Replicating Portfolio Approach to Capital Calculation

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Literature

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- FOPI (2006). Technical document on the Swiss Solvency Test. http://www.finma.ch/archiv/bpv/download/e/SST_techDok_061002_E_wo_Li_20070118.pdf.
- Swiss Life. Internal Documentation.
- and others ..

Outline

Outline

Purpose of Capital

- Economic capital: minimum amount of equity to ensure ongoing operations of the firm
- Solvency capital requirement (SCR, target capital): appropriate amount of capital to protect policyholders from (the consequences of) insolvency
- Examples: Solvency II, Swiss Solvency Test (SST): recent developments towards "new standard life model" (simplify life fallacy?)
- Focus of this lecture: life insurance

Risk Framework

- Economy with finite time horizon T
- Randomness modeled on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is real-world measure
- Total risk = financial risk and insurance risk
- Financial risk = equity, interest rates, spreads, volatility, ..
- Insurance risk = parameter risk and fluctuation risk

Financial Risk

- Includes all market risk factors such as equity and commodity prices, volatility, interest rates, credit spread, ..
- Generate a filtration $\mathcal{M}_t \subset \mathcal{F}$ for $0 \leq t \leq T$
- All values and cash flows are discounted by some numeraire
- Corresponding pricing measure $\mathbb{Q} \sim \mathbb{P}$

Parameter Risk

- Parameter risk refers to uncertainty in parameter specification and/or their permanent changes
- Includes
  - mortality
  - longevity: expected mortality improvement. Commonly expressed as
    $$q(x,t) = q(x,t_0)\mathrm{e}^{-\lambda_x(t-t_0)}$$
    where $q(x,t)$ = best estimate 1y death probability of an x-year old insured live in year t and λ_x = longevity parameter
  - disability, recovery, surrender/lapse, costs
  - capital option: policyholders choice to opt for a lump-sum settlement rather than an annuity payment
- Best estimate parameter ϑ
- Random "true" parameter Θ

Fluctuation Risk

- Risk of losses due to random fluctuations of actual rates around expected rates during the year under consideration
- Negligible for large and diversified portfolios in normal years
- Captures extreme events such as a pandemic in extreme years
- In life, fluctuation risk is small compared to parameter risk for large and diversified portfolios

Fluctuation Risk: Modelling

- Compound Poisson distribution $Z = \sum_{j=1}^{N} X_j$ where X_j's are i.i.d. random claims and N is the independent number of claims during the year under consideration
- Mean $\mu_Z = \mathbb{E}[Z] = \mathbb{E}[N]\mathbb{E}[X_1]$
- Variance $\sigma_Z^2 = \mathrm{var}[Z] = \mathbb{E}[N]\mathbb{E}[X_1^2]$
- Estimated in practice and used to calibrate an auxiliary distribution, such as a centered normal, for the random fluctuation

$$Y = Z - \mu_Z$$

Fluctuation Risk: Example Mortality

- Risk of random fluctuations of actual mortality rates around expected mortality rates during the year under consideration
- Estimates

$$\widehat{\mathbb{E}[N]} = \sum_{j=1}^{n} q_{x_j}, \quad \widehat{\mathbb{E}[X_1]} = \sum_{j=1}^{n} \frac{s_j}{n}, \quad \widehat{\mathbb{E}[X_1^2]} = \sum_{j=1}^{n} \frac{s_j^2}{n}$$

where

  - n = number of policies for which payment of benefits (lump sum or multiple payments) is contingent on mortality risk
  - q_x = best estimate death probability of x-year old insured live
  - s_j = (deterministic) sum-at-risk of insured live number j
- Estimated mean and variance

$$\hat{\mu}_Z = \sum_{j=1}^{n} q_{x_j} \sum_{j=1}^{n} \frac{s_j}{n}, \quad \hat{\sigma}_Z^2 = \sum_{j=1}^{n} q_{x_j} \sum_{j=1}^{n} \frac{s_j^2}{n}$$

Fluctuation Risk: Example Mortality Numerical Illustration

- Portfolio comprising $n = 1,000,000$ identical insured lives
- Best estimate mortality rate $\bar{q} = 0.01$, sum assured $\bar{s} = 50,000$
- Estimated standard deviation of the mortality fluctuation risk

$$\hat{\sigma}_Z = \sqrt{n\bar{q}\bar{s}^2} = 5 \times 10^6$$

- Estimated mean $\hat{\mu}_Z = n\bar{q}\bar{s} = 5 \times 10^8$
- Coefficient of variation $\hat{\sigma}_Z / \hat{\mu}_Z = 0.01$

Liability Cash Flows

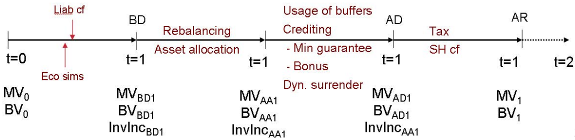
- Asset-liability portfolio with annual liability cash flows

$$Z_t, \quad t = 1 \dots T$$

- Arising from written insurance policies in force at $t = 0$
- May include anticipated new business of the following k years (typically, $k = 1$)
- Liability cash flows include
  - traditional life insurance obligations (with-profits products)
  - unit linked business (guarantees)
  - tax payments
  - cost
  - external reinsurance premium
  - ..

Liability Cash Flows: Illustration

Illustration of a typical cash flow calculation process with Prophet



Source: Swiss Life

1. BD (Before Decision): asset returns projected → direct investment income, new asset book values, liability cash flows
2. Asset Rebalancing: according to strategic asset allocation
3. Crediting: management rules → realise/build buffers (URG), guarantees, PH bonus, surrender, SH profits, tax payments, ..

Liability Cash Flows: Formalization

Liability cash flow

$$Z_t = Z_t^\vartheta = \underbrace{\mu_{Z,t}^\vartheta}_{\text{best estimate}} + \underbrace{Y_t}_{\text{fluctuation}}$$

is a (non-linear) function of

- time t
- insurance parameter ϑ
- market factors trajectory up to t: $\mu_{Z,t}^\vartheta$ is $\mathcal{M}_t$-measurable
- random fluctuation Y_t

Probabilistic Assumptions

- Financial risks $\mathcal{M}_T$ and insurance risks $\mathcal{I} = \sigma(\Theta, Y_1 \dots Y_T)$ are independent
- Density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is $\mathcal{M}_T$-measurable. Hence $\mathbb{Q} = \mathbb{P}$ on $\mathcal{I}$: no risk premium for insurance risk under $\mathbb{Q}$
- Fluctuations are centered $\mathbb{E}^{\mathbb{P}}[Y_t] = 0$.

Best Estimate Liabilities

- Define expected best estimate liability

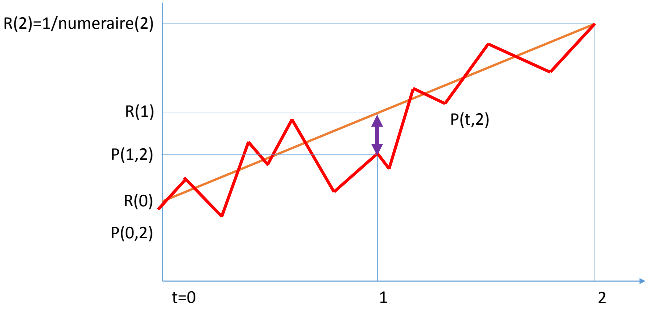
$$L_t = L_t^\vartheta = \mathbb{E}^{\mathbb{Q}}\left[\sum_{s=t+1}^{T} Z_s^\vartheta \mid \mathcal{M}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\sum_{s=t+1}^{T} \mu_{Z,s}^\vartheta \mid \mathcal{M}_t\right]$$

- L_t is not a prudential provision (LLN, no safety loading)
- Will add market value margin later ..

Best Estimate Liabilities: Example Surrender Option

- Endowment insurance with term $T = 2$
- Statutory reserve $R_t =$ surrender value, $R_2 = 1/\mathrm{numeraire}_2$
- Surrender option at $t = 1$
- Life time of insured τ

$P(t, T) =$ (discounted) T-bond price at t



Best Estimate Liabilities: Example Surrender Option ctd

Cash flows:

$$Z_1 = R_1 1_{\{\tau \le 1\}} + R_1 1_{\{R_1 > P(1,2)\}} 1_{\{\tau > 1\}}$$
$$Z_2 = R_2 1_{\{R_1 \le P(1,2)\}} 1_{\{\tau > 1\}}$$

Best estimate liability at $t = 0$:

$$\begin{aligned}
L_0 &= \mathbb{E}^{\mathbb{Q}}[Z_1 + Z_2] = \mathbb{E}^{\mathbb{Q}}\left[Z_1 + \mathbb{E}^{\mathbb{Q}}[Z_2 \mid \mathcal{M}_1]\right] \\
&= \mathbb{E}^{\mathbb{Q}}[R_1]\,\mathbb{Q}[\tau \le 1] + \mathbb{E}^{\mathbb{Q}}\left[R_1 1_{\{R_1 > P(1,2)\}}\right]\mathbb{Q}[\tau > 1] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[P(1,2)\left(1 - 1_{\{R_1 > P(1,2)\}}\right)\right]\mathbb{Q}[\tau > 1] \\
&= P(0,1) R_1^{\text{nominal}} \mathbb{P}[\tau \le 1] && \text{PV death by } t = 1 \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[(R_1 - P(1,2))^+\right]\mathbb{P}[\tau > 1] && \text{PV caplet} \\
&\quad + P(0,2)\mathbb{P}[\tau > 1] && \text{PV terminal cash flow}
\end{aligned}$$

Assets

- Liability cash flows depend (non-linearly) on performance of market and book values of asset portfolio
- Asset portfolio invested in m financial instruments (and numeraire) with gains processes

$$G_t = (G_{1t}, \ldots, G_{mt})^\top$$

- Rebalancing only end/beginning of year
- Asset allocation given by

$$a_t = (a_{1t}, \ldots, a_{mt})^\top$$

  where a_{it} = units of instrument i held in year $t = 1 \ldots T$
- a_t is fixed beginning of year t and $\mathcal{M}_{t-1}$-measurable

Total Asset Value Process

- Asset value change $\Delta A_t = A_t - A_{t-1}$ given by

$$\Delta A_t = a_t^\top \Delta G_t - Z_t - \Delta SH_t$$

where ΔSH_t = other cash flows (shareholder dividends)

- Total asset value process is

$$A_t = A_0 + \sum_{s=1}^{t} a_s^\top \Delta G_s - \sum_{s=1}^{t} Z_s - SH_t$$

where SH_t = accumulated other cash flows

Gross Available Capital

- Gross (of market value margin) available capital at t

$$\bar{C}_t = A_t - L_t$$

- Annual gross result $\Delta\bar{C}_t = \bar{C}_t - \bar{C}_{t-1}$ given by

$$\Delta\bar{C}_t = \underbrace{a_t^\top \Delta G_t}_{\text{financial result}} + \underbrace{L_{t-1} - Z_t - L_t}_{\text{insurance result}} \underbrace{-\Delta SH_t}_{\text{other cash flows}}$$

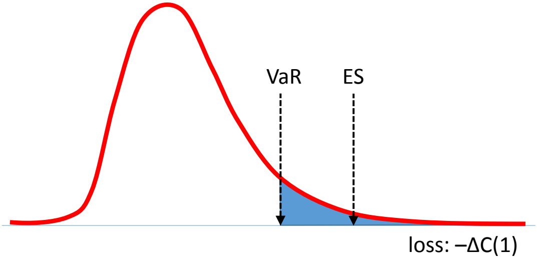
- Terminal condition: asset-liability portfolio is liquidated at T
- All remaining liabilities are subsumed in the last cash flow, $Z_T = Z_T + L_T$, such that $L_T = 0$ and $\bar{C}_T = A_T$.

Solvency Capital Requirement (SCR)

- Risk measure ρ = expected shortfall or Value at Risk
- Solvency II: $\rho = \mathrm{VaR}_{99.5\%}$
- SST: $\rho = \mathrm{ES}_{99\%}$
- Solvency capital requirement (SCR)

$$K = \rho[-\Delta C_1]$$

where C_t = net available capital (surplus) ..



Market Value Margin (MVM)

- Cost of capital approach: market value margin should cover cost of future “run-off” solvency capital requirements

$$\text{“}MVM_t = \mathrm{CoC} \times \sum_{s=t+1}^{T} \rho[-\Delta C_s^{\text{run-off}}]\text{”}$$

  for cost of capital rate CoC (e.g. 6% in Solvency II and SST)
- Terminal condition: $MVM_T = 0$
- “Market value” of liabilities

$$\bar{L}_t = L_t + MVM_t$$

- Net available capital

$$C_t = A_t - \bar{L}_t = \bar{C}_t - MVM_t$$

Simplifying Assumption

- Net annual result

$$\Delta C_t = \underbrace{a_t^\top \Delta G_t}_{\text{financial result}} + \underbrace{L_{t-1} - Z_t - L_t - \Delta MVM_t}_{\text{gross insurance result}} \underbrace{-\Delta SH_t}_{\text{other cash flows}}$$

- **Assume** that $-\Delta MVM_t = \Delta SH_t$ shareholder dividend
- Consequence: net annual result is simply given by

$$\Delta C_t = \underbrace{a_t^\top \Delta G_t}_{\text{financial result}} + \underbrace{L_{t-1} - Z_t - L_t}_{\text{net insurance result}}$$

SCR Revisited: Parameter Risk

- Is L_1 equal to L_1^ϑ or L_1^Θ ?
- In reality $L_1 = L_1^\vartheta$, or slightly (Bayesian) updated ϑ, ..
- .. but for SCR we also need to capture parameter risk
- Assume at $t = 1$ we know true parameter Θ, total 1y risk is

$$\Delta C_1 = C_1^\Theta - C_0^\vartheta = a_1^\top \Delta G_1 + L_0^\vartheta - Z_1^\Theta - L_1^\Theta = -M - P - F$$

with

- market loss

$$M = -a_1^\top \Delta G_1 - L_0^\vartheta + \mu_{Z,1}^\vartheta + L_1^\vartheta$$

- parameter loss

$$P = (\mu_{Z,1}^\Theta + L_1^\Theta) - (\mu_{Z,1}^\vartheta + L_1^\vartheta) \approx \mathbb{E}^\mathbb{Q}[P \mid \Theta] = L_0^\Theta - L_0^\vartheta$$

- fluctuation loss

$$F = Y_1$$

SCR Revisited: Parameter Risk Alternative

- Alternatively, consider "real" annual result

$$\Delta C_1 = C_1^{\vartheta} - C_0^{\vartheta} = a_1^{\top} \Delta G_1 + L_0^{\vartheta} - Z_1^{\vartheta} - L_1^{\vartheta} = -M - F$$

- To capture parameter risk, add $C_0^{\Theta} - C_0^{\vartheta} = -P$:

$$\Delta C_1 + C_0^{\Theta} - C_0^{\vartheta} = -M - P - F$$

- Note that parameter uncertainty is not linearly priced:

$$L_0^{\vartheta} \neq \mathbb{E}^{\mathbb{Q}}[L_0^{\Theta}]$$

- Future research: develop Bayesian framework, uncertainty quantification, ..

Summary of Assumptions

1. Financial $\mathcal{M}_T$ and insurance $\mathcal{I} = \sigma(\Theta, Y_1 \dots Y_T)$ independent
2. Density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is $\mathcal{M}_T$-measurable: no $\mathbb{Q}$-risk premium for $\mathcal{I}$
3. Liability cash flows

$$Z_t = Z_t^\vartheta = \mu_{Z,t}^\vartheta + Y_t$$

   where
   - Best estimate $\mu_{Z,t}^\vartheta$ is $\mathcal{M}_t$-measurable
   - Fluctuations are centered $\mathbb{E}^{\mathbb{P}}[Y_t] = 0$
4. Asset allocation a_t is $\mathcal{M}_{t-1}$-measurable

Hence: market loss M and insurance losses (P, F) independent

Discussion: for projections, should ..

- $\mu_{Z,2}^\vartheta \dots \mu_{Z,T}^\vartheta$ and $a_2 \dots a_T$ also depend on Y_1?
- more general, $\mu_{Z,t}^\vartheta$ and a_t also depend on $Y_1 \dots Y_{t-1}$?

Computational Issues

- Simulations of
  - $L_1^{\vartheta} = \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=2}^{T} \mu_{Z,t}^{\vartheta} \mid \mathcal{M}_1\right]$
  - $L_0^{\Theta} = \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \mu_{Z,t}^{\Theta} \mid \Theta\right]$

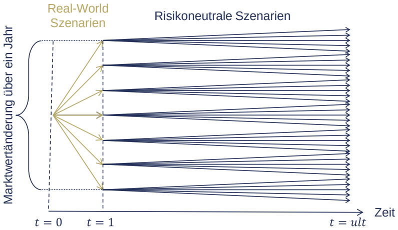
  are computationally extremely costly (nested simulations!)
- Despite assumption that for projections we neglect liability fluctuations $Y_1 \ldots Y_T$ ($\rightarrow$ only market samples needed)
- For every "outer" $\mathcal{M}_1$-sample generate "inner" projections for $\mu_{Z,2}^{\vartheta} \cdots \mu_{Z,T}^{\vartheta}$
- For every "outer" Θ-sample generate "inner" projections for $\mu_{Z,1}^{\Theta} \cdots \mu_{Z,T}^{\Theta}$
- Much more samples if $\mu_{Z,t}^{\vartheta}$ and a_t also depend on $Y_1 \ldots Y_{t-1}$

Problems

Simulating inner projections $\mu^{\vartheta}_{Z,1} \dots \mu^{\vartheta}_{Z,T}$ is costly:

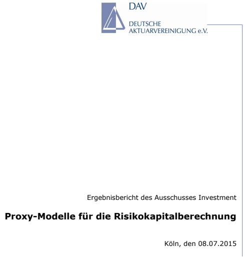
- large time horizon $T \geq 40$ years
- liability cash flows are strongly path dependent:
  - embedded options, e.g. minimum rate guarantees
  - management & regulatory rules, e.g. policyholder participation
  - policyholder behaviour, e.g. lapsing
- number of economic factors large $\sim 10^3$

Nested simulation is computationally extremely costly:



Source: DAV 2015

German Actuarial Society (DAV) Report 2015



DAV
DEUTSCHE
AKTUARVEREINIGUNG e.V.

Ergebnisbericht des Ausschusses Investment

Proxy-Modelle für die Risikokapitalberechnung

Köln, den 08.07.2015

- Compares proxy methods for capital calculation in life ins.
- Pros and cons for nested simulation (too slow), curve fitting, least-squares Monte Carlo (use abstract function classes), and replicating portfolio (uses financial instruments as special functions with a clear model-independent interpretation)
- Advise match terminal discounted value rather than cash flows
- In practice $1000 \leq n \leq 5000$ samples of $\mu^{\vartheta}_{Z,1} \dots \mu^{\vartheta}_{Z,T}$
- In practice $10 \leq m \leq 50$ financial instruments

Outline

Ingredients

- Fixed time horizon T
- $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with real-world measure $\mathbb{P}$
- All values and cashflows discounted by some numeraire
- Corresponding risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ has density process

$$D_t = \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right]$$

- $\mathbb{M}$ placeholder for either $\mathbb{P}$ or $\mathbb{Q}$

Standing technical assumption:

$$\|D_T\|_{L^2(\mathbb{P})}, \|D_T/D_1\|_{L^2(\mathbb{P})}, \|1/D_T\|_{L^2(\mathbb{Q})} < \infty$$

Capital Calculation in Stylized Form

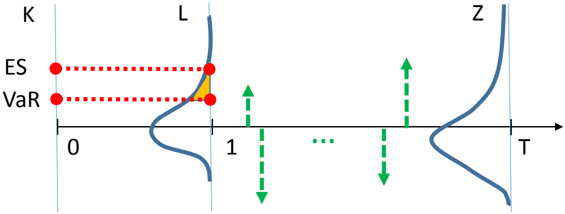
- Terminal loss of asset-liability portfolio $Z \in L^2(\mathbb{M})$ at $t = T$
- Portfolio is fairly price at $t = 0$ such that $\mathbb{E}^{\mathbb{Q}}[Z] = 0$
- One-year loss is given by

$$L = \mathbb{E}^{\mathbb{Q}}\left[Z \mid \mathcal{F}_1\right]$$

Goal: solvency capital calculation:

$$K = \rho[L]$$

where ρ is placeholder for either VaR_α or ES_α



Basic Lemma

- For $X, Y \in L^2(\mathbb{M})$:

$$\left\| \mathbb{E}^{\mathbb{Q}}\left[X \mid \mathcal{F}_1\right] - \mathbb{E}^{\mathbb{Q}}\left[Y \mid \mathcal{F}_1\right] \right\|_{L^1(\mathbb{P})} \leq C \|X - Y\|_{L^2(\mathbb{M})}$$

- For $X, Y \in L^1(\mathbb{P})$:

$$|\mathrm{ES}_\alpha[X] - \mathrm{ES}_\alpha[Y]| \leq \frac{1}{1-\alpha} \|X - Y\|_{L^1(\mathbb{P})}$$

- For $X_n \to X$ in law, such that $q_\alpha^-[X] = q_\alpha^+[X]$:

$$\lim_{n\to\infty} \mathrm{VaR}_\alpha[X_n] = \mathrm{VaR}_\alpha[X]$$

Replicating Portfolio Approach

Goal: approximate Z in $L^2(\mathbb{M})$, and thus L in $L^1(\mathbb{P})$, by a portfolio invested in m financial instruments

$$\boldsymbol{G}_t = (G_{1t}, \ldots, G_{mt})^\top$$

that can be efficiently simulated.

Candidate Replicating Instruments

- zero-coupon bonds
- inflation linked bonds
- equity index
- real estate index
- swaptions
- equity options
- ...

Bonds and options spot and forward starting

Dynamic Portfolio Strategies for $m = 1$ Instrument

- Fix partition $0 = t_0 < t_1 < \cdots < t_N = T$ containing $\{1 \ldots T\}$
- Write
$$\Delta G_j = G_{t_j} - G_{t_{j-1}}$$
- Chaos expansion: portfolio strategies are linear in the running product of gains ΔG_j
- $\mathcal{P}$ family of $\mathcal{J}$ where $\mathcal{J}$ is a subset of $\{1, \ldots, N\}$
- For any $\mathcal{J} \in \mathcal{P}$ define corresponding product of gains
$$\Delta \boldsymbol{G}_{\mathcal{J}} = \prod_{j \in \mathcal{J}} \Delta G_j$$
- Absence of arbitrage: $\boldsymbol{G}_t$ is a $\mathbb{Q}$-martingale:
$$\mathbb{E}^{\mathbb{Q}} \left[\Delta \boldsymbol{G}_{\mathcal{J}} \mid \mathcal{F}_{t_j} \right] = 0 \quad \text{for all } j < \min \mathcal{J}$$

Dynamic Portfolio Strategies for $m = 1$ Instrument

- Any choice of $\boldsymbol{\phi} = \{\phi_{\mathcal{J}} \mid \mathcal{J} \in \mathcal{P}\} \in \mathbb{R}^{|\mathcal{P}|}$ and initial wealth v gives self-financing portfolio with value process

$$V_t^{v,\boldsymbol{\phi}} = v + \sum_{\mathcal{J} \in \mathcal{P} | t_{\max \mathcal{J}} \leq t} \phi_{\mathcal{J}} \Delta \boldsymbol{G}_{\mathcal{J}}.$$

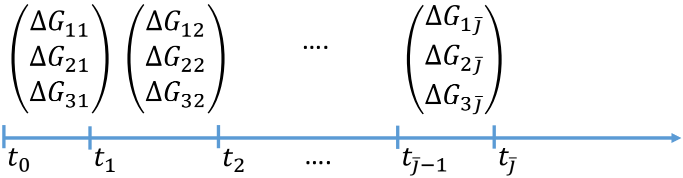
- Absence of arbitrage implies that $V_t^{v,\boldsymbol{\phi}}$ is a $\mathbb{Q}$-martingale.
- Positions in the instruments $\boldsymbol{G}_t$ path-dependent: $\bar{j} = \max \mathcal{J}$

$$\phi_{\mathcal{J}} \Delta \boldsymbol{G}_{\mathcal{J}} = \underbrace{\phi_{\mathcal{J}} \prod_{j \in \mathcal{J} \setminus \{\bar{j}\}} \Delta G_j}_{\text{position}} \times \underbrace{\left(G_{t_{\bar{j}}} - G_{t_{\bar{j}}-1}\right)}_{\text{gain over } (t_{\bar{j}-1}, t_{\bar{j}}]}$$

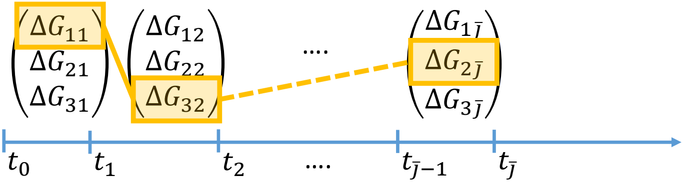
Dynamic Portfolio Strategies for $m > 1$

- Fix partition $0 = t_0 < t_1 < \cdots < t_N = T$ containing $t_j = 1$
- Write
$$\Delta G_{ij} = G_{i,t_j} - G_{i,t_{j-1}}$$
- Chaos expansion: portfolio strategies are linear in the running product of gains ΔG_{ij}
- $\mathcal{P}$ family of pairs $(\mathcal{I}, \mathcal{J})$ where $\mathcal{J}$ is a subset of $\{1, \ldots, N\}$ and $\mathcal{I} : \mathcal{J} \to \{1, \ldots, m\}$ is a mapping
- For any $(\mathcal{I}, \mathcal{J}) \in \mathcal{P}$ define corresponding product of gains
$$\Delta \boldsymbol{G}_{(\mathcal{I},\mathcal{J})} = \prod_{j \in \mathcal{J}} \Delta G_{\mathcal{I}(j)j}$$
- Absence of arbitrage: $\boldsymbol{G}_t$ is a $\mathbb{Q}$-martingale:
$$\mathbb{E}^{\mathbb{Q}} \left[\Delta \boldsymbol{G}_{(\mathcal{I},\mathcal{J})} \mid \mathcal{F}_{t_j} \right] = 0 \quad \text{for all } j < \min \mathcal{J}$$

Dynamic Portfolio Strategies for $m > 1$



Dynamic Portfolio Strategies for $m > 1$



Dynamic Portfolio Strategies for $m > 1$

- Any choice of $\boldsymbol{\phi} = \{\phi_{(\mathcal{I},\mathcal{J})} \mid (\mathcal{I},\mathcal{J}) \in \mathcal{P}\} \in \mathbb{R}^{|\mathcal{P}|}$ and initial wealth v gives self-financing portfolio with value process

$$V_t^{v,\boldsymbol{\phi}} = v + \sum_{(\mathcal{I},\mathcal{J})\in\mathcal{P} \mid t_{\max \mathcal{J}} \le t} \phi_{(\mathcal{I},\mathcal{J})} \Delta \boldsymbol{G}_{(\mathcal{I},\mathcal{J})}.$$

- Absence of arbitrage implies that $V_t^{v,\boldsymbol{\phi}}$ is a $\mathbb{Q}$-martingale.
- Positions in the instruments $\boldsymbol{G}_t$ path-dependent: $\bar{j} = \max \mathcal{J}$

$$\phi_{(\mathcal{I},\mathcal{J})} \Delta \boldsymbol{G}_{(\mathcal{I},\mathcal{J})} = \underbrace{\phi_{(\mathcal{I},\mathcal{J})} \prod_{j \in \mathcal{J} \setminus \{\bar{j}\}} \Delta G_{\mathcal{I}(j)j}}_{\text{position}} \times \underbrace{\left(G_{\mathcal{I}(\bar{j}),t_{\bar{j}}} - G_{\mathcal{I}(\bar{j}),t_{\bar{j}}-1} \right)}_{\text{gain over } (t_{\bar{j}-1}, t_{\bar{j}}]}$$

Example: first order portfolio

- Assume $|\mathcal{J}| = 1$ for all $(\mathcal{I}, \mathcal{J}) \in \mathcal{P}$
- Obtain first order portfolio with value process

$$V_t^{v,\phi} = v + \sum_{t_j \leq t} \sum_{i=1}^{m} \phi_{ij} \Delta G_{ij}$$

for the components $\phi_{ij} = \phi_{(\mathcal{I},\mathcal{J})}$ where $\mathcal{J} = \{j\}$ and $\mathcal{I}(j) = i$.

Example: univariate chaos expansion

- Assume one risky asset $m = 1$
- Omit trivial mapping $\mathcal{I}(j) \equiv 1$, and write $\phi^{\mathcal{I}}_{\mathcal{J}} = \phi_{\mathcal{J}}$.
- $\mathcal{P}$ becomes a family of subsets $\mathcal{J}$ of $\{1, \ldots, N\}$ and we write

$$\Delta \boldsymbol{G}_{\mathcal{J}} = \prod_{j \in \mathcal{J}} \Delta G_j$$

and

$$V_t^{v,\phi} = v + \sum_{\mathcal{J} \in \mathcal{P} | t_{\max \mathcal{J}} \leq t} \phi_{\mathcal{J}} \Delta \boldsymbol{G}_{\mathcal{J}}.$$

Simplifying Notation

- Portfolio gains up to one year

$$\boldsymbol{A} = \left(\Delta \boldsymbol{G}_{(\mathcal{I},\mathcal{J})} \mid t_{\max \mathcal{J}} \leq 1\right)^{\top}$$

- Portfolio gains up beyond one year

$$\boldsymbol{B} = \left(\Delta \boldsymbol{G}_{(\mathcal{I},\mathcal{J})} \mid t_{\max \mathcal{J}} > 1\right)^{\top}$$

- Portfolio values of $V_t^{v,\boldsymbol{\phi}}$ at $t = 0, 1, T$ become

$$V_0^{v,\boldsymbol{\phi}} = v, \quad V_1^{v,\boldsymbol{\phi}} = v + \phi_A^{\top}\boldsymbol{A}, \quad V_T^{v,\boldsymbol{\phi}} = v + \phi_A^{\top}\boldsymbol{A} + \phi_B^{\top}\boldsymbol{B}.$$

Replicating Portfolio

- Choose (v, ϕ) that solves the $\mathbb{M}$-least-squares problem

$$\min_{(v,\phi)\in\mathbb{R}^{1+|\mathcal{P}|}} \left\| Z - V_T^{v,\phi} \right\|_{L^2(\mathbb{M})} . \tag{P}$$

- Corresponding $V_t^{v,\phi}$ is called replicating portfolio (RP)

Solution for $\mathbb{M} = \mathbb{P}$

For $\mathbb{M} = \mathbb{P}$ the formal solution is given by

$$\begin{pmatrix} v^{\mathbb{P}} \\ \phi_A^{\mathbb{P}} \\ \phi_B^{\mathbb{P}} \end{pmatrix} = \mathcal{M}^{-1} \mathbb{E}^{\mathbb{P}} \left[\begin{pmatrix} Z \\ \boldsymbol{A} Z \\ \boldsymbol{B} Z \end{pmatrix} \right]$$

with Gram matrix

$$\mathcal{M} = \mathbb{E}^{\mathbb{P}} \left[\begin{pmatrix} 1 & \boldsymbol{A}^\top & \boldsymbol{B}^\top \\ \boldsymbol{A} & \boldsymbol{A}\boldsymbol{A}^\top & \boldsymbol{A}\boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{B}\boldsymbol{A}^\top & \boldsymbol{B}\boldsymbol{B}^\top \end{pmatrix} \right].$$

Solution for $\mathbb{M} = \mathbb{Q}$

$\mathbb{Q}$-martingale property: $\mathbb{E}^{\mathbb{Q}}[\boldsymbol{A}] = 0$, $\mathbb{E}^{\mathbb{Q}}[\boldsymbol{B}] = 0$, and $\mathbb{E}^{\mathbb{Q}}[\boldsymbol{A}\boldsymbol{B}^\top] = 0$.

For $\mathbb{M} = \mathbb{Q}$ the formal solution of (P) is thus given by

$$v^{\mathbb{Q}} = 0, \quad \begin{pmatrix} \phi_A^{\mathbb{Q}} \\ \phi_B^{\mathbb{Q}} \end{pmatrix} = \mathcal{N}^{-1} \mathbb{E}^{\mathbb{Q}} \left[\begin{pmatrix} \boldsymbol{A} Z \\ \boldsymbol{B} Z \end{pmatrix} \right]$$

with block-diagonal reduced Gram matrix

$$\mathcal{N} = \mathbb{E}^{\mathbb{Q}} \left[\begin{pmatrix} \boldsymbol{A}\boldsymbol{A}^\top & 0 \\ 0 & \boldsymbol{B}\boldsymbol{B}^\top \end{pmatrix} \right].$$

Problem: $\mathcal{M}$ and $\mathcal{N}$ may be close to singular due to possible strong correlation between the instruments $\boldsymbol{G}_t$.

- Numerical problems for their inverse.
- Closed form $\mathcal{M}$ and $\mathcal{N}$ preferred (e.g. polynomial models)

Capital Proxies

- Denote the residual from the $L^2(\mathbb{M})$-projection:

$$\epsilon^{\mathbb{M}} = Z - v^{\mathbb{M}} - \phi_A^{\mathbb{M}\top}\boldsymbol{A} - \phi_B^{\mathbb{M}\top}\boldsymbol{B}$$

- One-year loss:

$$L = v^{\mathbb{M}} + \phi_A^{\mathbb{M}\top}\boldsymbol{A} + \mathbb{E}^{\mathbb{Q}}\left[\epsilon^{\mathbb{M}} \mid \mathcal{F}_1\right]$$

- Two proxies for L:

$$L_1^{\mathbb{M}} = v^{\mathbb{M}} + \phi_A^{\mathbb{M}\top}\boldsymbol{A}$$
$$L_2^{\mathbb{M}} = v^{\mathbb{M}} + \phi_A^{\mathbb{M}\top}\boldsymbol{A} + \epsilon^{\mathbb{M}} = Z - \phi_B^{\mathbb{M}\top}\boldsymbol{B}$$

- Two proxies for capital requirement K:

$$K_1^{\mathbb{M}} = \rho[L_1^{\mathbb{M}}] = v^{\mathbb{M}} + \rho[\phi_A^{\mathbb{M}\top}\boldsymbol{A}]$$
$$K_2^{\mathbb{M}} = \rho[L_2^{\mathbb{M}}] = \rho[Z - \phi_B^{\mathbb{M}\top}\boldsymbol{B}]$$

Discussion

$L_2^{\mathbb{M}}$ is a good proxy for L under any of the equivalent conditions:

- $\mathbb{E}^{\mathbb{Q}}[\epsilon^{\mathbb{M}} \mid \mathcal{F}_1] = \epsilon^{\mathbb{M}}$
- $\epsilon^{\mathbb{M}}$ is $\mathcal{F}_1$-measurable
- Cash flows beyond $t = 1$ are spanned by the instruments $\boldsymbol{G}_t$,

$$Z - L = \phi_B^{\mathbb{M}\top} \boldsymbol{B}.$$

The Replicating Portfolio Approach Works

Basic Lemma implies: $\left\| L - L_i^{\mathbb{M}} \right\|_{L^1(\mathbb{P})} \leq C \left\| \epsilon^{\mathbb{M}} \right\|_{L^2(\mathbb{M})}$.

Meta Corollary: RP approach is asymptotically consistent if $\{1, \boldsymbol{A}, \boldsymbol{B}\}$ forms a basis of $L^2(\mathbb{M})$ asymptotically for number of factors $|\mathcal{P}| \to \infty$: capital approximation error

$$K_i^{\mathbb{M}} \to K.$$

Increasing number of factors $|\mathcal{P}|$ either by increasing

- number m of instruments $\boldsymbol{G}_t$
- number N of time steps or degree of path-dependence $|\mathcal{J}|$

Industry Standard Static First Order RP

Formal: $N = 2$, $\boldsymbol{A} = \boldsymbol{G}_1 - \boldsymbol{G}_0$, $\boldsymbol{B} = \boldsymbol{G}_T - \boldsymbol{G}_1$, $\phi_A = \phi_B = \psi$.

- $\mathbb{M}$-least-squares problem (P):

$$\min_{(v,\psi)\in\mathbb{R}^{1+m}} \left\| Z - v - \psi^\top(\boldsymbol{A} + \boldsymbol{B}) \right\|_{L^2(\mathbb{M})}$$

- For $\mathbb{M} = \mathbb{P}$ the formal solution is given by

$$\begin{pmatrix} \widetilde{v}^{\mathbb{P}} \\ \widetilde{\phi}^{\mathbb{P}}_A \end{pmatrix} = \widetilde{\mathcal{M}}^{-1}\mathbb{E}^{\mathbb{P}}\left[\begin{pmatrix} Z \\ (\boldsymbol{A}+\boldsymbol{B})Z \end{pmatrix}\right], \quad \widetilde{\phi}^{\mathbb{P}}_B = \widetilde{\phi}^{\mathbb{P}}_A,$$

  with Gram matrix

$$\widetilde{\mathcal{M}} = \mathbb{E}^{\mathbb{P}}\left[\begin{pmatrix} 1 & (\boldsymbol{A}+\boldsymbol{B})^\top \\ \boldsymbol{A}+\boldsymbol{B} & (\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}+\boldsymbol{B})^\top \end{pmatrix}\right].$$

- For $\mathbb{M} = \mathbb{Q}$ the formal solution is given by

$$\widetilde{v}^{\mathbb{Q}} = 0, \quad \widetilde{\phi}^{\mathbb{Q}}_A = \widetilde{\phi}^{\mathbb{Q}}_B = \widetilde{\mathcal{N}}^{-1}\mathbb{E}^{\mathbb{Q}}\left[(\boldsymbol{A}+\boldsymbol{B})Z\right]$$

  with reduced Gram matrix

$$\widetilde{\mathcal{N}} = \mathbb{E}^{\mathbb{Q}}\left[(\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}+\boldsymbol{B})^\top\right].$$

Choice of Projection Measure $\mathbb{M} = \mathbb{P}$ or $\mathbb{Q}$?

Arguments for $\mathbb{M} = \mathbb{Q}$:

- RP expressions simplify, no need to estimate $v^{\mathbb{Q}} = 0$
- Long-term projections: $\boldsymbol{G}_t$ is martingale under $\mathbb{Q}$, no need to specify market price of risk (propagating model error)
- Specification of insurance risks under $\mathbb{Q}$: usual assumption D_t and insurance risk cash flows $\mathbb{P}$-independent, such that $\mathbb{Q} = \mathbb{P}$ on (σ-field of) insurance risks
- Can also simulate $1/D_t$ for quantification of real-world likelihoods of risk-neutral scenarios.

Outline

Simulation-based $L^2(\mathbb{M})$ Projection (P)

- Assume $\boldsymbol{A}Z, \boldsymbol{B}Z \in L^2(\mathbb{M})$.
- Simulate n i.i.d. copies of $(\boldsymbol{A}, \boldsymbol{B}, Z)$ under $\mathbb{M}$:

$$\left(\boldsymbol{A}^{(j)}, \boldsymbol{B}^{(j)}, Z^{(j)}\right), \quad j = 1 \dots n$$

- For $\mathbb{M} = \mathbb{P}$ obtain unbiased estimators

$$\begin{pmatrix} \widehat{v^{\mathbb{P}}} \\ \widehat{\phi_A^{\mathbb{P}}} \\ \widehat{\phi_B^{\mathbb{P}}} \end{pmatrix} = \mathcal{M}^{-1} \frac{1}{n} \sum_{j=1}^{n} \begin{pmatrix} Z^{(j)} \\ \boldsymbol{A}^{(j)} Z^{(j)} \\ \boldsymbol{B}^{(j)} Z^{(j)} \end{pmatrix}$$

- LLN: $\left(\widehat{v^{\mathbb{P}}}, \widehat{\phi_A^{\mathbb{P}}}, \widehat{\phi_B^{\mathbb{P}}}\right) \to \left(v^{\mathbb{P}}, \phi_A^{\mathbb{P}}, \phi_B^{\mathbb{P}}\right)$ a.s. as $n \to \infty$.
- CLT ✓
- Similarly (simpler) for $\mathbb{M} = \mathbb{Q}$

Monte-Carlo Estimates of Capital Proxies

Estimators of the solvency capital proxies $K_i^{\mathbb{M}}$:

$$\widehat{K_1^{\mathbb{M}}} = \widehat{v^{\mathbb{M}}} + \rho\left[\widehat{\phi_A^{\mathbb{M}}}^{\top}\boldsymbol{A} \mid \mathcal{G}\right]$$
$$\widehat{K_2^{\mathbb{M}}} = \rho\left[Z - \widehat{\phi_B^{\mathbb{M}}}^{\top}\boldsymbol{B} \mid \mathcal{G}\right]$$

where $\mathcal{G}$ is σ-algebra generated by the sample $(\boldsymbol{A}^{(j)}, \boldsymbol{B}^{(j)}, Z^{(j)})$.

Theorem: Monte–Carlo estimates asymptotically consistent:

$$\widehat{K_i^{\mathbb{M}}} \to K_i^{\mathbb{M}} \text{ a.s. as } n \to \infty$$

Monte-Carlo Error

The total capital estimation error amounts to

$$\left\| K - \widehat{K_i^{\mathbb{M}}} \right\|_{L^2(\mathbb{M})} \leq \underbrace{\left| K - K_i^{\mathbb{M}} \right|}_{\text{approximation error}} + \underbrace{\left\| K_i^{\mathbb{M}} - \widehat{K_i^{\mathbb{M}}} \right\|_{L^2(\mathbb{M})}}_{\text{Monte-Carlo error}}$$

Theorem: For $\rho = \mathrm{ES}_\alpha$, asymptotically for large n:

$$\left\| K_i^{\mathbb{M}} - \widehat{K_i^{\mathbb{M}}} \right\|_{L^2(\mathbb{M})} \leq \sqrt{\frac{1}{n}} \times \underbrace{\mathrm{MCE}_i^{\mathbb{M}}}_{\text{constant}}$$

Numerical examples: Monte-Carlo error always dominated by approximation error ($\rightarrow$ prefer more factors over less factors).

Estimation of Value at Risk and Expected Shortfall

Law-invariance: $\rho[X] = \rho[\mu]$ function of $\mathbb{P}$-distribution μ of X.
Theorem: Let X be a random variable satisfying

$$\begin{cases} q_\alpha^-[X] = q_\alpha^+[X], & \text{if } \rho = \mathrm{VaR}_\alpha, \\ X \in L^1(\mathbb{P}), & \text{if } \rho = \mathrm{ES}_\alpha. \end{cases}$$

Let $(X^{(j)}, d\mathbb{P}/d\mathbb{M}^{(j)})$, $j \geq 1$, be an i.i.d. sequence of random variables with the same $\mathbb{M}$-law as $(X, d\mathbb{P}/d\mathbb{M})$. Define weights

$$w^{(j)} = \frac{d\mathbb{P}/d\mathbb{M}^{(j)}}{\sum_{k=1}^n d\mathbb{P}/d\mathbb{M}^{(k)}} \quad (= 1/n \text{ if } \mathbb{M} = \mathbb{P})$$

and the empirical $\mathbb{P}$-distribution of $X^{(1)}, \dots, X^{(n)}$,

$$\widehat{\mu}_n = \textstyle\sum_{j=1}^n w^{(j)} \delta_{X^{(j)}}.$$

Then $\rho[\widehat{\mu}_n] \to \rho[X]$ a.s. as $n \to \infty$.

Outline

Two Sources of Incompleteness of the Insurance Market

Two sources for incompleteness under static hedging with the underlying financial instruments:

- More factors driving insurance cash flows than traded financial instruments for their replication (RP cannot help ..)
- Insurance liability cash flows are nonlinear functions of the financial instruments (RP works!)

Both effects superpose in practice. In the following two examples, we disentangle these effects.

Example 1: Arithmetic Brownian Motion

- $\mathbb{P}$-Brownian motion $W_t = (W_{1t}, \ldots, W_{dt})^\top$ with $d = 5$
- Constant market price of risk $\gamma = 0.1 \times \mathbf{1} \in \mathbb{R}^d$
- Time partition $t_0 = 0$, $t_1 = 1$, $t_2 = T = 5$
- Two volatility regimes $\lambda_A, \lambda_B \in \mathbb{R}^d$ such that

$$\text{one-year loss:} \quad L = \lambda_A^\top (W_1 + \gamma)$$
$$\text{terminal loss:} \quad Z = L + \lambda_B^\top (W_T - W_1 + \gamma(T-1))$$

- $m \leq d$ financial instruments with gains processes

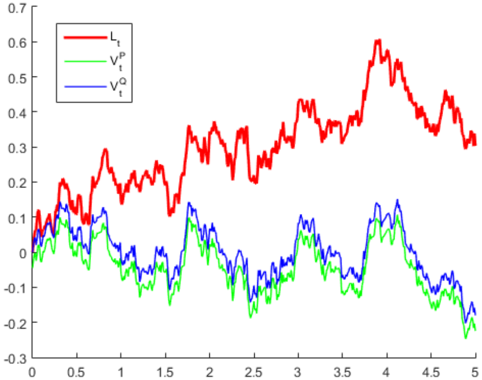
$$\boldsymbol{G}_t = W_{1\ldots m,t} + \gamma_{1\ldots m} t$$

- Closed form capital proxies

Example 1: Arithmetic Brownian Motion

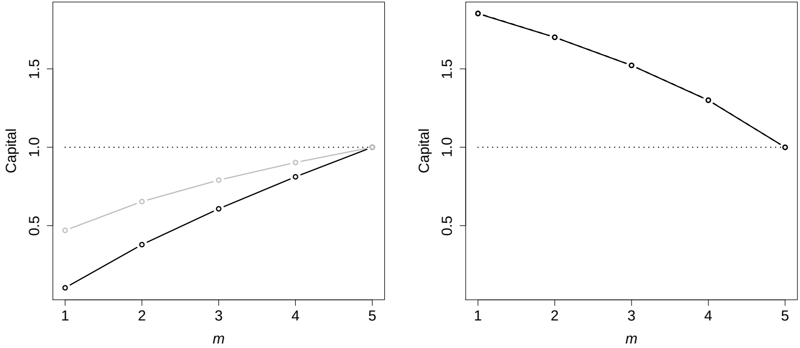
- Two cases ($\lambda_A, \lambda_B \leq 0$ because $\mathbb{E}^{\mathbb{P}}[Z] < 0$):
  1. $\lambda_A = \lambda_B = -0.2 \times \mathbf{1}/\sqrt{d}$: constant volatility, $\widetilde{L}_i^{\mathbb{M}} = L_i^{\mathbb{M}}$.
  2. $\lambda_A = -0.2 \times \mathbf{1}/\sqrt{d}$, $\lambda_B = 0$: no cash flows beyond $t = 1$, $L_2^{\mathbb{P}} = L_2^{\mathbb{Q}} = L$ are exact proxies.
- Risk measure $\rho = \mathrm{ES}_{99\%}$
- Capital requirements and proxies normalised: $K = 1$

Loss & RP Trajectories Case 1: $\lambda_A = \lambda_B = -0.2 \times \frac{1}{\sqrt{d}}$



Loss trajectory $L_t = \mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_t]$ and RPs $V_t^{\mathbb{P}}$ and $V_t^{\mathbb{Q}}$ for $m = 4$

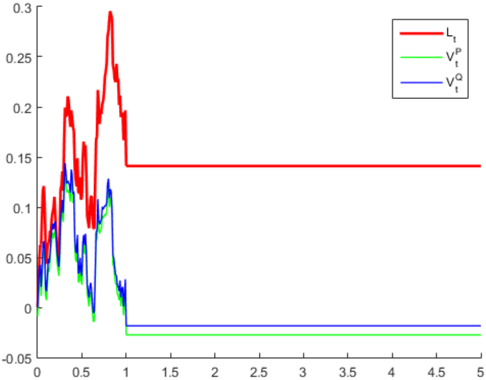
Capital Proxies Case 1: $\lambda_A = \lambda_B = -0.2 \times \frac{1}{\sqrt{d}}$



Left panel: $K = 1$ (dotted), $K_1^{\mathbb{P}} = \widetilde{K}_1^{\mathbb{P}}$ (black), $K_1^{\mathbb{Q}} = \widetilde{K}_1^{\mathbb{Q}}$ (grey).

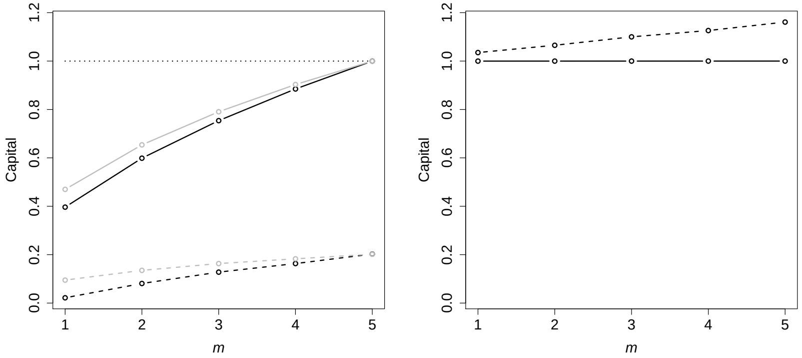
Right panel: $K = 1$ (dotted), $K_2^{\mathbb{P}} = K_2^{\mathbb{Q}} = \widetilde{K}_2^{\mathbb{P}} = \widetilde{K}_2^{\mathbb{Q}}$ (black).

Loss & RP Trajectories Case 2: $\lambda_A = -0.2 \times \frac{1}{\sqrt{d}}$, $\lambda_B = 0$



Loss trajectory $L_t = \mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_t]$ and RPs $V_t^{\mathbb{P}}$ and $V_t^{\mathbb{Q}}$ for $m = 4$

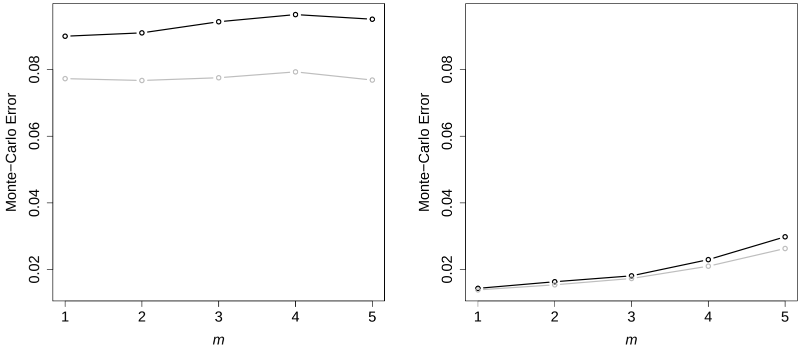
Capital Proxies Case 2: $\lambda_A = -0.2 \times \frac{1}{\sqrt{d}}$, $\lambda_B = 0$



Left panel: $K = 1$ (dotted), $K_1^{\mathbb{P}}$ (black solid), $K_1^{\mathbb{Q}}$ (grey solid), $\widetilde{K}_1^{\mathbb{P}}$ (black dashed), $\widetilde{K}_1^{\mathbb{Q}}$ (grey dashed).

Right panel: $K = 1 = K_2^{\mathbb{P}} = K_2^{\mathbb{Q}}$ (solid), $\widetilde{K}_2^{\mathbb{P}} = \widetilde{K}_2^{\mathbb{Q}}$ (dashed).

Monte-Carlo Error Case 1: $\lambda_A = \lambda_B = -0.2 \times \frac{1}{\sqrt{d}}$

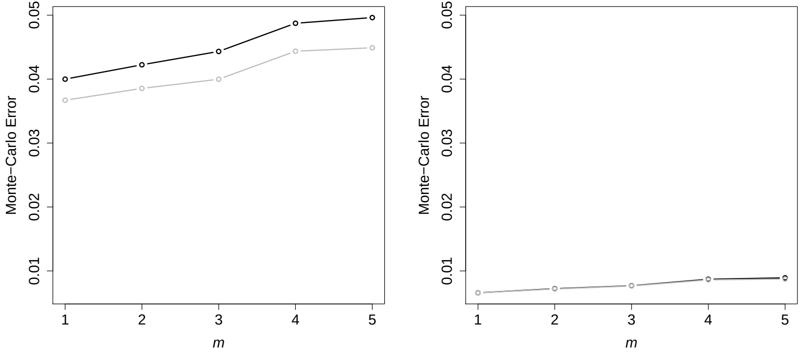


Left panel: MC error for $K_1^{\mathbb{P}}$ (black) and $K_1^{\mathbb{Q}}$ (grey).

Right panel: MC error for $K_2^{\mathbb{P}}$ (black) and $K_2^{\mathbb{Q}}$ (grey).

Sample size $n = 1000$

Monte-Carlo Error Case 2: $\lambda_A = -0.2 \times \frac{1}{\sqrt{d}}$, $\lambda_B = 0$



Left panel: MC error for $K_1^{\mathbb{P}}$ (black) and $K_1^{\mathbb{Q}}$ (grey).

Right panel: MC error for $K_2^{\mathbb{P}}$ (black) and $K_2^{\mathbb{Q}}$ (grey).

Sample size $n = 1000$

Example 2: Geometric Brownian Motion

- In this example only risk-neutral projection measure $\mathbb{M} = \mathbb{Q}$.
- Scalar $\mathbb{P}$-Brownian motion W_t
- Constant market price of risk $\gamma = 0.1$
- $m = 1$ financial instrument with gains process

$$G_t = W_t + \gamma t$$

- Define $\mathbb{Q}$-martingale, with volatility $\lambda = -0.2$,

$$M_t = \exp\left(\lambda G_t - \frac{\lambda^2}{2} t\right)$$

and assume

$$\text{one-year loss:} \quad L = M_1 - 1$$
$$\text{terminal loss:} \quad Z = M_T - 1$$

- Risk measure $\rho = \mathrm{ES}_{99\%}$
- Capital requirements and proxies normalised: $K = 1$

Wiener Chaos Expansion

Wiener chaos expansion theory: orthogonal series in $L^2(\mathbb{Q})$

$$M_t - 1 = \sum_{k=1}^{\infty} \int_{0<s_1<\cdots<s_k\le t} \lambda^k \, dG_{s_1} dG_{s_2} \cdots dG_{s_k} = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} t^{k/2} \underbrace{H_k\left(\frac{G_t}{\sqrt{t}}\right)}_{\text{Hermite poly}}.$$

Comparing with

$$V_t^{v,\phi} = v + \sum_{\mathcal{J}\in\mathcal{P} | t_{\max \mathcal{J}} \le t} \phi_{\mathcal{J}} \Delta \boldsymbol{G}_{\mathcal{J}} = v + \sum_{\mathcal{J}\in\mathcal{P} | t_{\max \mathcal{J}} \le t} \phi_{\mathcal{J}} \prod_{j\in\mathcal{J}} \Delta G_j$$

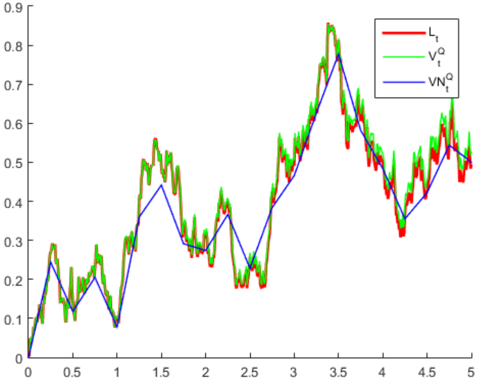
suggests that $v = 0$ and $\phi_{\mathcal{J}} = \lambda^{|\mathcal{J}|}$, asymptotically for $N \to \infty$.

We obtain, for $t = 1$,

$$L_1^{\mathbb{Q}} = \sum_{k=1}^{J} \frac{\lambda^k}{k!} H_k(G_1), \quad L_2^{\mathbb{Q}} = L_1^{\mathbb{Q}} + \epsilon^{\mathbb{Q}}$$

with $\epsilon^{\mathbb{Q}} = M_T - 1 - \sum_{k=1}^{J} \frac{\lambda^k}{k!} T^{k/2} H_k\left(\frac{G_T}{\sqrt{T}}\right)$ for $|\mathcal{J}| \le J = 1, 2 \ldots$

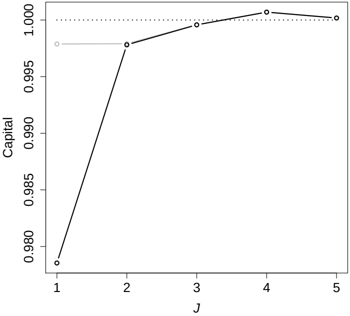
Loss & RP Trajectories



Loss trajectory $L_t = \mathbb{E}^{\mathbb{Q}}[Z \mid \mathcal{F}_t]$, theoretical RP $V_t^{\mathbb{Q}}$, and numerical RP $VN_t^{\mathbb{Q}}$ with quarterly rebalancing for $J = 2 \rightarrow$ very good fit!

Capital Proxies

2nd order RPs capture nonlinearities of liability cash flows significantly better than first order industry standard static RP:



- $K = 1$ (dotted), $K_1^{\mathbb{Q}}$ (black), $K_2^{\mathbb{Q}}$ (grey).
- Industry standard static proxies $\widetilde{K}_1^{\mathbb{Q}}$, $\widetilde{K}_2^{\mathbb{Q}}$ correspond to $J = 1$.

Conclusion

- Dynamic path-dependent RP for capital calculation captures nonlinear path-dependence of liability cash flows very well.
- RP cannot help to overcome incompleteness due to lack of financial instruments (no other numerical method can!)
- VaR and ES capital estimates asymptotically consistent under $\mathbb{P}$ and $\mathbb{Q}$ sampling if chaotic representation property holds.
- Numerical examples illustrate that dynamic path-dependent RP outperforms industry standard static RP.
- Approximation error dominates MC error: more factors preferred. In practice we can always assume complete market.
- Could be readily built into existing projection tools in practice.
- Future research: importance sampling, real-world study, ..