Credit Derivatives in an Affine Framework

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Abstract

An efficient method for valuing credit derivatives based on three entities is developed in an affine framework. This includes interdependence of market and credit risk, joint credit migration and counterparty default risk of three firms. As an application we provide closed form expressions for the joint distribution of default times, default correlations, and default swap spreads in the presence of counterparty default risk.

Key words: affine intensity based models, counterparty risk, credit derivatives, default dependence

1 Introduction

The rapid growth of the credit derivatives market generates an upsurge for valuation models of various credit derivatives, including credit default swaps (CDSs). This requires analytically tractable models which incorporate an appropriate dependence structure between market and credit risk, credit migration and default risk of multiple firms. These aspects are inevitable for accurately pricing credit derivatives and efficient model calibration.

In this paper we present a method to valuate default-sensitive securities based on three entities in an affine intensity-based framework. We model risk-free rates and the credit states of the firms jointly as an affine state process. Due to a elementary but rigorous mathematical trick, which allows to replace indicator variables by exponential-affine functions of the state process, we obtain closed form expressions for the conditional expectations of a variety of joint credit events. This allows us to derive closed form expressions for joint distributions of default times, default correlations, and CDS spreads in the presence of counterparty default risk. Using an affine approximation technique, which goes back to Singleton and Umantsev [23], we also obtain analytically tractable expressions for swaption prices.

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The state of each firm is expressed by a tuple consisting of a credit index and credit indicator. The credit index, as mentioned in [16], is regarded as the firm's credit score, which can be related to its asset value or its credit rating. For example, it can in principle be obtained by a monotone transformation of the actual credit rating given by Moody's or S&P; i.e., $\mathbb{R}_+$ is decomposed into finitely many non-overlapping intervals I_{Aaa}, $I_{Aa}, \dots$ with the credit index in I_R meaning that the firm is R-rated, $R \in \{Aaa, Aa, \dots\}$. Or it can also be determined by the distance to default variable estimated from the firm's asset information as shown in [8]. In [4], the authors propose a way to determine this credit index variable using the corresponding corporate bond spread. It is further assumed that the higher the credit index value, the worse the firm's financial situation and zero-value of the corresponding credit index implies the perfect financial health of the firm.

The indicator variable is defined to follow a simple point process starting at zero with a constant jump size one and intensity given by the credit index process. The first jump of this process indicates the default of the corresponding firm. This method is originally proposed in [1] and specified to a doubly stochastic setup in [18]. To model risk-free rates, for simplicity, here we only employ a one-factor affine model and define the factor as the short rate. It is straightforward to include an affine multi-factor interest rate model. Additional, e.g. industry specific, factors can easily be built in, as long as they comply with the affine structure.

In contrast to the usual probabilistic approach in doubly stochastic intensity based credit risk models (see e.g. [10]), we use rigorous analytic methods, involving Laplace transforms and ODEs, to obtain the joint distribution of the future default events conditional on the current state of the world. As a result, we provide an efficient intensity based valuation method for credit derivatives in an affine framework. We also point out that we allow for simultaneous default of several firms. A very rare, but realistic event, which is often ignored by other models.

As for the recovery issue of a credit derivative, we adopt the convention of recovery at default and assume that the recovery rate is a random variable depending on both risk-free rates and the credit index of the default firm, which is more reasonable than assuming recovery at maturity, as e.g. in [17], or that the recovery rate is stochastically independent of default probability and risk-free rates, as e.g. in [15].

The literature on credit risk modeling is huge and fastly growing. We do not intend to provide a comprehensive reference list. Instead, we refer to the recent books [2, 10, 14, 19, 20, 21] for an overview.

The remainder of the paper is organized as follows. Section 2 introduces the basic three-firm model, based on affine diffusion and simple point processes. In Section 3 we discuss and illustrate the joint distribution of default times, the density function and default correlation. In Section 4 we derive closed form expressions for the valuation of a CDS in the presence of counterparty risk. In Section 5 we sketch how to price a swaption by affine approximation. Section 6 concludes. For the sake of readability we have postponed some technical parts to the appendix.

2 The Basic Three-Firm Model

In this section we describe the basic model incorporating three firms and a one-factor short rate model. We note that the extension to a multi-factor interest rate and multi-firm model along the following lines is straightforward.

For background and theory of affine processes we refer the reader to [11]. With e_i we denote the i-th standard basis vector in $\mathbb{R}^7$, $i = 0, 1, \dots, 6$. Moreover, we shall frequently use the multi-index notation

$$\mathbf{p} = (p_4, p_5, p_6),\ \mathbf{q} = (q_4, q_5, q_6) \in \mathbf{I} := \{0,1\}^3.$$

The scalar product of two vectors x and y is denoted by $\langle x, y\rangle$.

We now consider the affine jump-diffusion process $X = (X^0, \dots, X^6)$ in $\mathbb{R}^7_+$ with generator

$$\begin{aligned}\mathcal{A}f(x) &= \sum_{i=0}^{3} \alpha_i x_i \partial^2_{x_i} f(x) + \sum_{i=0}^{3} (b_i + \langle \beta_i, x\rangle)\, \partial_{x_i} f(x) \\ &\quad + \sum_{\mathbf{p}\in\mathbf{I}} (f(x + p_4 e_4 + p_5 e_5 + p_6 e_6) - f(x))\, (\ell_{\mathbf{p}} + \langle \lambda_{\mathbf{p}}, x\rangle)\,,\end{aligned} \tag{2.1}$$

where

$$\alpha_i,\ b_i \geq 0, \quad \beta_i = (\beta_{i0}, ..., \beta_{i6}) \in \mathbb{R}^7 \text{ with } \beta_{ij} \geq 0,\ \forall\ j \neq i, \quad \ell_{\mathbf{p}} \geq 0, \quad \lambda_{\mathbf{p}} \in \mathbb{R}^7_+.$$

We let X be realized on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions (e.g. the "canonical" space of càdlàg paths in $\mathbb{R}^7_+$). Depending on the context, $\mathbb{P}$ stands either for the real-world or risk-neutral measure. In the latter case, prices are computed as $\mathbb{P}$-(conditional) expectations. Equivalent measure changes which preserve the respective affine structures exist and are feasible. For a discussion we refer to [6] and [7].

X^0 denotes the short rate process. The pair (X^i, X^{3+i}) represents the credit state of firm i, where X^i denotes the credit index and X^{i+3} represents the default indicator of firm i, and we further assume that $X_0^{3+i} = 0$ for $i = 1, 2, 3$. Then the first jump time

$$\tau_i := \inf\{t \mid X_t^{3+i} > 0\}$$

of X^{3+i} models the default time of firm i. We see from (2.1) that the firms can default simultaneously in all possible combinations, since we sum up over all jumps in the directions of $p_4 e_4 + p_5 e_5 + p_6 e_6$, for $\mathbf{p} \in \mathbf{I}$ (to exclude one of these combinations, simply set the corresponding intensity coefficients ($\ell_{\mathbf{p}}$ and $\lambda_{\mathbf{p}}$) to be zero). This is a rare but realistic event. X^i plays the role of measuring the financial health for firm i. As mentioned before, the larger X^i the more likely is a default of firm i (this effect can be achieved by an appropriate choice of the model parameters $\lambda_{\mathbf{p},i}$).

The generator (2.1) implies a rich interdependence structure between the components X^i:

- The interest rates, X^0, influence all credit risk related variables, $X^1, \dots,$ X^6, by β_{i0} (mean-reversion level of X^i) and the respective $\lambda_{\mathbf{p},0}$ (jump intensity of X^{3+i}).

- The credit index of firm i, X^i, $i = 1, 2, 3$, drives the intensities for (joint) defaults of firms 1, 2 and 3 by the respective $\lambda_{\mathbf{p},i}$. This type of correlation has already been used by [12].

  X^i also influences the mean reversion level for X^j by β_{ji}, $j = 0, \dots, 3$ (however, typically we let the short rates evolve autonomously, that is, we set $\beta_{0i} = 0$).

- The counter process for firm i, X^{3+i}, $i = 1, 2, 3$, influences the intensities for (joint) defaults of firms 1,2 and 3 by the respective $\lambda_{\mathbf{p},3+i}$. Note that this introduces "infectious defaults" or a "loop dependent default risk structure" as proposed in [9] and [17], respectively: the default of either firm increases the default intensity of the other firm. See also Example 2 below.

  X^{3+i} also influences the mean reversion level for X^j by $\beta_{j,3+i}$, $j = 0, \dots, 3$, another form of default contagion.

Remark 2.1. *Prior to the default of firm $i \in \{1, 2, 3\}$, X^i and X^{i+3} represent the credit index and default indicator of the firm. Once firm i defaults, these two state variables become latent factors that drive the credit index and default dynamics of the remaining firms. One could mitigate the impact of X^{i+3} on the remaining firms after default of firm i by adding a zero-mean reverting drift term, say $-\beta_{i+3}x_{i+3}\partial_{x_{i+3}}$ for some $\beta_{i+3} > 0$, to the generator* (2.1)*. For simplicity of exposure we omit this here.*

Remark 2.2. *The present framework may be modified to include multiple defaults with constant fractional recovery $r \in (0, 1)$ of the nominal value. Simply let X_t^{i+3} be the counter of the number of defaults of firm i by time t. At each default the payoff of the firm's bond is reduced to r times its previous value. The terminal payoff, at time T say, is then reduced by the factor $r^{X_T^{i+3}} = \exp(X_T^{i+3} \log r)$, which is an affine function of X_T and hence analytically tractable. See [21, Section 6.1.3] for a more detailed discussion.*

Remark 2.3. *With regard to the parameters in* (2.1)*, it is worth mentioning how the model calibration, say under the risk neutral measure, works. First as shown in [4], the parameters $\{\alpha_i, b_i, \beta_i\}_{0 \le i \le 3}$, the short rate X^0 and individual credit index values $\{X^i\}_{1 \le i \le 3}$ can be estimated using market observations of treasury and corporate bond yields. The remaining parameters that are used to characterize the joint credit migration correlations between different firms can then be calibrated using, e.g., credit default swap (CDS) data in combination with the explicit formula for CDS spreads in Section 4 below.*

Since market price of risk specifications which preserve the affine structure (2.1) *exist ([6, 7]), it is also possible to calibrate the model to real-world default correlations, from which risk premiums can be inferred.*

Fix $\delta \geq 0$. The basic affine property of this process reads (see [11])

$$\mathbb{E}\left[e^{-\delta \int_t^T X_s^0 ds} e^{\langle v, X_t \rangle} \mid \mathcal{F}_t\right] = e^{\phi(T-t,v;\delta) + \langle \psi(T-t,v;\delta), X_t \rangle} \tag{2.2}$$

for all $v \in \mathbb{R}_-^7$, where the $\mathbb{R}_-$-valued functions $\phi = \phi(t, v; \delta)$ and $\psi_i = \psi_i(t, v; \delta)$ solve the following generalized Riccati equations (GREs)

$$\begin{aligned}
\partial_t \phi &= \sum_{k=0}^{3} b_k \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \ell_{\mathbf{p}} \left(e^{p_4\psi_4 + p_5\psi_5 + p_6\psi_6} - 1\right), \\
\phi(0, v; \delta) &= 0, \\
\partial_t \psi_i &= \alpha_i \psi_i^2 + \sum_{k=0}^{3} \beta_{ki} \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \lambda_{\mathbf{p},i} \left(e^{p_4\psi_4 + p_5\psi_5 + p_6\psi_6} - 1\right) - \delta 1_{\{i=0\}} \\
\psi_i(0, v; \delta) &= v_i, \\
\partial_t \psi_j &= \sum_{k=0}^{3} \beta_{kj} \psi_k + \sum_{\mathbf{p} \in \mathbf{I}} \lambda_{\mathbf{p},j} \left(e^{p_4\psi_4 + p_5\psi_5 + p_6\psi_6} - 1\right), \\
\psi_j(0, v; \delta) &= v_j
\end{aligned} \tag{2.3}$$

for $i = 0, 1, 2, 3$ and $j = 4, 5, 6$.

2.1 Basic Trick

The following basic trick allows to express a variety of possible joint credit events as (limits) of exponential functions of X: let $i, j \in \{1, 2, 3\}$, then

$$\begin{aligned}
1_{\{t<\tau_i\}} &= \lim_{k\to\infty} e^{-kX_t^{3+i}}, \\
1_{\{s<\tau_i\leq t\}} &= 1_{\{s<\tau_i\}} - 1_{\{t<\tau_i\}} = \lim_{k\to\infty} \left(e^{-kX_s^{3+i}} - e^{-kX_t^{3+i}}\right), \quad s < t, \\
1_{\{s<\tau_i,\, t<\tau_j\}} &= 1_{\{s<\tau_i\}} 1_{\{t<\tau_j\}} = \lim_{k\to\infty} e^{-k(X_s^{3+i} + X_t^{3+j})}, \\
&\text{etc.}
\end{aligned} \tag{2.4}$$

This asks for the following general result, the proof of which is postponed to Section A.

Proposition 2.4. *For* $t \leq T$, $v \in \mathbb{R}_-^7$, $\delta \geq 0$ *and* $\mathbf{p} \in \mathbf{I}$ *we have*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\delta \int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} \lim_{k\to\infty} e^{-k(p_4 X_T^4 + p_5 X_T^5 + p_6 X_T^6)} \mid \mathcal{F}_t\right] \\
&\qquad = e^{\Phi(T-t,v;\delta;\mathbf{p}) + \sum_{i\in\{0,\ldots,3\}\cup J_0(\mathbf{p})} \Psi_i(T-t,v;\delta;\mathbf{p}) X_t^i} \prod_{j\in J_1(\mathbf{p})} 1_{\{X_t^j = 0\}},
\end{aligned} \tag{2.5}$$

where $J_0(\mathbf{p}) := \{4 \le j \le 6 \mid p_j = 0\}$*,* $J_1(\mathbf{p}) := \{4 \le j \le 6 \mid p_j = 1\}$ *and the* $\mathbb{R}_-$*-valued functions*

$$\Phi = \Phi(t, v; \delta; \mathbf{p}) \quad \textit{and} \quad \Psi_i = \Psi_i(t, v; \delta; \mathbf{p})$$

satisfy

$$\begin{aligned}
\partial_t \Phi &= \sum_{k=0}^{3} b_k \Psi_k + \sum_{\mathbf{q} \in \mathbf{I}_0(\mathbf{p})} \ell_{\mathbf{q}} \left(e^{q_4 \Psi_4 + q_5 \Psi_5 + q_6 \Psi_6} - 1 \right) - \sum_{\mathbf{q} \in \mathbf{I}_1(\mathbf{p})} \ell_{\mathbf{q}}, \\
\Phi(0, v; \delta; \mathbf{p}) &= 0, \\
\partial_t \Psi_i &= \alpha_i \Psi_i^2 + \sum_{k=0}^{3} \beta_{ki} \Psi_k + \sum_{\mathbf{q} \in \mathbf{I}_0(\mathbf{p})} \lambda_{\mathbf{q},i} \left(e^{q_4 \Psi_4 + q_5 \Psi_5 + q_6 \Psi_6} - 1 \right) \\
&\quad - \sum_{\mathbf{q} \in \mathbf{I}_1(\mathbf{p})} \lambda_{\mathbf{q},i} - \delta 1_{\{i=0\}}, \\
\Psi_i(0, v; \delta; \mathbf{p}) &= v_i, \\
\partial_t \Psi_j &= \sum_{k=0}^{3} \beta_{kj} \Psi_k + \sum_{\mathbf{q} \in \mathbf{I}_0(\mathbf{p})} \lambda_{\mathbf{q},j} \left(e^{q_4 \Psi_4 + q_5 \Psi_5 + q_6 \Psi_6} - 1 \right) - \sum_{\mathbf{q} \in \mathbf{I}_1(\mathbf{p})} \lambda_{\mathbf{q},j}, \\
\Psi_j(0, v; \delta; \mathbf{p}) &= v_j
\end{aligned} \tag{2.6}$$

for $i = 0, 1, 2, 3$ *and* $j \in J_0(\mathbf{p})$*, where* $\mathbf{I}_0(\mathbf{p}) := \{\mathbf{q} \in \mathbf{I} \mid q_j = 0 \ \forall j \in J_1(\mathbf{p})\}$ *and* $\mathbf{I}_1(\mathbf{p}) := \mathbf{I} \setminus \mathbf{I}_0(\mathbf{p}) = \{\mathbf{q} \in \mathbf{I} \mid q_j = 1 \text{ for some } j \in J_1(\mathbf{p})\}$*.*

Remark 2.5. *Notice that* $\Phi(t, v; \delta; \mathbf{p})$ *and* $\Psi_i(t, v; \delta; \mathbf{p})$ *in Proposition 2.4 do in fact not depend on* v_j *for* $j \in J_1(\mathbf{p})$*, as becomes clear from the GREs* (2.6) *and the definition of* $\mathbf{I}_0(\mathbf{p})$*.*

Example 1 Let $t \le T$. The $\mathcal{F}_t$-conditional Laplace transform of X_T with respect to the T-forward measure conditional on $\{T < \tau_1 \wedge \tau_2\}$ is

$$\frac{\mathbb{E}\left[e^{-\int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} 1_{\{T < \tau_1 \wedge \tau_2\}} \mid \mathcal{F}_t \right]}{\mathbb{E}\left[e^{-\int_t^T X_s^0 ds} 1_{\{T < \tau_1 \wedge \tau_2\}} \mid \mathcal{F}_t \right]}, \quad v \in \mathbb{R}_-^7,$$

where

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} 1_{\{T < \tau_1 \wedge \tau_2\}} \mid \mathcal{F}_t \right] \\
&= \mathbb{E}\left[e^{-\int_t^T X_s^0 ds} e^{\langle v, X_T \rangle} \lim_{k \to \infty} e^{-k(X_T^4 + X_T^5)} \mid \mathcal{F}_t \right] \\
&= e^{\Phi(T-t, v; 1; 1, 1, 0) + \sum_{i \in \{0, \dots, 3, 6\}} \Psi_i(T-t, v; 1; 1, 1, 0) X_t^i} 1_{\{X_t^4 = X_t^5 = 0\}}.
\end{aligned}$$

3 Joint Distribution of Default Times

With the aid of (2.4) and Proposition 2.4 we now discuss the dependence structure of the default times τ_1 and τ_2.

Fix $s \geq 0$. For the $\mathcal{F}_s$-conditional joint distribution of (τ_1, τ_2) we have

$$\begin{aligned} F(t,T) &= \mathbb{P}[\tau_1 \leq t,\, \tau_2 \leq T \mid \mathcal{F}_s] \\ &= 1 - \mathbb{E}[1_{\{t<\tau_1\}} \mid \mathcal{F}_s] - \mathbb{E}[1_{\{T<\tau_2\}} \mid \mathcal{F}_s] + \mathbb{E}[1_{\{t<\tau_1\}} 1_{\{T<\tau_2\}} \mid \mathcal{F}_s], \end{aligned} \tag{3.1}$$

for $t, T \geq s$. The terms involved are

$$\begin{aligned} \mathbb{E}[1_{\{t<\tau_1\}} \mid \mathcal{F}_s] &= \mathbb{E}\left[\lim_{k\to\infty} e^{-kX_t^4} \mid \mathcal{F}_s\right] \\ &= e^{\Phi(t-s,0;0;1,0,0)+\sum_{i\in\{0,\dots,3,5,6\}} \Psi_i(t-s,0;0;1,0,0)X_s^i} 1_{\{X_s^4=0\}}, \\ \mathbb{E}[1_{\{T<\tau_2\}} \mid \mathcal{F}_s] &= \mathbb{E}\left[\lim_{k\to\infty} e^{-kX_T^5} \mid \mathcal{F}_s\right] \\ &= e^{\Phi(T-s,0;0;0,1,0)+\sum_{i\in\{0,\dots,3,4,6\}} \Psi_i(T-s,0;0;0,1,0)X_s^i} 1_{\{X_s^5=0\}}, \end{aligned}$$

and, for $t \leq T$,

$$\begin{aligned} &\mathbb{E}[1_{\{t<\tau_1\}} 1_{\{T<\tau_2\}} \mid \mathcal{F}_s] \\ &= \mathbb{E}\left[\lim_{k\to\infty} e^{-kX_t^4} \mathbb{E}\left[\lim_{l\to\infty} e^{-lX_T^5} \mid \mathcal{F}_t\right] \mid \mathcal{F}_s\right] \\ &= e^{\Phi(T-t,0;0;0,1,0)} \mathbb{E}\left[\lim_{k\to\infty} e^{-k(X_t^4+X_t^5)} e^{\sum_{i\in\{0,\dots,3,4,6\}} \Psi_i(T-t,0;0;0,1,0)X_t^i} \mid \mathcal{F}_s\right] \\ &= e^{\Phi(T-t,0;0;0,1,0)+\Phi\left(t-s,\sum_{i\in\{0,\dots,3,6\}} \Psi_i(T-t,0;0;0,1,0)e_i;0;1,1,0\right)} \\ &\quad \times e^{\sum_{j\in\{0,\dots,3,6\}} \Psi_j\left(t-s,\sum_{i\in\{0,\dots,3,6\}} \Psi_i(T-t,0;0;0,1,0)e_i;0;1,1,0\right)X_s^j} 1_{\{X_s^4=X_s^5=0\}}, \end{aligned} \tag{3.2}$$

and similarly for $t \geq T$,

$$\begin{aligned} &\mathbb{E}[1_{\{t<\tau_1\}} 1_{\{T<\tau_2\}} \mid \mathcal{F}_s] \\ &= e^{\Phi(t-T,0;0;1,0,0)+\Phi\left(T-s,\sum_{i\in\{0,\dots,3,6\}} \Psi_i(t-T,0;0;1,0,0)e_i;0;1,1,0\right)} \\ &\quad \times e^{\sum_{j\in\{0,\dots,3,6\}} \Psi_j\left(T-s,\sum_{i\in\{0,\dots,3,6\}} \Psi_i(t-T,0;0;1,0,0)e_i;0;1,1,0\right)X_s^j} 1_{\{X_s^4=X_s^5=0\}}. \end{aligned} \tag{3.3}$$

Remark 3.1. *For simplicity, we will set $s = 0$ in what follows and use the convention $X_0^j = 0$ for $j = 4, 5, 6$. All results carry over after a slight modification to the general case $s \geq 0$.*

3.1 Example of a Continuous Joint Density

Notice that the joint distribution function (3.1) is twice continuously differentiable in (t, T) for $t \neq T$, but not on the diagonal $\Delta := \{(t,t) \mid t \geq 0\}$ in general.

In this section, we work out an example where the jointly continuous density function f with

$$F(t,T) = \int_0^t \int_0^T f(u,v)\, dudv, \quad \forall (t,T) \in \mathbb{R}_+^2, \tag{3.4}$$

exists. In the working paper version [5], we also illustrate the two cases where f exists but is only piecewise continuous (infectious defaults), and where the entire mass of the distribution is concentrated on Δ (simultaneous defaults) and hence a density does not exist.

We let the generator (2.1) be of the form

$$\begin{aligned}\mathcal{A}f(x) = {} & \alpha_0 x_0 \partial_{x_0}^2 f(x) + (b_0 + \beta_{00} x_0) \partial_{x_0} f(x) \\ & + \sum_{i=1}^2 \alpha_i x_i \partial_{x_i}^2 f(x) + \sum_{i=1}^2 (b_i + \beta_{i0} x_0 + \beta_{ii} x_i)\, \partial_{x_i} f(x) \\ & + (f(x+e_4) - f(x)) \left(\lambda_{(1,0,0),0} x_0 + \lambda_{(1,0,0),1} x_1 + \lambda_{(1,0,0),2} x_2\right) \\ & + (f(x+e_5) - f(x)) \left(\lambda_{(0,1,0),0} x_0 + \lambda_{(0,1,0),1} x_1 + \lambda_{(0,1,0),2} x_2\right),\end{aligned} \tag{3.5}$$

with the symmetric structure

$$\begin{gathered}\alpha_1 = \alpha_2, \quad b_1 = b_2, \quad \beta_{10} = \beta_{20}, \quad \beta_{11} = \beta_{22}, \\ \lambda_{(1,0,0),0} = \lambda_{(0,1,0),0}, \quad \lambda_{(1,0,0),1} = \lambda_{(0,1,0),2}, \quad \lambda_{(1,0,0),2} = \lambda_{(0,1,0),1}.\end{gathered} \tag{3.6}$$

Note that τ_1 and τ_2 are conditionally independent given the information $\mathcal{G} = \sigma(X_t^0, \dots, X_t^3 \mid t \geq 0)$ generated by $X^0, \dots, X^3$. In other words, the default times τ_1 and τ_2 are doubly stochastic driven by the factors $(X^0, \dots, X^3)$, see e.g. [10]. Hence we have

$$\mathbb{P}[\tau_1 \leq t,\, \tau_2 \leq T \mid \mathcal{G}] = \mathbb{P}[\tau_1 \leq t \mid \mathcal{G}] \cdot \mathbb{P}[\tau_2 \leq T \mid \mathcal{G}]$$

and both of the $\mathcal{G}$-conditional distribution functions on the right hand side have a $\mathcal{G}$-measurable continuous density. It is thus rather obvious that $F(t,T) = \mathbb{E}\left[\mathbb{P}[\tau_1 \leq t,\, \tau_2 \leq T \mid \mathcal{G}]\right]$ admits a continuous density. We now give a more formal argument as follows.

We write short $\Phi(v;\mathbf{p}) = \Phi(t,v;0;\mathbf{p})$ and $\Psi_i(v,\mathbf{p}) = \Psi_i(t,v;0;\mathbf{p})$. The relevant ODEs (2.6) are

$$\begin{aligned}\partial_t \Phi(0;\mathbf{p}) &= b_0 \Psi_0(0;\mathbf{p}) + b_1 \left(\Psi_1(0;\mathbf{p}) + \Psi_2(0;\mathbf{p})\right), \\ \partial_t \Psi_0(0;\mathbf{p}) &= \alpha_0 \Psi_0^2(0;\mathbf{p}) + \beta_{00} \Psi_0(0;\mathbf{p}) + \beta_{10} \left(\Psi_1(0;\mathbf{p}) + \Psi_2(0;\mathbf{p})\right) - \lambda_{\mathbf{p},0}, \\ \partial_t \Psi_i(0;\mathbf{p}) &= \alpha_1 \Psi_i^2(0;\mathbf{p}) + \beta_{11} \Psi_i(0;\mathbf{p}) - \lambda_{\mathbf{p},i}, \quad i = 1,2,\end{aligned}$$

for $\mathbf{p} = (1,0,0)$, $(0,1,0)$, and

$$\begin{aligned}\partial_t \Phi(v;1,1,0) &= b_0 \Psi_0(v;1,1,0) + 2b_1 \Psi_1(v;1,1,0), \\ \partial_t \Psi_0(v;1,1,0) &= \alpha_0 \Psi_0^2(v;1,1,0) + \beta_{00} \Psi_0(v;1,1,0) + 2\beta_{10} \Psi_1(v;1,1,0) - 2\lambda_{\mathbf{p},0}, \\ \partial_t \Psi_1(v;1,1,0) &= \alpha_1 \Psi_1^2(v;1,1,0) + \beta_{11} \Psi_1(v;1,1,0) - \lambda_{(1,0,0),1} - \lambda_{(0,1,0),1},\end{aligned}$$

with $\Psi_2(v;1,1,0) = \Psi_1(v;1,1,0)$, by symmetry.

Note that Ψ_1 and Ψ_2 above solve autonomous Riccati equations. The following solution formula is well know:

Lemma 3.2. *The function*

$$G = G(t, r_0) = -\frac{2C\,(e^{\rho t}-1) - (\rho\,(e^{\rho t}+1) + B\,(e^{\rho t}-1))\,r_0}{\rho\,(e^{\rho t}+1) - B\,(e^{\rho t}-1) - 2A\,(e^{\rho t}-1)\,r_0} \tag{3.7}$$

with $\rho := \sqrt{B^2 + 4AC}$ *is the unique solution of the Riccati equations*

$$\partial_t G = AG^2 + BG - C, \quad G(0, r_0) = r_0,$$

for $A, C \geq 0$, $B \in \mathbb{R}$ *and* $r_0 \leq 0$.

With formula (3.7) at hand it is indeed possible—but somehow cumbersome (we used Mathematica for the formal calculations)—to formally show that $F(t,T)$, given by (3.1)–(3.3), admits a continuous density. Figures 1–3 show this density function for

$$\begin{gathered} \alpha_0 = 1.736 \times 10^{-5}, \quad \alpha_1 = 3.2648, \\ b_0 = 0.01167, \quad b_1 = 1.6328 \times 10^{-5}, \\ \beta_{00} = -0.15492, \quad \beta_{10} = 0.23006, \quad \beta_{11} = -1.472, \\ \lambda_{(1,0,0),0} = 0.26365, \quad \lambda_{(1,0,0),1} = 0.10613, \\ X_0^0 = 0.0105, \quad X_0^1 = X_0^2 = 0.07, \quad X_0^3 = 0, \end{gathered} \tag{3.8}$$

and different values for $\lambda_{(1,0,0),2}$, the impact of the second firm's credit rating, X_t^2, on the default intensity of firm 1, and vice versa. The larger $\lambda_{(1,0,0),2}$, the stronger the dependence of the default times, which is best seen on the diagonal of the domain in Figures 2 and 3.

Remark 3.3. *The above parameters were obtained by the single name model calibration in [4] and the symmetry assumption* (3.6). $X_0^1 = 0.07$ *corresponds to Moody's rating class Aaa. For further calibration details we refer to [4].*

3.2 Default Correlation

The joint distribution function (3.1) contains all the information about the dependence of the default times τ_1 and τ_2. A first (but not sufficient) indicator for this dependence is the correlation of the events $\{\tau_1 \leq T\}$ and $\{\tau_2 \leq T\}$,

$$corr(T) = \frac{Cov_{12}(T)}{\sqrt{Cov_{11}(T)Cov_{22}(T)}} \tag{3.9}$$

with

$$\begin{aligned} Cov_{ij}(T) &:= \mathbb{E}[1_{\{\tau_i \leq T\}} 1_{\{\tau_j \leq T\}}] - \mathbb{E}[1_{\{\tau_i \leq T\}}]\mathbb{E}[1_{\{\tau_j \leq T\}}] \\ &= \begin{cases} \mathbb{E}[1_{\{\tau_i \leq T\}}] - \left(\mathbb{E}[1_{\{\tau_i \leq T\}}]\right)^2, & i = j, \\ F(T,T) - \mathbb{E}[1_{\{\tau_i \leq T\}}]\mathbb{E}[1_{\{\tau_j \leq T\}}], & i \neq j, \end{cases} \end{aligned}$$

for varying $T \geq 0$. The terms involved are

$$\mathbb{E}[1_{\{\tau_i \leq T\}}] = 1 - \mathbb{E}\left[\lim_{k\to\infty} e^{-kX_T^{3+i}}\right] = 1 - e^{\Phi(T,0;0;\mathbf{p}(i))+\sum_{j=0}^{3}\Psi_j(T,0;0;\mathbf{p}(i))X_0^j},$$

where $\mathbf{p}(1) := (1,0,0)$ and $\mathbf{p}(2) := (0,1,0)$.

As documented in [22], the default correlations introduced by correlated default rating processes are typically too low to match the empirical correlations observed from markets. However, since in our model the default intensity of one firm depends explicitly, by factor $\lambda_{(1,0,0),2}$, on the rating process of the other firm, the resulting default correlation can reach the level of market observations (see [24]). Figure 4 shows the term structure of default correlations (3.9) for the model (3.5), (3.6), and (3.8) with different values for $\lambda_{(1,0,0),2}$. This illustrates once more the flexibility of our approach to account for dependence of default.

4 Valuing Credit Default Swaps

We now consider the valuation of a plain vanilla credit default swap (CDS) with notional principal \$1. The *seller* (firm 3) of a CDS contract provides the *buyer* (firm 2) insurance against the risk of default of a third party called the *reference entity* (firm 1). In return, the buyer makes periodic payments to the seller. We denote by T_0 the start date of the CDS and the payment dates by $T_1, \dots, T_n$. We assume that $T_k - T_{k-1} \equiv \Delta$ for all $k = 1, \dots, n$. We consider a Bermudan setup. That is, cashflows take place at dates T_k only, given the events that happened in the preceding periods $(T_{j-1}, T_j]$, $j = 1, \dots, k$.

At time T_k:

- if no default has occurred yet ($T_k < \tau_1 \wedge \tau_2 \wedge \tau_3$) then the buyer pays to the seller a fixed rate c;
- if the reference entity has defaulted in period $(T_{k-1}, T_k]$ ($T_{k-1} < \tau_1 \leq T_k$) *and* the seller has not defaulted yet ($T_k < \tau_3$) *and* the buyer has not defaulted by T_{k-1} ($T_{k-1} < \tau_2$) then the seller pays $1 - G(X_{T_k})$ and the contract terminates, where

$$G(x) = e^{r+\langle \rho, x\rangle} \leq 1$$

denotes the recovery rate for the bond issued by the reference entity, for some $r \in \mathbb{R}_-$ and $\rho \in \mathbb{R}^7_-$;
- in all other cases there is no payment and the contract terminates.

The value at time $t \le T_0$ of the buyer's payments accordingly is cB_t, where

$$\begin{aligned} B_t &= \mathbb{E}\left[\sum_{k=1}^n e^{-\int_t^{T_k} X_s^0 ds} \Delta 1_{\{T_k < \tau_1 \wedge \tau_2 \wedge \tau_3\}} \mid \mathcal{F}_t\right] \\ &= \Delta \sum_{k=1}^n \mathbb{E}\left[e^{-\int_t^{T_k} X_s^0 ds} \lim_{l\to\infty} e^{-l(X_{T_k}^4 + X_{T_k}^5 + X_{T_k}^6)} \mid \mathcal{F}_t\right] \\ &= \Delta \sum_{k=1}^n e^{\Phi(T_k - t, 0;1;1,1,1) + \sum_{i=0}^3 \Psi_i(T_k - t, 0;1;1,1,1) X_t^i} 1_{\{X_t^4 = X_t^5 = X_t^6 = 0\}}. \end{aligned} \tag{4.1}$$

The value at time $t \le T_0$ of the seller's payment is

$$\begin{aligned} S_t &= \mathbb{E}\left[\sum_{k=1}^n e^{-\int_t^{T_k} X_s^0 ds} (1 - G(X_{T_k})) 1_{\{T_{k-1} < \tau_1 \le T_k\}} 1_{\{T_{k-1} < \tau_2\}} 1_{\{T_k < \tau_3\}} \mid \mathcal{F}_t\right] \\ &= \sum_{k=1}^n \mathbb{E}\Big[e^{-\int_t^{T_k} X_s^0 ds} (1 - G(X_{T_k})) \\ &\qquad \times \lim_{l,m\to\infty} \left(e^{-l X_{T_{k-1}}^4} - e^{-m X_{T_k}^4}\right) e^{-l X_{T_{k-1}}^5 - m X_{T_k}^6} \mid \mathcal{F}_t\Big] \\ &= \sum_{k=1}^n S_t^{1k} - S_t^{2k} - S_t^{3k} + S_t^{4k}, \end{aligned} \tag{4.2}$$

for some exponential affine terms $S_t^{1k}, \ldots, S_t^{4k}$. For sake of readability, the detailed expressions are given in Section B below.

The forward CDS spread $C(t)$ at time $t \le T_0$ is the fixed rate at which we have $C(t)B_t = S_t$. From the above, we obtain

Lemma 4.1. *The forward CDS spread is given by*

$$C(t) = \frac{S_t}{B_t} = \frac{\sum_{k=1}^n S_t^{1k} - S_t^{2k} - S_t^{3k} + S_t^{4k}}{\Delta \sum_{k=1}^n e^{\Phi(T_k - t, 0;1;1,1,1) + \sum_{i=0}^3 \Psi_i(T_k - t, 0;1;1,1,1) X_t^i}}, \tag{4.3}$$

where the terms B_t and S_t^{ik} are given by (4.1) *and* (B.1)–(B.4)*, respectively.*

Figure 5 shows the CDS spread $C(T_0)$ for different CDS lengths, $T_n - T_0$, for the case of single-party risk (only the reference entity can default, that is, $X^2 = X^3 = 0$) with different rating classes: $X_{T_0}^1 = 0.07$ (Moody's Aaa), 0.13102 (Aa), 0.465 (A), 0.80907 (Baa) and $X_{T_0}^4 = 0$ (no default by T_0). The recovery rate is one, that is, $r = 0$ and $\rho = 0$. The remaining parameters are according to (3.8).

5 Swaption Pricing by Affine Approximation

In this section we sketch a method for approximating swaption prices as proposed by Singleton and Umantsev [23]. Consider a call option (swaption) on the above CDS with strike rate K and expiry date T_0. Its payoff at T_0 is

$$(S_{T_0} - KB_{T_0})^+ = S_{T_0} 1_{\{C(T_0)>K\}} - KB_{T_0} 1_{\{C(T_0)>K\}}$$

and the price at time $t < T_0$, accordingly,

$$P_{\text{swpt}}(t) = \mathbb{E}\left[e^{-\int_t^{T_0} X_s^0 ds} S_{T_0} 1_{\{C(T_0)>K\}} \mid \mathcal{F}_t\right] \\ - K\mathbb{E}\left[e^{-\int_t^{T_0} X_s^0 ds} B_{T_0} 1_{\{C(T_0)>K\}} \mid \mathcal{F}_t\right].$$

Note that S_{T_0} and B_{T_0} are sums of exponential-affine functions in X_{T_0}. Define the CDS spread function

$$c(x) := C(T_0, X_{T_0} = x),$$

see (4.3). The idea is to approximate the exercise boundary $\partial D(K) := \{c = K\}$ by a hyperplane in $\mathbb{R}^7$, hence to linearize c. That is, for a fixed average strike rate K^*, we write

$$c(x) \approx K^* + \langle \nabla c(x^*), x - x^* \rangle \tag{5.1}$$

for some $x^* \in \partial D(K^*)$. The exercise domain $D(K) := \{c > K\}$ is accordingly replaced by the half-space

$$\{x \mid \langle \nabla c(x^*), x \rangle > K - K^* + \langle \nabla c(x^*), x^* \rangle\}.$$

The computation of $P_{\text{swap}}(t)$ then boils down to the Fourier-inversion methods for conditional distributions with Laplace transforms of the form (2.2) with $\delta = 1$ and $T = T_0$, as discussed in [13].

To illustrate the effectiveness of this approach, we show in Figure 6 the level sets for different levels, K, on the two-dimensional cross-sectional surface $(x_0, x_1) \mapsto c(x_0, x_1, 0, \dots, 0)$. The cross-sections of the corresponding exercise boundaries, $\partial D(K)$, are visible in the (x_0, x_1)-plane. The model parameters are as at the end of Section 4. Figure 6 suggests that the linear approximation (5.1) will yield accurate swaption prices. A more detailed empirical study is left for future research.

6 Conclusion

In this paper we provided some basic and efficient techniques for valuating credit derivatives in an affine intensity based framework. In particular, the usual doubly stochastic methods (see e.g. [10]) have been replaced by Laplace transformations and ODEs. This resulted in a rigorous analytically tractable framework which is flexible enough to capture complex dependence structures in the presence of counterparty credit risk.

We demonstrated the efficiency of this approach by explicitly calculating the joint distribution and density of default times, default correlations, and CDS spreads in the presence of counter-party default risk. Also we sketched the pricing of swaptions by using an affine approximation technique, as proposed by Singleton and Umantsev [23].

Our results, for simplicity of exposure, are based on affine diffusion and simple point processes for a three-firm model. Extensions towards large credit portfolios and more general affine jump-diffusion processes, including multi-factor interest rate models and additional industry specific factors, are possible (see the working paper version [5]) and left for future research.

A Proof of Proposition 2.4

By dominated convergence, the left hand side of (2.5) equals

$$\begin{aligned}&\lim_{k\to\infty}\mathbb{E}\left[e^{-\delta\int_t^T X_s^0 ds}e^{\langle v,X_T\rangle}e^{-k(p_4X_T^4+p_5X_T^5+p_6X_T^6)}\mid\mathcal{F}_t\right]\\&\qquad=\lim_{k\to\infty}e^{\phi(T-t,v-k(p_4e_4+p_5e_5+p_6e_6);\delta)+\langle\psi(T-t,v-k(p_4e_4+p_5e_5+p_6e_6);\delta),X_t\rangle}.\end{aligned}$$

Let $j\in\{4,5,6\}$. Since $\partial_t\psi_j\leq 0$, we have that

$$\begin{aligned}&\psi_j(t,v-k(p_4e_4+p_5e_5+p_6e_6);\delta)\\&\qquad\leq\psi_j(0,v-k(p_4e_4+p_5e_5+p_6e_6);\delta)=v_j-k1_{\{p_j=1\}},\quad\forall t\geq 0.\end{aligned}$$

By classical results on inhomogeneous ODEs (see e.g. [3]), we conclude that the right hand sides of the GREs (2.3) which correspond to ϕ, $i=0,\dots,3$ and $j\in J_0(\mathbf{p})$ converge uniformly on compacts in t, ϕ, ψ_i, $i=0,\dots,3$, and ψ_j, $j\in J_0(\mathbf{p})$, to the respective right hand sides of (2.6), for $k\to\infty$. This proves the proposition.

B Detailed Expressions for (4.2)

Taking into account Remark 2.5, we derive

$$\begin{aligned}S_t^{1k}&=\mathbb{E}\left[e^{-\int_t^{T_k}X_s^0ds}\lim_{l,m\to\infty}e^{-l(X_{T_{k-1}}^4+X_{T_{k-1}}^5)-mX_{T_k}^6}\mid\mathcal{F}_t\right]\\&=\mathbb{E}\Bigg[e^{-\int_t^{T_{k-1}}X_s^0ds}\,\mathbb{E}\left[e^{-\int_{T_{k-1}}^{T_k}X_s^0ds}\lim_{m\to\infty}e^{-mX_{T_k}^6}\mid\mathcal{F}_{T_{k-1}}\right]\\&\qquad\qquad\times\lim_{l\to\infty}e^{-l(X_{T_{k-1}}^4+X_{T_{k-1}}^5)}\mid\mathcal{F}_t\Bigg]\\&=e^{\Phi(\Delta,0;1;0;0,1)}\mathbb{E}\Bigg[e^{-\int_t^{T_{k-1}}X_s^0ds}e^{\sum_{i=0}^5\Psi_i(\Delta,0;1;0;0,1)X_{T_{k-1}}^i}\\&\qquad\qquad\qquad\times\lim_{l\to\infty}e^{-l(X_{T_{k-1}}^4+X_{T_{k-1}}^5+X_{T_{k-1}}^6)}\mid\mathcal{F}_t\Bigg]\\&=e^{\Phi(\Delta,0;1;0;0,1)+\Phi\left(T_{k-1}-t,\sum_{i=0}^3\Psi_i(\Delta,0;1;0;0,1)e_i;1;1,1,1\right)}\\&\quad\times e^{\sum_{j=0}^3\Psi_j\left(T_{k-1}-t,\sum_{i=0}^3\Psi_i(\Delta,0;1;0;0,1)e_i;1;1,1,1\right)X_t^j}1_{\{X_t^4=X_t^5=X_t^6=0\}},\end{aligned}\tag{B.1}$$

$$
\begin{aligned}
S_t^{2k} &= \mathbb{E}\left[e^{-\int_t^{T_k} X_s^0 ds} e^{r+\langle \rho, X_{T_k}\rangle} \lim_{l,m\to\infty} e^{-l(X^4_{T_{k-1}}+X^5_{T_{k-1}})-mX^6_{T_k}} \mid \mathcal{F}_t \right] \\
&= e^r \mathbb{E}\Bigg[e^{-\int_t^{T_{k-1}} X_s^0 ds}\, \mathbb{E}\left[e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} e^{\langle \rho, X_{T_k}\rangle} \lim_{m\to\infty} e^{-mX^6_{T_k}} \mid \mathcal{F}_{T_{k-1}} \right] \\
&\qquad \times \lim_{l\to\infty} e^{-l(X^4_{T_{k-1}}+X^5_{T_{k-1}})} \mid \mathcal{F}_t \Bigg] \\
&= e^{r+\Phi(\Delta,\rho;1;0;0,1)} \mathbb{E}\Bigg[e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i=0}^5 \Psi_i(\Delta,\rho;1;0;0,1) X^i_{T_{k-1}}} \\
&\qquad \times \lim_{l\to\infty} e^{-l(X^4_{T_{k-1}}+X^5_{T_{k-1}}+X^6_{T_{k-1}})} \mid \mathcal{F}_t \Bigg] \\
&= e^{r+\Phi(\Delta,\rho;1;0;0,1)+\Phi\left(T_{k-1}-t, \sum_{i=0}^3 \Psi_i(\Delta,\rho;1;0;0,1)e_i;1;1,1,1\right)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j\left(T_{k-1}-t, \sum_{i=0}^3 \Psi_i(\Delta,\rho;1;0;0,1)e_i;1;1,1,1\right) X_t^j} 1_{\{X_t^4=X_t^5=X_t^6=0\}},
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
S_t^{3k} &= \mathbb{E}\left[e^{-\int_t^{T_k} X_s^0 ds} \lim_{l,m\to\infty} e^{-lX^5_{T_{k-1}}-m(X^4_{T_k}+X^6_{T_k})} \mid \mathcal{F}_t \right] \\
&= \mathbb{E}\Bigg[e^{-\int_t^{T_{k-1}} X_s^0 ds}\, \mathbb{E}\left[e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} \lim_{m\to\infty} e^{-m(X^4_{T_k}+X^6_{T_k})} \mid \mathcal{F}_{T_{k-1}} \right] \\
&\qquad \times \lim_{l\to\infty} e^{-lX^5_{T_{k-1}}} \mid \mathcal{F}_t \Bigg] \\
&= e^{\Phi(\Delta,0;1;1,0,1)} \mathbb{E}\Bigg[e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i\in\{0,\dots,3,5\}} \Psi_i(\Delta,0;1;1,0,1) X^i_{T_{k-1}}} \\
&\qquad \times \lim_{l\to\infty} e^{-l(X^4_{T_{k-1}}+X^5_{T_{k-1}}+X^6_{T_{k-1}})} \mid \mathcal{F}_t \Bigg] \\
&= e^{\Phi(\Delta,0;1;1,0,1)+\Phi\left(T_{k-1}-t, \sum_{i=0}^3 \Psi_i(\Delta,0;1;1,0,1)e_i;1;1,1,1\right)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j\left(T_{k-1}-t, \sum_{i=0}^3 \Psi_i(\Delta,0;1;1,0,1)e_i;1;1,1,1\right) X_t^j} 1_{\{X_t^4=X_t^5=X_t^6=0\}},
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
S_t^{4k} &= \mathbb{E}\left[e^{-\int_t^{T_k} X_s^0 ds} e^{r+\langle \rho, X_{T_k}\rangle} \lim_{l,m\to\infty} e^{-lX^5_{T_{k-1}}-m(X^4_{T_k}+X^6_{T_k})} \mid \mathcal{F}_t \right] \\
&= e^r \mathbb{E}\Bigg[e^{-\int_t^{T_{k-1}} X_s^0 ds}\, \mathbb{E}\left[e^{-\int_{T_{k-1}}^{T_k} X_s^0 ds} e^{\langle \rho, X_{T_k}\rangle} \lim_{m\to\infty} e^{-m(X^4_{T_k}+X^6_{T_k})} \mid \mathcal{F}_{T_{k-1}} \right] \\
&\qquad \times \lim_{l\to\infty} e^{-lX^5_{T_{k-1}}} \mid \mathcal{F}_t \Bigg] \\
&= e^{r+\Phi(\Delta,\rho;1;1,0,1)} \mathbb{E}\Bigg[e^{-\int_t^{T_{k-1}} X_s^0 ds} e^{\sum_{i\in\{0,\dots,3,5\}} \Psi_i(\Delta,\rho;1;1,0,1) X^i_{T_{k-1}}} \\
&\qquad \times \lim_{l\to\infty} e^{-l(X^4_{T_{k-1}}+X^5_{T_{k-1}}+X^6_{T_{k-1}})} \mid \mathcal{F}_t \Bigg] \\
&= e^{r+\Phi(\Delta,\rho;1;1,0,1)+\Phi\left(T_{k-1}-t, \sum_{i=0}^3 \Psi_i(\Delta,\rho;1;1,0,1)e_i;1;1,1,1\right)} \\
&\quad \times e^{\sum_{j=0}^3 \Psi_j\left(T_{k-1}-t, \sum_{i=0}^3 \Psi_i(\Delta,\rho;1;1,0,1)e_i;1;1,1,1\right) X_t^j} 1_{\{X_t^4=X_t^5=X_t^6=0\}}.
\end{aligned} \tag{B.4}
$$

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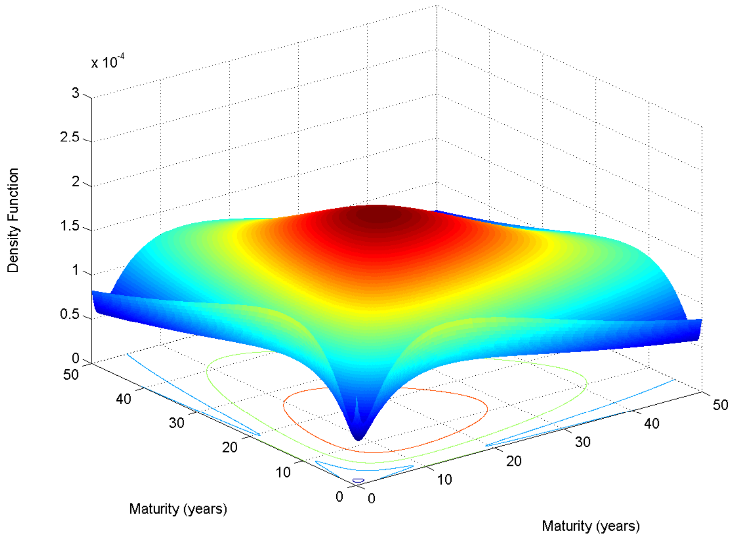

Figure 1: Density function of (τ_1, τ_2) for $\lambda_{(1,0,0),2} = 0$.

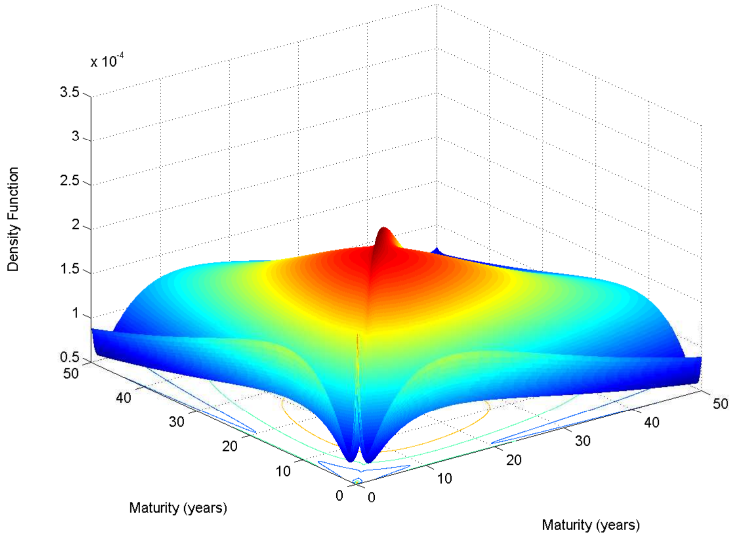

Figure 2: Density function of (τ_1, τ_2) for $\lambda_{(1,0,0),2} = 0.01$.

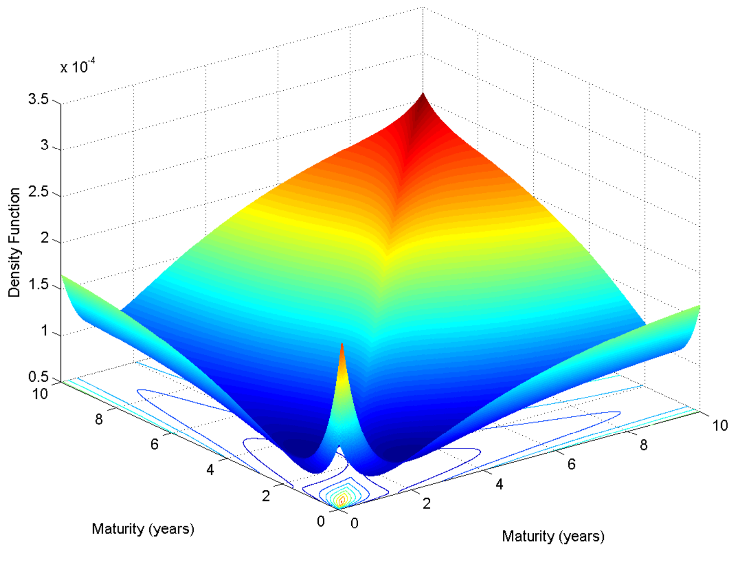

Figure 3: Density function of (τ_1, τ_2) for $\lambda_{(1,0,0),2} = 0.01$ (zoomed).

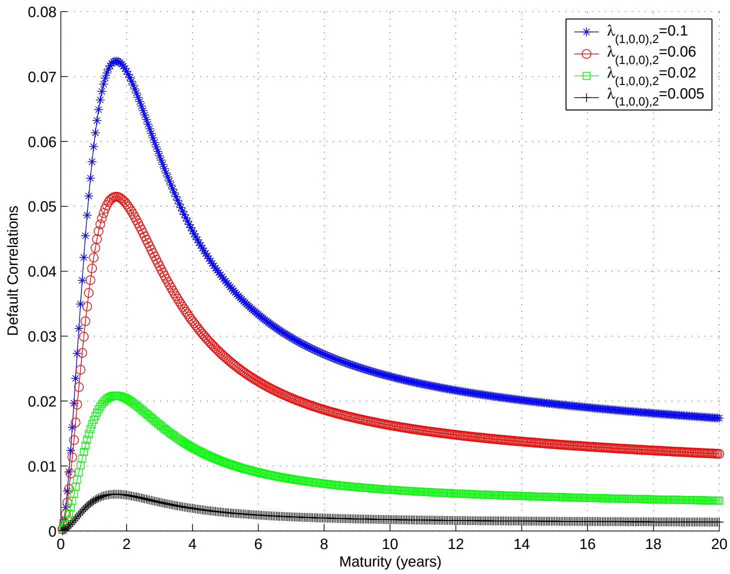

Figure 4: Term structure $T \mapsto corr(T)$ of default correlations (3.9) between two 'Aaa' rated firms for the model (3.5), (3.6) and (3.8) and different values for $\lambda_{(1,0,0),2}$.

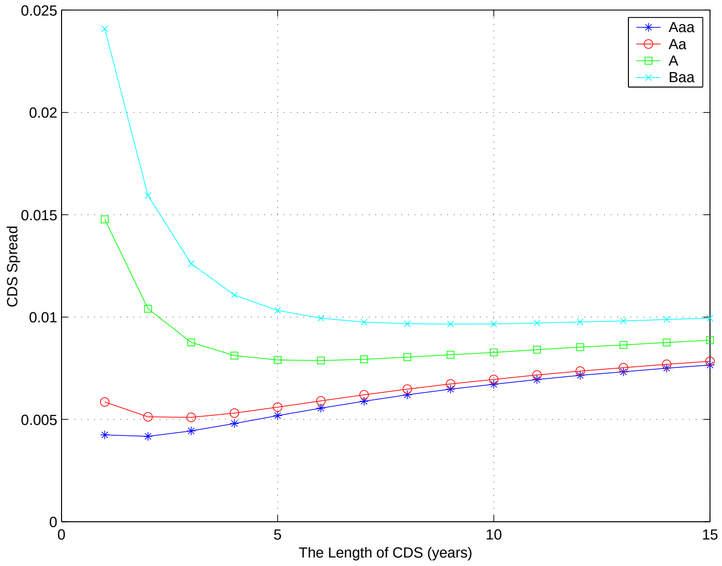

Figure 5: CDS spreads with single-party risk (reference entity)

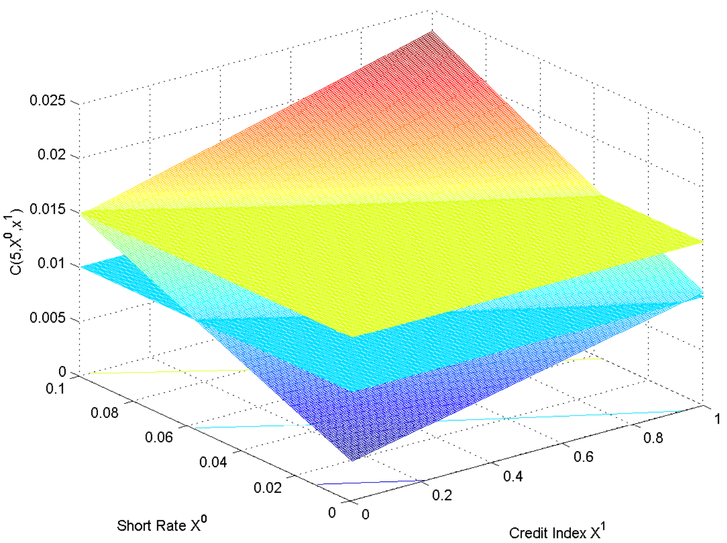

Figure 6: Exercising boundaries (dotted lines in the (X^0, X^1)-plane) of a default swaption with maturity 5 years.