}$  represent the portfolio (8) of business unit  $i$ . We call an *indifference valuation principle for business unit  $i$  with respect to  $x_i$*  any linear functional  $V : E \rightarrow \mathbb{R}$  such that adding positions  $z \in \mathbb{R}^{n+1}$  to  $x_i$  is less optimal (that is, requires more capital) than adding the value equivalent cash amount of

$$\sum_{j=0}^n z^j V(Z_j) = p \cdot z,$$

where the value vector  $p = p(V) \in \mathbb{R}^{n+1}$  is defined as  $p^j := V(Z_j)$ . Formally, this means

$$u_i(x_i + z) \geq u_i(x_i + (p \cdot z)e_0) \quad \forall z \in \mathbb{R}^{n+1}. \quad (23)$$

There is a strict correspondence between such indifference valuation principles and the subgradients of  $u_i$  at  $x_i$ :

**Lemma 5.3.**  $V$  is an indifference valuation principle for business unit  $i$  with respect to  $x_i$  if and only if  $-p \in \partial u_i(x_i)$ . In particular, we then have

$$p \cdot e_0 = p^0 = 1. \quad (24)$$

Hence the value of a unit of cash is one.

*Proof.* Follows from (23), in view of the cash-invariance property (20) and (21).  $\square$

Notice that  $p$  depends on  $i$  and  $x_i$ . Consistent valuation across the business units therefore can only take place at C&R transfers  $(x_1, \dots, x_m)$  where

$$\partial u_1(x_1) \cap \dots \cap \partial u_m(x_m) \neq \emptyset.$$

It turns out that this is just the first order condition for the optimization problem (16).

**Theorem 5.4.** Let  $(\hat{x}_1, \dots, \hat{x}_m) \in \prod_{i=1}^m W_i$  be an attainable C&R transfer. The following are equivalent:

- (i)  $(\hat{x}_1, \dots, \hat{x}_m)$  is a minimizer for (16).
- (ii)  $(\hat{x}_1, \dots, \hat{x}_m)$  is Pareto optimal, in the sense that: for any attainable C&R transfer  $(x_1, \dots, x_m)$  which satisfies  $u_i(x_i) \leq u_i(\hat{x}_i)$  for all  $i$ , we have  $u_i(x_i) = u_i(\hat{x}_i)$  for all  $i$ .
- (iii) There exists a consistent valuation principle  $p = p(V) \in \mathbb{R}^{n+1}$  with

$$-p \in \mathcal{P} \quad (\text{positivity}) \quad (25)$$

$$-p \in \bigcap_{i=1}^m \partial u_i(\hat{x}_i) \quad (\text{first order condition}) \quad (26)$$

$$p \cdot \sum_{i=1}^m \hat{x}_i = 0 \quad (\text{value clearing}). \quad (27)$$

- (iv)  $(\hat{x}_i^j, i = 1, \dots, m, j = 1, \dots, n) \in \mathbb{R}^{m \times n}$  is a minimizer for the unconstrained  $m \times n$ -dimensional convex optimization problem

$$\inf_{(x_i^j) \in \mathbb{R}^{m \times n}} \left( \sum_{i=1}^m u_i(0, x_i^1, \dots, x_i^n) + \text{ess sup} \sum_{i=1}^m \sum_{j=1}^n x_i^j Z_j \right), \quad (28)$$

and the optimal net cash flow satisfies

$$\sum_{i=1}^m \hat{x}_i^0 = -\text{ess sup} \sum_{i=1}^m \sum_{j=1}^n \hat{x}_i^j Z_j. \quad (29)$$

Notice that (25) is equivalent to  $p \cdot z \geq 0 \forall z \succeq 0$ .

We will give sufficient conditions for the existence of a minimizer for (16) below in Section 7. Suppose, for the moment, that an optimal C&R transfer  $(\hat{x}_1, \dots, \hat{x}_m)$  exists and let  $p$  be of the form (25)–(27). Then the modified asset-liability portfolio of business unit  $i$  becomes

$$C_i + \sum_{j=0}^n \hat{x}_i^j Z_j. \quad (30)$$

For the valuation of this modified portfolio, we now assume that the initial asset-liability portfolio,  $C_i$ , is more diverse than any portfolio consisting solely of the instruments  $Z_0, \dots, Z_n$ . We believe that this is a realistic assumption, which can be expressed by an arbitrarily high (e.g. infinite) dimension of the model space  $E$ . See for an example Section 8. Formally, this means

**(PF9)**  $C_i$  does not lie in the linear span of  $Z_0, \dots, Z_n$ , for all  $i = 1, \dots, m$ .

As a consequence, any indifference valuation principle  $V$ , which is characterized by Lemma 5.3 on the linear span of  $Z_0, \dots, Z_n$ , can be freely specified at  $C_i$ . In other words, the valuation of the C&R transfer  $x_i$  can be made consistent with any prior valuation principle for  $C_i$  by setting

$$V(C_i) := c_i. \quad (31)$$

Hence it is enough to specify  $p$ , as it is done in Theorem 5.4.

In view of (31), the value of (30) (=the modified available capital) is

$$\hat{c}_i = c_i + p \cdot \hat{x}_i, \quad (32)$$

and the modified required capital becomes

$$\hat{k}_i = \hat{c}_i + u_i(\hat{x}_i). \quad (33)$$

In view of (27), the group required capital is then

$$k_{\text{group}} = \sum_{i=1}^m \hat{k}_i = \sum_{i=1}^m c_i + \sum_{i=1}^m u_i(\hat{x}_i). \quad (34)$$

Notice that  $\hat{k}_i$  obtained in (33) is the capital allocated to business unit  $i$ , as outlined in (4). Due to our bottom-up approach we do not need an exogenous capital allocation method as in (UF5).

Theorem 5.4 states that the optimal C&R transfer  $(\hat{x}_1, \dots, \hat{x}_m)$  necessarily clears in value (27). However, as we shall see in the example in Section 8, it is possible that  $\sum_{i=1}^m \hat{x}_i \neq 0$ . Thus it may be optimal for the group to “throw away” the non-trivial remainder portfolio  $\hat{x}_0 := -\sum_{i=1}^m \hat{x}_i$ , which bears no downside risk,  $\hat{x}_0 \succeq 0$ ! This includes the net cash flow (29) in particular.

To explain this seeming paradox, we have to distinguish between the current value of a portfolio, its required capital and its realized value at the future reference date.

Indeed, the current value of the portfolio  $\hat{x}_0$  is zero,  $p \cdot \hat{x}_0 = 0$ , and it requires no regulatory capital charge since it bears no downside risk,  $\hat{x}_0 \succeq 0$ . As for its realized value, consider a state of the world  $\omega \in \Omega$ . The business units first realize the values  $C_i(\omega)$  corresponding to their initial asset-liability portfolios. According to the legally enforceable C&R transfer  $(\hat{x}_1, \dots, \hat{x}_m)$ , these values are then reallocated across the business units. The net remainder of this reallocation is  $\sum_{j=0}^n \hat{x}_0^j Z_j(\omega) \geq 0$ , which has to be shared with somebody in order that the C&R transfer actually clears.

**(PF10)** We assume that the net remainder portfolio,  $\hat{x}_0$ , can be transferred to some third party, such as the holding company of the group, or the shareholders.

This assumption is justified by the fact that in our context, the remainder portfolio,  $\hat{x}_0$ , does not interfere with the current balance sheet of the group, since  $p \cdot \hat{x}_0 = 0$  and  $\hat{x}_0 \succeq 0$ . On the other hand, it allows the group to realize a legally enforceable optimal C&R transfer which minimizes the regulatory capital requirements. Furthermore, there is a third party, e.g. the holding company or the shareholders, which can assume a non-trivial position without any downside risk. A genuine win-win situation!

Note, however, that the remainder portfolio  $\hat{x}_0$  always satisfies

$$\text{ess inf} \sum_{j=0}^n \hat{x}_0^j Z_j = 0, \quad (35)$$

which is a consequence of (29). Hence there is no sure strict positive gain for the holder of  $\hat{x}_0$ .

**Remark 5.5.** In fact, if for some business unit there are no feasibility constraints, e.g.  $W_1 = \mathbb{R}^{n+1}$ , then it is always possible to find a clearing C&R transfer among the optimal ones. Indeed, since  $\hat{x}_0 + \hat{x}_1 \in W_1$ , the remainder portfolio  $\hat{x}_0$  can be allocated to business unit 1. This does not alter its risk measure,  $u_1(\hat{x}_1) = u_1(\hat{x}_0 + \hat{x}_1)$ , since otherwise the optimality of  $(\hat{x}_1, \dots, \hat{x}_m)$  would be violated. Hence  $(\hat{x}_0 + \hat{x}_1, \hat{x}_2, \dots, \hat{x}_m) \in \prod_{i=1}^m W_i$  is indeed a clearing optimal C&R transfer, as  $\sum_{i=0}^m \hat{x}_i = 0$ .

On the other hand, if the feasibility constraints are strict,  $W_i \neq \mathbb{R}^{n+1}$ , then there may only exist non-clearing optimal C&R transfers. See [15] for an example.

## 6 Equilibrium C&R Transfers

We now distinguish a particular optimal C&R transfer which does not affect the business units' individual available capitals. We will then respond to the fallacy mentioned in Section 3.

First observe that, by the cash-invariance (20), there is no unique solution of (16). Indeed, let  $(\hat{x}_1, \dots, \hat{x}_m)$  be a minimizer for (16) and  $r_1, \dots, r_m \in \mathbb{R}$

with  $\sum_{i=1}^m r_i = 0$ . Then

$$(\bar{x}_1, \dots, \bar{x}_m) := (\hat{x}_1 + r_1 e_0, \dots, \hat{x}_m + r_m e_0) \quad (36)$$

is also attainable and a minimizer for (16), since  $\sum_{i=1}^m u_i(\bar{x}_i) = \sum_{i=1}^m u_i(\hat{x}_i)$ . With this rebalancing of cash we can find a particular optimal C&R transfer.

**Definition 6.1.** *An attainable C&R transfer  $(x_1, \dots, x_m)$  together with a valuation principle  $p \in \mathbb{R}^{n+1}$  of the form (24) and (25) is called an equilibrium C&R transfer if, for every  $i$ ,*

$$p \cdot x_i \leq 0 \quad \text{and} \quad u_i(x_i) = \inf_{p \cdot z \leq 0} u_i(z). \quad (37)$$

It turns out that an equilibrium C&R transfer is optimal and does not affect the business units' individual available capitals.

**Theorem 6.2.** *Let  $(\hat{x}_1, \dots, \hat{x}_m)$  and  $p$  be an equilibrium C&R transfer. Then  $(\hat{x}_1, \dots, \hat{x}_m)$  is a minimizer for (16), and  $p \cdot \hat{x}_i = 0$  for all  $i = 1, \dots, m$ .*

*Conversely, let  $(x_1, \dots, x_m)$  be a minimizer for (16) and  $p$  be a valuation principle of the form (25) and (26). Then*

$$(\hat{x}_1, \dots, \hat{x}_m) := (x_1 - (p \cdot x_1)e_0, \dots, x_m - (p \cdot x_m)e_0) \quad (38)$$

*and  $p$  form an equilibrium C&R transfer.*

The next theorem states that an equilibrium C&R transfer is fair in the sense that no sub-group (or “sub-level of diversification”) can overturn the diversification benefit of the entire group. This is a convenient type of “fairness” for capital allocation methods, which in some sense responds to the fallacy mentioned in Section 3. Note, however, that the effects on a single business unit level as described in Example 3.2 cannot be avoided.

**Theorem 6.3.** *An equilibrium C&R transfer  $(\hat{x}_1, \dots, \hat{x}_m)$  is fair in the sense that*

$$\sum_{i \in I} u_i(\hat{x}_i) \leq \inf_{0 \geq \sum_{i \in I} x_i} \sum_{i \in I} u_i(x_i)$$

*for every level of diversification  $I \subset \{1, \dots, m\}$ .*

*In this sense,  $\hat{k}_i = c_i + u_i(\hat{x}_i)$  is a fair capital allocation (4).*

**Remark 6.4.** In view of Theorems 6.2 and 6.3, we recommend equilibrium C&R transfers for optimizing the business structure of an insurance group.

## 7 Existence of Optimal C&R Transfers

In this section we provide sufficient conditions for the existence of optimal C&R transfers. Recall the definitions (47) and (48) of the recession cone  $0^+W_i$  and recession function  $u_i 0^+$ . In view of (14) and since  $0 \in \text{dom}(u_i)$ , we have

$$u_i 0^+(x) = \lim_{\lambda \downarrow 0} \lambda v_i(\lambda^{-1}x) + \delta(x \mid 0^+W_i). \quad (39)$$

Here is our main existence result.

**Theorem 7.1.** *Suppose that  $u_i 0^+ = u_1 0^+ =: v$  for all  $i = 1, \dots, m$ , and that for every  $x \neq 0$  with  $0 \succeq x$  there exists some  $q \in \text{dom}(v^*)$  such that  $q \cdot x > 0$ . Then there exists an optimal C&R transfer.*

The following version is most useful for applications. It assumes, in particular, that the risk measures  $\rho_i$  are coherent and coincide on the linear span of  $Z_0, \dots, Z_n$ .

**Corollary 7.2.** *Suppose that  $0^+ W_i \equiv 0^+ W_1$  and  $\rho_i$  are coherent risk measures satisfying*

$$\lim_{\lambda \downarrow 0} \rho_i \left( \lambda C_i + \sum_{j=0}^n x^j Z_j \right) = \rho_i \left( \sum_{j=0}^n x^j Z_j \right) \equiv \rho \left( \sum_{j=0}^n x^j Z_j \right) \quad \forall x \in \mathbb{R}^{n+1},$$

for some risk measure  $\rho$  which is relevant with respect to  $Z_0, \dots, Z_n$ :

$$\rho(X) > 0 \quad \text{for all } X = \sum_{j=0}^n x^j Z_j \leq 0 \text{ with } \mathbb{P}[X < 0] > 0.$$

Then there exists an optimal C&R transfer.

*Proof.* From the assumptions and (39) we deduce  $u_i 0^+(x) \equiv \rho \left( \sum_{j=0}^n x^j Z_j \right) + \delta(x \mid 0^+ W_1) =: v(x)$ . Let  $x \neq 0$  with  $0 \succeq x$ . Then  $X := \sum_{j=0}^n x^j Z_j$  satisfies  $X \leq 0$  and  $\mathbb{P}[X < 0] > 0$ . Thus  $v(x) = \rho(X) + \delta(x \mid 0^+ W_1) > 0$ . In view of (44), there must therefore exist some  $q \in \text{dom}(v^*)$  with  $q \cdot x > 0$ , and the corollary follows from Theorem 7.1.  $\square$

## 7.1 Example for Non-Existence

The following example is taken from [23]. Let  $D \geq 0$  be a random variable with  $\mathbb{E}[D] = 1$ ,  $\mathbb{P}[D = 0] > 0$ ,  $\mathbb{E}[D \log D] < \infty$ . We define  $E$  as the space of random variables  $X$  with  $\mathbb{E}[|DX|] < \infty$ , and define the convex risk measures  $\rho_i : E \rightarrow (-\infty, +\infty]$  as

$$\rho_1(X) := \log \mathbb{E}[\exp(-X)], \quad \rho_2(X) := -\mathbb{E}[DX].$$

The following inequality is fundamental for the relative entropy of  $D$ , see Lemma 3.29 [19],

$$\log \mathbb{E}[e^{-X}] + \mathbb{E}[DX] > -\mathbb{E}[D \log D] \quad \forall X \in E. \quad (40)$$

Equality in (40) could hold only if  $e^{-X} = D$ , which is impossible by the finiteness of  $X$ .

Consider  $m = 2$  business units with no feasibility constraints,  $W_1 = W_2 = \mathbb{R}^{n+1}$ . Let  $(x_1, x_2) \in \mathbb{R}^{2 \times (n+1)}$  be an admissible C&R transfer, and write  $Y(x) := \sum_{j=0}^n x^j Z_j$ . Then  $Y(x_1) + Y(x_2) \leq 0$ , and we obtain from (40)

$$\begin{aligned} u_1(x_1) + u_2(x_2) &= \log \mathbb{E}[e^{-(C_1 + Y(x_1))}] + \mathbb{E}[D(C_1 + Y(x_1))] \\ &\quad - \mathbb{E}[D(C_1 + C_2 + Y(x_1) + Y(x_2))] \\ &> -\mathbb{E}[D \log D] - \mathbb{E}[D(C_1 + C_2)]. \end{aligned} \quad (41)$$

Hence in particular  $\inf_{0 \geq x_1 + x_2} (u_1(x_1) + u_2(x_2)) > -\infty$ .

Suppose now there exists a sequence  $(x_p)$  in  $\mathbb{R}^{n+1}$  with

$$\lim_p \mathbb{E}[e^{-(C_1 + Y(x_p))}] = \mathbb{E}[D] = 1 \quad \text{and} \quad \lim_p \mathbb{E}[D(C_1 + Y(x_p))] = -\mathbb{E}[D \log D].$$

(Disregarding (PF9) for a moment, this holds, for instance, if the linear span of  $Z_0, \dots, Z_n$  lies dense in  $L^\infty$ .) Then

$$\begin{aligned} \inf_{0 \geq x_1 + x_2} (u_1(x_1) + u_2(x_2)) &= \lim_p (u_1(x_p) + u_2(-x_p)) \\ &= -\mathbb{E}[D \log D] - \mathbb{E}[D(C_1 + C_2)]. \end{aligned} \quad (42)$$

Hence, in view of (41) and (42), an optimal C&R transfer cannot exist.

## 8 Example

The European Solvency II framework envisages two levels of capitalization: the simple rules-based Minimum Capital Requirement (MCR) and the risk-sensitive principles-based Solvency Capital Requirement (SCR). The SCR should be interpreted as target capital level and reflects the company's risk profile, while the MCR is a strict minimum level, below which the company is considered as insolvent. In our model framework, the SCR can be identified with the required capital (3), while the MCR can be considered as some firm specific but exogenously given value.

We thus assume that every business unit  $i$  faces some  $MCR_i > 0$  which is specified by local legislation. As a first approach, it seems reasonable to assume that the local regulators will agree upon legal enforceability of contingent capital notes with payoffs of the form  $(C_i - MCR_i)^+$ . That is, business units can mutually share their excess capital beyond  $MCR_i$ . Hence we set  $n = m$ , the number of business units, and

$$Z_j := (C_j - MCR_j)^+, \quad j = 1, \dots, m.$$

We assume no feasibility constraints,  $W_i = \mathbb{R}^{n+1}$ , for all  $i = 1, \dots, m$ .

Unless the model is degenerate, (PF9) is satisfied. That is,  $C_i$  does not lie in the linear span of  $1, (C_1 - MCR_1)^+, \dots, (C_m - MCR_m)^+$ . This will allow for consistent valuation.

As a first simple numerical example, let  $\Omega = \{\omega_1, \dots, \omega_{100}\}$ ,  $\mathcal{F} = 2^\Omega$ , and  $\mathbb{P}[\omega_i] = 1/100$ . Furthermore,  $E = L^0 \equiv \mathbb{R}^{100}$ . We consider  $m = 2$  business units, and let  $\rho_1 = \rho_2 = \rho$  be the expected shortfall with confidence level 99%. That is,

$$\rho(X) = -\min_{\omega \in \Omega} X(\omega).$$

We model  $(C_1, C_2)$  as 100 sample points of a joint normal distribution with mean  $\mathbb{E}[C_i] = 100$  and covariance matrix  $\sigma^2 \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}$ , where  $\sigma$  is determined such

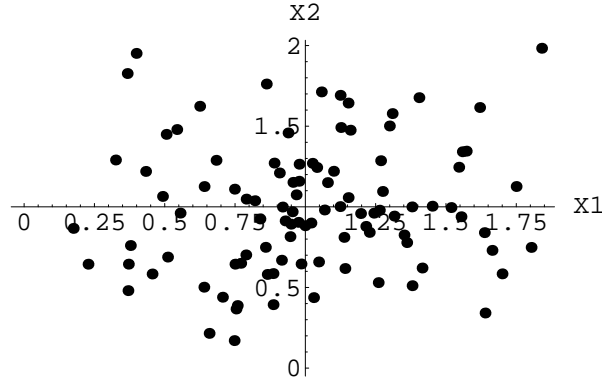


that — theoretically — available capital,  $c_i := \mathbb{E}[C_i]$ , equals required capital,  $k_i = c_i$ . That is,

$$0 = \rho(C_i) = \sigma\rho(N) - 100,$$

where  $N$  is standard normal distributed. It follows that  $\sigma = 100/\rho(N) = 37.52$ . The realized sample — that is, our model for  $(C_1, C_2)$  — is pictured in Figure 1.

Figure 1: Model for  $(X_1, X_2) = (C_1, C_2)/100$



For this sample model we obtain

$$k_1 = 100 + \rho(C_1) = 82.5, \quad k_2 = 100 + \rho(C_2) = 82.9.$$

The fully diversified group required capital via usual method (UF3) becomes

$$k_{\text{group,UF}} = 200 + \rho(C_1 + C_2) = 115.$$

We now let the MCRs be given as

$$MCR_1 := 0.4 \times k_1 = 33, \quad MCR_2 := 0.4 \times k_2 = 33.2.$$

This is in line with the Swiss Solvency Test parameters if we identify MCR with the risk margin to finance the regulatory capital for the run-off of the in force business. This risk margin is typically between 10%-40% of the required capital, see [14]. The C&R transfer instruments thus become

$$Z_0 = 1, \quad Z_1 = (C_1 - 33)^+, \quad Z_2 = (C_2 - 33.2)^+,$$

and the objective functions are

$$u_i(x_i) = \rho(C_i + \sum_{j=0}^2 x_i^j Z_j) = -\min_{\omega \in \Omega} (C_i(\omega) + x_i^1 Z_1(\omega) + x_i^2 Z_2(\omega)) - x_i^0.$$

The conditions of Corollary 7.2 are satisfied, and we find an optimal C&R transfer by solving (28) numerically (using Mathematica) of the form

$$(\hat{x}_1^1, \hat{x}_1^2) = (-0.58, 0.75), \quad (\hat{x}_2^1, \hat{x}_2^2) = (0.50, -0.81).$$

The optimal net cash flow is given by (29)

$$\sum_{i=1}^2 \hat{x}_i^0 = -\text{ess sup} \sum_{i=1}^2 \sum_{j=1}^2 \hat{x}_i^j Z_j = 1.22. \quad (43)$$

The subgradient  $p = -\nabla u_1(0, \hat{x}_1^1, \hat{x}_1^2) = -\nabla u_2(0, \hat{x}_2^1, \hat{x}_2^2)$  ( $u_i$  is numerically differentiable at  $(0, \hat{x}_i^1, \hat{x}_i^2)$ ) implies the valuation principle  $V$ ,

$$V(Z_0) = p^0 = 1, \quad V(Z_1) = p^1 = 4, \quad V(Z_2) = p^2 = 15,$$

while  $V(C_i) = c_i$ .

The equilibrium C&R transfer (38) is then  $\hat{x}_1 = (-8.93, -0.58, 0.75)$  and  $\hat{x}_2 = (10.15, 0.50, -0.81)$ , and the corresponding required capitals (33) become

$$\begin{aligned} \hat{k}_1 &= 100 + \rho(C_1 - 0.58(Z_1 - 4) + 0.75(Z_2 - 15)) = 63 \\ \hat{k}_2 &= 100 + \rho(C_2 + 0.50(Z_1 - 4) - 0.81(Z_2 - 15)) = 52. \end{aligned}$$

In sum

$$k_{\text{group}} = \hat{k}_1 + \hat{k}_2 = 115,$$

which is the same as the fully diversified group required capital,  $k_{\text{group,UF}}$ ! This does not always happen in general though. But it shows that the benefits of an optimal C&R transfer can be very close to the designated diversification benefits implied by the usual (unrealizable) method.

The 100 realizations of the remainder portfolio,

$$\hat{x}_0 = -\hat{x}_1 - \hat{x}_2 = (-1.22, 0.08, 0.06),$$

are plotted as a function of  $\omega \in \Omega$  in Figure 2 (to be scaled by 100). It illustrates well the fact that  $\hat{x}_0 \succeq 0$  while (35) holds.

Following Remark 5.5 we could allocate the remainder portfolio to e.g. business unit 1, in which case we obtain a clearing equilibrium C&R transfer

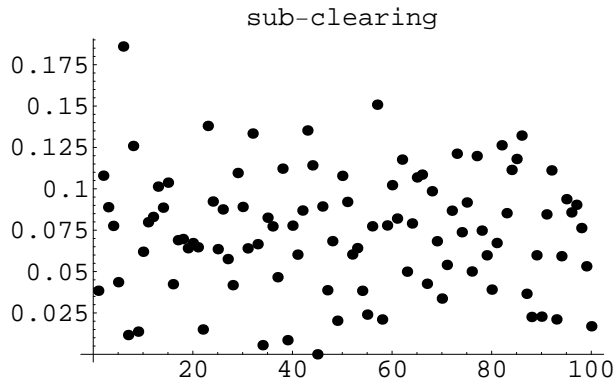
$$\bar{x}_1 = (-10.15, -0.50, 0.81), \quad \bar{x}_2 = (10.15, 0.50, -0.81).$$

This does neither alter  $\hat{k}_1$  nor  $\hat{k}_2$ , though. From the shareholder value point of view, the C&R transfer  $(\hat{x}_1, \hat{x}_2)$  is thus preferable.

## 9 Conclusion

We have provided a framework which allows to realizing the diversification benefits of an insurance group or other financial conglomerate. First, we have

Figure 2: The function  $\omega \mapsto \sum_{j=0}^2 \hat{x}_0^j Z_j(\omega)/100$



replaced the usual risk aggregation (5), which assumes full fungibility of capital, by means of some legally enforceable C&R transfers (12), which lead to a constrained optimization problem (13). Then we have characterized the optimal C&R transfers and obtained, as byproducts, a consistent valuation principle (32) and a capital allocation method (33). It turned out that an optimal C&R transfer minimizes the group required capital, while leaving the current group balance sheet invariant. But it does not necessarily clear. A third party, e.g. the shareholders, can assume the downside risk-free remainder portfolio. Hence, in one go, our method optimizes the capital and risk structure of the group in two regards. First, the regulatory capital requirements are minimized. Second, additional value for the shareholders can be created.

## A Some Facts from Convex Analysis

Most of the following results from general principles in convex analysis, which can be found e.g. in [25]. For the convenience of the reader we briefly recall here some of the main definitions and facts, which are used throughout the text.

Let  $f : \mathbb{R}^d \rightarrow (-\infty, +\infty]$  be a lower semi-continuous convex function. Its *conjugate*,

$$f^*(q) := \sup_{x \in \mathbb{R}^d} (q \cdot x - f(x)),$$

is again a lower semi-continuous convex function  $f^* : \mathbb{R}^d \rightarrow (-\infty, +\infty]$ , and  $f^{**} = f$  (see Theorem 12.2 in [25]). The *effective domain* of  $f$  is defined as

$$\text{dom}(f) = \{q \mid f(q) < \infty\}.$$

If, in addition,  $f$  is *positively homogeneous* ( $f(\lambda x) = \lambda f(x)$  for  $\lambda \geq 0$ ) then

$$f^*(q) \geq \lambda(q \cdot x - f(x)) \quad \forall x \in \mathbb{R}^d \quad \forall \lambda \geq 0.$$

In this case,  $f^*$  can only take the values 0 or  $+\infty$ , and  $\text{dom}(f^*) = \{q \mid f^*(q) = 0\}$ . It follows that

$$f(x) = f^{**}(x) = \sup_{q \in \text{dom}(f^*)} q \cdot x. \quad (44)$$

The *subgradients* of  $f$  form a (possibly empty) convex set

$$\partial f(x) = \{q \in \mathbb{R}^d \mid f(x+z) \geq f(x) + q \cdot z \ \forall z \in \mathbb{R}^d\},$$

and are characterized by

$$q \in \partial f(x) \iff f(x) + f^*(q) = q \cdot x, \quad (45)$$

see Theorem 23.5 in [25]. Furthermore,  $\partial f(x)$  consists of a single element if and only if  $f$  is differentiable at  $x$ . In this case  $\partial f(x) = \{\nabla f(x)\}$ , see Theorem 25.1 in [25].

For a non-empty convex set  $W$  in  $\mathbb{R}^d$  we define its *indicator function*

$$\delta(x \mid W) := \begin{cases} 0, & \text{if } x \in W \\ +\infty, & \text{otherwise.} \end{cases} \quad (46)$$

The *interior* and *relative interior* of  $W$  is denoted by  $\text{int}(W)$  and  $\text{ri}(W)$ , respectively. The *recession cone* of  $W$  is defined as

$$0^+W := \{y \in \mathbb{R}^d \mid x + \lambda y \in W, \forall \lambda \geq 0, \forall x \in W\}. \quad (47)$$

The *recession function* of  $f$  is, for any  $x \in \text{dom}(f)$ ,

$$f0^+(y) := \lim_{\lambda \downarrow 0} \lambda f(x + \lambda^{-1}y). \quad (48)$$

Theorem 8.5 in [25] states that, for all  $y \in \mathbb{R}^{n+1}$ , the limit in (48) exists in  $(-\infty, +\infty]$  and is independent of  $x \in \text{dom}(f)$ . Moreover,  $f0^+ : \mathbb{R}^{n+1} \rightarrow (-\infty, +\infty]$  is a lower semi-continuous convex and positively homogeneous function.

## B Proof of Theorem 5.4

We write  $u_0 := \delta(\cdot \mid \mathcal{P}^\circ) : \mathbb{R}^{n+1} \rightarrow (-\infty, +\infty]$  for the indicator function of  $\mathcal{P}^\circ$ , which obviously is lower semi-continuous and convex. Furthermore, its conjugate

$$u_0^*(q) = \sup_{x \in \mathcal{P}^\circ} q \cdot x = \begin{cases} 0, & \text{if } q \in \mathcal{P} \\ +\infty, & \text{otherwise,} \end{cases} \quad (49)$$

is the indicator function of  $\mathcal{P}$ .

It follows by Theorem 5.4 in [25] that

$$u(y) := \inf_{y = \sum_{i=0}^m x_i} \sum_{i=0}^m u_i(x_i) = \inf_{y \succeq \sum_{i=1}^m x_i} \sum_{i=1}^m u_i(x_i) \quad (50)$$

defines a convex function  $u : \mathbb{R}^{n+1} \rightarrow [-\infty, +\infty]$ . Theorem 16.4 in [25] states that its conjugate satisfies

$$u^*(q) = \sum_{i=0}^m u_i^*(q) \quad \forall q \in \mathbb{R}^{n+1} \quad (51)$$

(we do not need finiteness of  $u$  here).

It follows by inspection that the effective domain of  $u$  satisfies  $\text{dom}(u) \supseteq \mathcal{P}^\circ + W_1 + \dots + W_m$ , which in view of (PF7) contains an open neighborhood of 0. Moreover,  $u$  is locally monotone on  $\text{dom}(u)$  and cash-invariant; that is, satisfies (19) and (20) in lieu of  $u_i$  and for  $W_i$  replaced by  $\text{dom}(u)$ . Furthermore, the constrained optimization problem (16) is equivalent to (50) for  $y = 0$ . That is,  $(\hat{x}_1, \dots, \hat{x}_m)$  is a solution of (16) if and only if  $(\hat{x}_0, \hat{x}_1, \dots, \hat{x}_m)$  is a solution of (50) for  $y = 0$  with  $\hat{x}_0 = -\sum_{i=1}^m \hat{x}_i$ .

(i) $\Rightarrow$ (ii). This follows by inspection.

(ii) $\Rightarrow$ (i). Suppose  $(\hat{x}_1, \dots, \hat{x}_m)$  is not a minimizer for (16). Then there exists an attainable  $(x_1, \dots, x_m)$  with  $\sum_{i=1}^m u_i(x_i) < \sum_{i=1}^m u_i(\hat{x}_i)$ . By rebalancing the cash, as in (36), one finds an attainable  $(x'_1, \dots, x'_m)$  with  $u_i(x'_1) < u_i(\hat{x}_1)$  and  $u_i(x'_i) \leq u_i(\hat{x}_i)$  for all  $i$ . But then  $(\hat{x}_1, \dots, \hat{x}_m)$  is not Pareto optimal.

(i) $\Rightarrow$ (iii). Since  $u(0) = \sum_{i=1}^m u_i(\hat{x}_i) > -\infty$  and  $0 \in \text{int}(\text{dom } u) = \text{ri}(\text{dom } u)$ , it follows from Theorem 7.2 in [25] that  $u > -\infty$ . In view of Theorem 23.4 in [25], there exists a  $q \in \partial u(0)$ . Moreover, since  $\mathcal{P}^\circ \subset \text{dom}(u)$ , we have  $q \cdot z \leq u(z) - u(0) \leq 0$  for all  $z \succeq 0$ . Hence  $q \in \mathcal{P}$ . Now set  $p := -q$ . In view of (45), (49) and (51), and since  $-\sum_{i=1}^m \hat{x}_i \succeq 0$ , we conclude that

$$u(0) = -u^*(-p) = \sum_{i=1}^m -u_i^*(-p) \leq \sum_{i=1}^m (-p \cdot \hat{x}_i - u_i^*(-p)) \leq \sum_{i=1}^m u_i(\hat{x}_i) = u(0),$$

and therefore  $-p \in \bigcap_{i=1}^m \partial u_i(\hat{x}_i)$  and  $p \cdot \sum_{i=1}^m \hat{x}_i = 0$ .

(iii) $\Rightarrow$ (i). In view of (45), (49) and (51), it follows that

$$u(0) \geq -u^*(-p) = \sum_{i=1}^m (-p \cdot \hat{x}_i - u_i^*(-p)) = \sum_{i=1}^m u_i(\hat{x}_i).$$

Hence  $(\hat{x}_1, \dots, \hat{x}_m)$  is a minimizer for (16).

(i) $\Rightarrow$ (iv). First observe that, for all  $(x_1, \dots, x_m) \in \mathbb{R}^{m \times (n+1)}$ , we have

$$0 \succeq \sum_{i=1}^m x_i \quad \Leftrightarrow \quad -\sum_{i=1}^m x_i^0 \geq \text{ess sup} \sum_{i=1}^m \sum_{j=1}^n x_i^j Z_j. \quad (52)$$

Indeed,  $0 \succeq \sum_{i=1}^m x_i$  holds if and only if  $-\sum_{i=1}^m x_i^0 Z_0 \geq \sum_{i=1}^m \sum_{j=1}^n x_i^j Z_j$  a.s. which is equivalent to the right hand side of (52).

Now let  $(\hat{x}_1, \dots, \hat{x}_m)$  be a minimizer for (16), and define the attainable C&R transfer  $(\bar{x}_1, \dots, \bar{x}_m) \in \mathbb{R}^{m \times (n+1)}$  by

$$\bar{x}_i^j := \begin{cases} -\frac{1}{m} \text{ess sup} \sum_{i=1}^m \sum_{j=1}^n \hat{x}_i^j Z_j, & j = 0, \\ \hat{x}_i^j, & j = 1, \dots, n. \end{cases} \quad (53)$$

In view of the cash-invariance of  $u_i$  and (52), we obtain

$$\begin{aligned} \sum_{i=1}^m u_i(\hat{x}_i) &= \sum_{i=1}^m u_i(0, \hat{x}_i^1, \dots, \hat{x}_i^n) - \sum_{i=1}^m \hat{x}_i^0 \\ &\geq \sum_{i=1}^m u_i(0, \hat{x}_i^1, \dots, \hat{x}_i^n) + \text{ess sup} \sum_{i=1}^m \sum_{j=1}^n \hat{x}_i^j Z_j = \sum_{i=1}^m u_i(\bar{x}_i) \geq \sum_{i=1}^m u_i(\hat{x}_i). \end{aligned}$$

This proves the net cash flow condition (29). Now suppose there exists a  $(x_i^j) \in \mathbb{R}^{m \times n}$  such that

$$\begin{aligned} \sum_{i=1}^m u_i(0, x_i^1, \dots, x_i^n) + \text{ess sup} \sum_{i=1}^m \sum_{j=1}^n x_i^j Z_j \\ < \sum_{i=1}^m u_i(0, \hat{x}_i^1, \dots, \hat{x}_i^n) + \text{ess sup} \sum_{i=1}^m \sum_{j=1}^n \hat{x}_i^j Z_j = \sum_{i=1}^m u_i(\hat{x}_i). \end{aligned}$$

As in (53), replacing  $\hat{x}_i^j$  by  $x_i^j$ , one can define an attainable C&R transfer  $(\bar{x}_1, \dots, \bar{x}_m)$ , which then satisfies

$$\sum_{i=1}^m u_i(\bar{x}_i) < \sum_{i=1}^m u_i(\hat{x}_i),$$

contradicting the optimality of  $(\hat{x}_1, \dots, \hat{x}_m)$ . Hence (iv) follows.

(iv) $\Rightarrow$ (i). Let  $(\hat{x}_1, \dots, \hat{x}_m) \in \mathbb{R}^{m \times (n+1)}$  be given as in (iv). In view of (52), this is an attainable C&R transfer. Now suppose there exists an attainable C&R transfer  $(x_1, \dots, x_m)$  with

$$\sum_{i=1}^m u_i(x_i) < \sum_{i=1}^m u_i(\hat{x}_i).$$

Together with (52) and the cash-invariance of  $u_i$ , this implies that

$$\begin{aligned} \sum_{i=1}^m u_i(0, x_i^1, \dots, x_i^n) + \text{ess sup} \sum_{i=1}^m \sum_{j=1}^n x_i^j Z_j \\ \leq \sum_{i=1}^m u_i(x_i) < \sum_{i=1}^m u_i(\hat{x}_i) = \sum_{i=1}^m u_i(0, \hat{x}_i^1, \dots, \hat{x}_i^n) + \text{ess sup} \sum_{i=1}^m \sum_{j=1}^n \hat{x}_i^j Z_j, \end{aligned}$$

contradicting the optimality of  $(\hat{x}_1, \dots, \hat{x}_m)$ . Hence (i) follows.

## C Proof of Theorem 6.2

An equilibrium  $(\hat{x}_1, \dots, \hat{x}_m)$  and  $p$  satisfies in fact  $p \cdot \hat{x}_i = 0$  for all  $i$ . Indeed, otherwise use (24) and simply add to  $\hat{x}_i$  the positive cash position  $-(p \cdot \hat{x}_i)e_0$ ,

which reduces  $u_i$  by  $-p \cdot \hat{x}_i > 0$ , contradicting the optimality (37) of  $\hat{x}_i$ . Hence

$$\begin{aligned} \inf_{p \cdot y \leq 0} u_i(y) &= \inf_{p \cdot y = 0} u_i(y) = \inf_{y \in \mathbb{R}^{n+1}} u_i(y - (p \cdot y)e_0) \\ &= \inf_{y \in \mathbb{R}^{n+1}} (u_i(y) + p \cdot y) \\ &= - \sup_{y \in \mathbb{R}^{n+1}} (-p \cdot y - u_i(y)) = -u_i^*(-p), \end{aligned} \quad (54)$$

and we deduce

$$u(\hat{x}_i) = -u_i^*(-p) = -p \cdot \hat{x}_i - u_i^*(-p),$$

whence  $-p \in \partial u_i(\hat{x}_i)$ , see (45). In view of (iii) of Theorem 5.4,  $(\hat{x}_1, \dots, \hat{x}_m)$  is a minimizer for (16).

Conversely, let  $(x_1, \dots, x_m)$  be a minimizer for (16) and  $p$  be a valuation principle of the form (25)–(26) and hence satisfying (24) (Lemma 5.3). Let  $\hat{x}_i$  be given by (38). Then  $p \cdot \hat{x}_i = 0$  and, in view of (26), (45) and (54), we conclude

$$\begin{aligned} \inf_{p \cdot z \leq 0} u_i(z) &= -u_i^*(-p) = p \cdot x_i - u_i^*(-p) - p \cdot x_i = p \cdot x_i + u_i(x_i) \\ &= u_i(x_i - (p \cdot x_i)e_0) = u_i(\hat{x}_i), \end{aligned}$$

hence the theorem is proved.

## D Proof of Theorem 6.3

Let  $(\hat{x}_1, \dots, \hat{x}_m)$  be an equilibrium C&R transfer, and let  $I \subset \{1, \dots, m\}$  be a level of diversification,  $I \neq \emptyset$ . Theorem 6.2 implies that there exists a valuation principle  $p$  satisfying (25), (26) and  $p \cdot \hat{x}_i = 0$ . In view of (45) therefore

$$u_i(\hat{x}_i) = -u_i^*(-p).$$

We thus obtain

$$\begin{aligned} \sum_{i \in I} u_i(\hat{x}_i) &= \sum_{i \in I} (-u_i^*(-p)) \\ &\leq \inf_{0 \geq \sum_{i \in I} x_i} \sum_{i \in I} (-p \cdot x_i - u_i^*(-p)) \leq \inf_{0 \geq \sum_{i \in I} x_i} \sum_{i \in I} u_i(x_i), \end{aligned}$$

and the theorem is proved.

## E Proof of Theorem 7.1

Notice that the indicator function,  $u_0 = \delta(\cdot | \mathcal{P}^\circ)$ , of  $\mathcal{P}^\circ$  is positively homogeneous. Hence it coincides with its recession function,  $u_0 0^+ = u_0$ .

We will use Lemma E.1 below, which is a citation of Corollary 9.2.1 in [25], to complete the proof of the theorem. Let  $x_0 + \dots + x_m = 0$ . We distinguish the two cases  $x_0 = 0$  and  $x_0 \neq 0$ .

**Case 1:**  $x_0 = 0$ , and hence  $u_0 0^+(x_0) = 0$ . Suppose that (55) holds, that is,  $\sum_{i=1}^m v(x_i) \leq 0$ . Let  $q \in \text{dom}(v^*)$ . In view of (44), we deduce

$$0 = q \cdot \sum_{i=1}^m x_i = \sum_{i=1}^m q \cdot x_i \leq \sum_{i=1}^m v(x_i) \leq 0,$$

and hence  $v(x_i) = q \cdot x_i$ . It follows that

$$v(-x_i) = \sup_{q \in \text{dom}(v^*)} q \cdot (-x_i) = - \inf_{q \in \text{dom}(v^*)} q \cdot x_i = -v(x_i),$$

and therefore (56) is satisfied.

**Case 2:**  $x_0 \neq 0$ . We show that (55) cannot occur. If  $x_0 \notin \mathcal{P}^\circ$  then  $u_0 0^+(x_0) = +\infty$ , and there is nothing to prove. Hence suppose  $x_0 \succeq 0$ . Then  $u_0 0^+(x_0) = 0$  and, by (44),

$$\sum_{i=1}^m v(x_i) \geq q \cdot \sum_{i=1}^m x_i \quad \forall q \in \text{dom}(v^*).$$

But  $0 \succeq \sum_{i=1}^m x_i$ . By assumption, there exists some  $q \in \text{dom}(v^*)$  such that  $q \cdot \sum_{i=1}^m x_i > 0$ , whence  $\sum_{i=1}^m v(x_i) > 0$ , and the theorem is proved.

**Lemma E.1 (Corollary 9.2.1 in [25]).** *Assume that for all  $x_0 + \dots + x_m = 0$  with*

$$u_0 0^+(x_0) + \dots + u_m 0^+(x_m) \leq 0 \tag{55}$$

*we have*

$$u_0 0^+(-x_0) + \dots + u_m 0^+(-x_m) \leq 0. \tag{56}$$

*Then the infimum in (50) is attained, for all  $y \in \mathbb{R}^{m \times (n+1)}$ .*

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