A NOTE ON THE NELSON–SIEGEL FAMILY

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Abstract. We study a problem posed in Björk and Christensen (1999): does there exist any nontrivial interest rate model which is consistent with the Nelson–Siegel family? They show that within the HJM framework with deterministic volatility structure the answer is no.

In this paper we give a generalized version of this result including stochastic volatility structure. For that purpose we introduce the class of *consistent state space processes*, which have the property to provide an arbitrage-free interest rate model when representing the parameters of the Nelson–Siegel family. We characterize the consistent state space Itô processes in terms of their drift and diffusion coefficients. By solving an inverse problem we find their explicit form. It turns out that there exists no nontrivial interest rate model driven by a consistent state space Itô process.

1. Introduction

Björk and Christensen (1999) introduce the following concept: let $\mathcal{M}$ be an interest rate model and $\mathcal{G}$ a parameterized family of forward curves. $\mathcal{M}$ and $\mathcal{G}$ are called consistent, if all forward rate curves which may be produced by $\mathcal{M}$ are contained within $\mathcal{G}$, provided that the initial forward rate curve lies in $\mathcal{G}$. Under the assumption of a deterministic volatility structure and working under a martingale measure, they show that within the Heath–Jarrow–Morton (henceforth HJM) framework there exists no nontrivial forward rate model, consistent with the Nelson–Siegel family $\{F(\,.\,,z)\}$. The curve shape of $F(\,.\,,z)$ is given by the well known expression

$$F(x,z) = z_1 + z_2 e^{-z_4 x} + z_3 x e^{-z_4 x}, \tag{1}$$

introduced by Nelson and Siegel (1987).

For an optimal todays choice of the parameter $z \in \mathbb{R}^4$, expression (1) represents the current term structure of interest rates, i.e. $x \geq 0$ denotes time to maturity. This method of fitting the forward curve is widely used among central banks, see the BIS (1999) documentation.

From an economic point of view it seems reasonable to restrict z to the state space $\mathcal{Z} := \{z = (z_1, \dots, z_4) \in \mathbb{R}^4 \mid z_4 > 0\}$.

The corresponding term structure of the bond prices is given by

$$G(x,z) := \exp\left(-\int_0^x F(\eta, z)\, d\eta\right).$$

Then $G \in C^\infty([0,\infty) \times \mathcal{Z})$.

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In order to imply a stochastic evolution of the forward rates, we introduce in Section 2 some state space process $Z = (Z_t)_{0 \le t < \infty}$ with values in $\mathcal{Z}$ and ask whether $F(\,.\,, Z)$ provides an arbitrage-free interest rate model. We call Z consistent, if the corresponding discounted bond prices are martingales, see Section 3. Solving an inverse problem we characterize in Section 4 the class of consistent state space Itô processes. Since a diffusion is a special Itô process, the very important class of consistent state space diffusion processes is characterized as well. Still we are able to derive a more general result. It turns out that all consistent Itô processes have essentially deterministic dynamics. The corresponding interest rate models are in turn trivial.

Consistent state space Itô processes are, by definition, specified under a martingale measure. This seems to be a restriction at first and one may ask wether there exists any Itô process Z under some objective probability measure inducing a nontrivial arbitrage free interest rate model $F(\,.\,, Z)$. However if the underlying filtration is not too large we show in Section 5 that our (negative) result holds for Itô processes modelled under any probability measure, provided that there exists an equivalent martingale measure. Hence under the requirement of absence of arbitrage there exists no nontrivial interest rate model driven by Itô processes and consistent with the Nelson–Siegel family.

Using the same ideas, still larger classes of consistent processes like Itô processes with jumps could be characterized.

2. The interest rate model

For the stochastic background and notations we refer to Revuz and Yor (1994) and Jacod and Shiryaev (1987). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t < \infty}, \mathbb{P})$ be a filtered complete probability space, satisfying the usual conditions, and let $W = (W_t^1, \dots, W_t^d)_{0 \le t < \infty}$ denote a standard d-dimensional $(\mathcal{F}_t)$-Brownian motion, $d \ge 1$.

We assume as given, an Itô process $Z = (Z^1, \dots, Z^4)$ with values in the state space $\mathcal{Z}$ of the form

$$Z_t^i = Z_0^i + \int_0^t b_s^i \, ds + \sum_{j=1}^d \int_0^t \sigma_s^{ij} \, dW_s^j, \quad i = 1, \dots, 4, \quad 0 \le t < \infty, \tag{2}$$

where Z_0 is $\mathcal{F}_0$-measurable, and b, σ are progressively measurable processes with values in $\mathbb{R}^4$, resp. $\mathbb{R}^{4 \times d}$, such that

$$\int_0^t \left(|b_s| + |\sigma_s|^2 \right) ds < \infty, \quad \mathbb{P}\text{-a.s.,} \quad \text{for all finite } t. \tag{3}$$

Z could be for instance the (weak) solution of a stochastic differential equation, but in general Z is not Markov.

Define by

$$r(t, x) := F(x, Z_t)$$

the *instantaneous forward rate* at time t for date $t + x$.

It is shown in Delbaen and Schachermayer (1994), Section 7, that traded assets have to follow semimartingales. Hence it is of importance for us to observe that the price at time t for a *zero coupon bond* with maturity T

$$P(t, T) := G(T - t, Z_t), \quad 0 \le t \le T < \infty,$$

and the *short rates*

$$r(t,0) = \lim_{x\to 0} r(t,x) = F(0,Z_t) = -\frac{\partial}{\partial x} G(0,Z_t), \quad 0 \le t < \infty,$$

form continuous semimartingales, by the smoothness of F and G. Therefore the same holds for the process of the *savings account*

$$B(t) := \exp\left(\int_0^t r(s,0)\,ds\right), \quad 0 \le t < \infty.$$

3. Consistent state space processes

We are going to define consistency in our context, which slightly differs from that in Björk and Christensen (1999). We focus on the state space process Z, which follows an Itô process or may follow some more general process.

Definition 3.1. *Z is called* consistent *with the Nelson–Siegel family, if*

$$\left(\frac{P(t,T)}{B(t)}\right)_{0\le t\le T}$$

is a $\mathbb{P}$-martingale, for all $T < \infty$.

The next proposition is folklore in case that Z follows a diffusion process, i.e. if $b_t(\omega) = b(t, Z_t(\omega))$ and $\sigma_t(\omega) = \sigma(t, Z_t(\omega))$ for Borel mappings b and σ from $[0,\infty)\times\mathcal{Z}$ into the corresponding spaces. That case usually leads to a PDE including the generator of Z. The standard procedure is then to find a solution u (the term structure of bond prices) to this PDE on $(0,\infty)\times\mathcal{Z}$ with initial condition $u(0,\,.\,) = 1$. It is well known in the financial literature that Z is necessarily consistent with the corresponding forward rate curve family. In contrast we ask the other way round and are more general what concerns Z. Our aim is, *given* F, to derive *conditions on b and σ* being *necessary* for consistency of Z with $\{F(\,.\,,z)\}_{z\in\mathcal{Z}}$. But the coefficients b and σ are progressively measurable processes. Hence Z given by (2) is not Markov, i.e. there is no infinitesimal generator. By the nature of b and σ such conditions can therefore only be stated $dt\otimes d\mathbb{P}$-a.s. (note that equation (4) below is not a PDE). On the other hand the argument mentioned in the diffusion case works just in one direction: consistency of Z with a forward curve family $\mathcal{G} = \{v(\,.\,,z)\}_{z\in\mathcal{Z}}$ does not imply validity of the PDE condition for $u(x,z) = \exp(-\int_0^x v(\eta,z)\,d\eta)$ in general.

Actually one could re-parameterize $f(t,T) := r(t,T-t)$, for $0 \le t \le T < \infty$, and work within the HJM framework. Equation (4) below corresponds to the well known HJM drift condition for $(f(t,T))_{0\le t\le T}$. But as soon as Z is not an Itô process anymore, this connection fails and one has to proceed like in the following proof (see Remark 3.3), which therefore is given in its full form.

Set $a := \sigma\sigma^*$, where σ^* denotes the transpose of σ, i.e. $a_t^{ij} = \sum_{k=1}^d \sigma_t^{ik}\sigma_t^{jk}$, for $1 \le i,j \le 4$ and $0 \le t < \infty$. Then a is a progressively measurable process with values in the symmetric nonnegative definite 4×4-matrices.

Proposition 3.2. *Z is consistent with the Nelson–Siegel family only if*

$$\begin{aligned}\frac{\partial}{\partial x}F(x,Z) &= \sum_{i=1}^{4} b^i \frac{\partial}{\partial z_i}F(x,Z) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{4} a^{ij}\left(\frac{\partial^2}{\partial z_i \partial z_j}F(x,Z) - \frac{\partial}{\partial z_i}F(x,Z)\int_0^x \frac{\partial}{\partial z_j}F(\eta,Z)\,d\eta\right. \\ &\quad \left. - \frac{\partial}{\partial z_j}F(x,Z)\int_0^x \frac{\partial}{\partial z_i}F(\eta,Z)\,d\eta\right),\end{aligned} \tag{4}$$

for all $x \geq 0$, $dt \otimes d\mathbb{P}$-a.s.

Proof. For $f \in C^2(\mathcal{Z})$ we set

$$\mathcal{A}_t(\omega)f(z) := \sum_{i=1}^{4} b_t^i(\omega)\frac{\partial f}{\partial z_i}(z) + \frac{1}{2}\sum_{i,j=1}^{4} a_t^{ij}(\omega)\frac{\partial^2 f}{\partial z_i \partial z_j}(z), \quad 0 \leq t < \infty, \quad z \in \mathcal{Z}.$$

Using Itô's formula we get for $T < \infty$

$$\begin{aligned}P(t,T) &= P(0,T) + \int_0^t \left(\mathcal{A}_s G(T-s,Z_s) - \frac{\partial}{\partial x}G(T-s,Z_s)\right) ds \\ &\quad + \int_0^t \sigma_s^* \nabla_z G(T-s,Z_s)\,dW_s, \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

where ∇_z denotes the gradient with respect to (z_1, z_2, z_3, z_4), and

$$\frac{1}{B(t)} = 1 + \int_0^t \frac{1}{B(s)}\frac{\partial}{\partial x}G(0,Z_s)\,ds, \quad 0 \leq t < \infty, \quad \mathbb{P}\text{-a.s.}$$

For $0 \leq t \leq T$ define

$$H(t,T) := \frac{1}{B(t)}\left(\mathcal{A}_t G(T-t,Z_t) - \frac{\partial}{\partial x}G(T-t,Z_t) + \frac{\partial}{\partial x}G(0,Z_t)G(T-t,Z_t)\right)$$

and the local martingale

$$M(t,T) := \int_0^t \frac{1}{B(s)}\sigma_s^* \nabla_z G(T-s,Z_s)\,dW_s.$$

Integration by parts then yields

$$\frac{1}{B(t)}P(t,T) = P(0,T) + \int_0^t H(s,T)\,ds + M(t,T), \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.}$$

Let's suppose now that Z is consistent. Then necessarily for $T < \infty$

$$\int_0^t H(s,T)\,ds = 0, \quad \forall t \in [0,T], \quad \mathbb{P}\text{-a.s.} \tag{5}$$

Since b and σ are progressive and G is smooth, $H(\,.\,,T)$ is progressively measurable on $[0,T] \times \Omega$. We claim that (5) yields

$$H(\,.\,,T) = 0, \quad \text{on } [0,T] \times \Omega, \quad dt \otimes d\mathbb{P}\text{-a.s.} \tag{6}$$

Proof of (6). Define $N := \{(t,\omega) \in [0,T] \times \Omega \mid H(t,T)(\omega) > 0\}$. Then N is a $\mathcal{B}\otimes\mathcal{F}$-measurable set. Since $H(\,.\,,T)$ is positive on N we can use Tonelli's theorem

$$\int_N H(t,T)(\omega)\,dt \otimes d\mathbb{P} = \int_\Omega \left(\int_{N_\omega} H(t,T)(\omega)\,dt\right) d\mathbb{P}(\omega) = 0,$$

where $N_\omega := \{t \mid (t,\omega) \in N\} \in \mathcal{B}$ and we have used (5) and the inner regularity of the measure dt. We therefore conclude that N has $dt \otimes d\mathbb{P}$-measure zero. By using a similar argument for $-H(\,.\,,T)$ we have proved (6).

Note that (6) holds for all $T < \infty$, where the $dt \otimes d\mathbb{P}$-nullset depends on T. But since $H(t,T)$ is continuous in T, a standard argument yields

$$H(t,t+x)(\omega) = 0, \quad \forall x \geq 0, \quad \text{for } dt \otimes d\mathbb{P}\text{-a.e. } (t,\omega).$$

Multiplying this equation with $B(t)$ and using again the full form this reads

$$\mathcal{A}\,G(x,Z) - \frac{\partial}{\partial x}G(x,Z) + \frac{\partial}{\partial x}G(0,Z)G(x,Z) = 0, \quad \forall x \geq 0, \quad dt \otimes d\mathbb{P}\text{-a.s.} \tag{7}$$

Differentiation gives

$$\int_0^x \mathcal{A}\,F(\eta,Z)\,d\eta - \frac{1}{2}\sum_{i,j=1}^4 a^{ij}\left(\int_0^x \frac{\partial}{\partial z_i}F(\eta,Z)\,d\eta\right)\left(\int_0^x \frac{\partial}{\partial z_j}F(\eta,Z)\,d\eta\right)$$
$$- F(x,Z) + F(0,Z) = 0, \quad \forall x \geq 0, \quad dt \otimes d\mathbb{P}\text{-a.s.},$$

where we have divided by $G(x,Z)$, since $G > 0$ on $[0,\infty) \times \mathcal{Z}$. Differentiating with respect to x finally yields (4). □

Remark 3.3. *Definition 3.1 can be extended in a natural way to a wider class of state space processes Z. We mention here just two possible directions:*

a) *Z a* time homogeneous Markov process with infinitesimal generator $\mathcal{L}$. *The corresponding version of Proposition 3.2 can be formulated in terms of equation (7), where $\mathcal{A}$ has to be replaced by $\mathcal{L}$. The difficulty here consists of checking whether G goes well together with Z, i.e. $x \mapsto G(x,\,.\,)$ has to be a nice mapping from $[0,\infty)$ into the domain of $\mathcal{L}$.*
b) *Z an* Itô process with jumps. *Again one could reformulate Proposition 3.2 on the basis of equation (7) by adding the corresponding stochastic integral with respect to the compensator of the jump measure, which we assume to be absolutely continuous with respect to dt. The jump measure could be implied for example by a homogeneous Poisson process.*

4. The class of consistent Itô processes

Equation (4) characterizes b and a, resp. σ, just up to a $dt \otimes d\mathbb{P}$-nullset. But the stochastic integral in (2) is (up to indistinguishability) defined on the equivalence classes with respect to the $dt \otimes d\mathbb{P}$-nullsets. Hence this is enough to determine the process Z, given Z_0, uniquely (up to indistinguishability). On the other hand Z cannot be represented in the form (2) with integrands that differ from b and σ on a set with $dt \otimes d\mathbb{P}$-measure strictly greater than zero (the characteristics of Z are unique up to indistinguishability). Therefore it makes sense to pose the following *inverse problem* on the basis of equation (4): For which choices of coefficients b and σ do we get a consistent state space Itô process Z starting in Z_0?

The answer is rather remarkable:

Theorem 4.1. *Let Z be a consistent Itô process. Then Z is of the form*

$$\begin{aligned}
Z_t^1 &= Z_0^1 \\
Z_t^2 &= Z_0^2 e^{-Z_0^4 t} + Z_0^3 t e^{-Z_0^4 t} \\
Z_t^3 &= Z_0^3 e^{-Z_0^4 t} \\
Z_t^4 &= Z_0^4 + \left(\int_0^t b_s^4 \, ds + \sum_{j=1}^d \int_0^t \sigma_s^{4j} \, dW_s^j \right) 1_{\Omega_0}
\end{aligned}$$

for all $0 \leq t < \infty$, where $\Omega_0 := \{Z_0^2 = Z_0^3 = 0\}$.

Remark 4.2. *On Ω_0 the processes Z^2 and Z^3 are zero. Hence $F(x, Z_t) = Z_0^1$ on Ω_0, i.e. the process Z^4 has no influence on $F(\,.\,, Z_t)$ on Ω_0. So it holds that the corresponding interest rate model is of the form*

$$\begin{aligned}
r(t,x) &= F(x, Z_t) \\
&= Z_0^1 + \left(Z_0^2 e^{-Z_0^4 t} + Z_0^3 t e^{-Z_0^4 t} \right) e^{-Z_0^4 x} + Z_0^3 e^{-Z_0^4 t} x e^{-Z_0^4 x} \\
&= Z_0^1 + Z_0^2 e^{-Z_0^4 (t+x)} + Z_0^3 (t+x) e^{-Z_0^4 (t+x)} \\
&= r(0, t+x), \quad \forall t, x \geq 0,
\end{aligned}$$

and is therefore quasi deterministic, i.e. all randomness remains $\mathcal{F}_0$-measurable.

Remark 4.3. *One can show a similar (negative) result even for the wider class of state space Itô processes with jumps on a finite or countable mark space, see Remark 3.3. However allowing for more general exponential-polynomial families $\{F(\,.\,, z)\}$ there exist (although very restricted) consistent Itô processes providing a nontrivial interest rate model, see Filipović (1998).*

Proof. Let Z be a consistent Itô process given by equation (2). The proof of the theorem relies on expanding equation (4).

First of all we subtract $\frac{\partial}{\partial x} F(x, Z)$ from both sides of (4) to obtain a null equation. Fix then a point (t, ω) in $[0, \infty) \times \Omega$. For simplicity we write (z_1, z_2, z_3, z_4) for $Z_t(\omega)$, a_{ij} for $a_t^{ij}(\omega)$ and b_i for $b_t^i(\omega)$. Notice that $Z_t(\omega) \in \mathcal{Z}$, i.e. $z_4 > 0$. Observe then that our null equation is in fact of the form

$$p_1(x) + p_2(x) e^{-z_4 x} + p_3(x) e^{-2 z_4 x} = 0, \tag{8}$$

which has to hold simultaneously for all $x \geq 0$. The expressions p_1, p_2 and p_3 denote some polynomials in x, which depend on the z_i's, b_i's and a_{ij}'s. Since the functions $\{1, e^{-z_4 x}, e^{-2 z_4 x}\}$ are independent over the ring of polynomials, (8) can only be satisfied if each of the p_i's is 0. This again yields that all coefficients of the p_i's have to be zero. To proceed in our analysis we list all terms appearing in (4):

$$\begin{aligned}
\frac{\partial}{\partial x} F(x,z) &= (-z_2 z_4 + z_3 - z_3 z_4 x) e^{-z_4 x}, \\
\nabla_z F(x,z) &= \left(1, e^{-z_4 x}, x e^{-z_4 x}, (-z_2 x - z_3 x^2) e^{-z_4 x}\right), \\
\frac{\partial^2}{\partial z_i \partial z_j} F(x,z) &= 0, \quad \text{for } 1 \leq i, j \leq 3, \\
\frac{\partial}{\partial z_4} \nabla_z F(x,z) &= \left(0, -x e^{-z_4 x}, -x^2 e^{-z_4 x}, (z_2 x^2 + z_3 x^3) e^{-z_4 x}\right).
\end{aligned}$$

Finally we need the relation

$$\int_0^x \eta^m e^{-z_4\eta}\, d\eta = -q_m(x)e^{-z_4 x} + \frac{m!}{z_4^{m+1}}, \quad m = 0, 1, 2, \dots,$$

where $q_m(x) = \sum_{k=0}^m \frac{m!}{(m-k)!} \frac{x^{m-k}}{z_4^{k+1}}$ is a polynomial in x of order m.

First we shall analyze p_1. The terms that contribute to p_1 are those containing $\frac{\partial}{\partial z_1}F(x,z)$ and $\frac{\partial}{\partial z_1}F(x,z)\int_0^x \frac{\partial}{\partial z_j}F(\eta,z)\,d\eta$, for $1 \le j \le 4$. Actually p_1 is of the form

$$p_1(x) = a_{11}x + \cdots + b_1,$$

where ... stands for terms of zero order in x containing the factors $a_{1j} = a_{j1}$, for $1 \le j \le 4$. It follows that $a_{11} = 0$. But the matrix (a_{ij}) has to be nonnegative definite, so necessarily

$$a_{1j} = a_{j1} = 0, \quad \text{for all } 1 \le j \le 4,$$

and therefore also $b_1 = 0$. Thus p_1 is done.

The contributing terms to p_3 are those containing $\frac{\partial}{\partial z_i}F(x,z)\int_0^x \frac{\partial}{\partial z_j}F(\eta,z)\,d\eta$, for $2 \le i,j \le 4$. But observe that the degree of p_3 and p_2 depends on whether z_2 or z_3 are equal to zero or not. Hence we have to distinguish between the four cases

i) $z_2 \ne 0,\ z_3 \ne 0$ **iii)** $z_2 = 0,\ z_3 \ne 0$
ii) $z_2 \ne 0,\ z_3 = 0$ **iv)** $z_2 = z_3 = 0$.

case i): The degree of p_3 is 4. The fourth order coefficient contains a_{44}, i.e.

$$p_3(x) = a_{44}\frac{z_3^2}{z_4}x^4 + \dots,$$

where ... stands for terms of lower order in x. Hence

$$a_{4j} = a_{j4} = 0, \quad \text{for } 1 \le j \le 4.$$

The degree of p_3 reduces to 2. The second order coefficient is $\frac{a_{33}}{z_4}$. Hence $a_{3j} = a_{j3} = 0$, for $1 \le j \le 4$. It remains $p_3(x) = \frac{a_{22}}{z_4}$. Thus the diffusion matrix (a_{ij}) is zero. This implies that (σ_{ij}) is zero, independent of the choice of d (the number of Brownian motions in (2)). Now we can write down p_2:

$$p_2(x) = -b_4 z_3 x^2 + (b_3 - b_4 z_2 + z_3 z_4)x + b_2 + z_2 z_4 - z_3.$$

It follows that $b_4 = 0$ and

$$b_2 = z_3 - z_2 z_4,$$
$$b_3 = -z_3 z_4.$$

For the other three cases we will need the following lemma, which is a direct consequence of the occupation times formula, see Revuz and Yor (1994), Corollary (1.6), Chapter VI.

Lemma 4.4. *Using the same notation as in (2), it holds for $1 \le i \le 4$ that*

$$a^{ii}1_{\{Z^i=0\}} = b^i 1_{\{Z^i=0\}} = 0, \quad dt \otimes d\mathbb{P}\text{-a.s.}$$

As a consequence, since we are characterizing a and b up to a $dt \otimes d\mathbb{P}$-nullset, we may and will assume that $z_i = 0$ implies $a_{ij} = a_{ji} = b_i = 0$, for $1 \le j \le 4$ and $i = 2, 3$.

case ii): We have $\nabla_z F(x,z) = (1, e^{-z_4 x}, x e^{-z_4 x}, -z_2 x e^{-z_4 x})$. Hence the degree of p_3 is 2. Since $a_{3j} = a_{j3} = b_3 = 0$, for $1 \le j \le 4$, the second order coefficient comes from

$$-a_{44} \frac{\partial}{\partial z_4} F(x,z) \int_0^x \frac{\partial}{\partial z_4} F(\eta, z)\, d\eta = a_{44} \frac{z_2^2}{z_4} x^2 e^{-2z_4 x} + \dots,$$

where ... denotes terms of lower order in x. Hence $a_{4j} = a_{j4} = 0$, for $1 \le j \le 4$. The polynomial p_3 reduces to $p_3(x) = \frac{a_{22}}{z_4}$. It follows that also in this case the diffusion matrix (a_{ij}) is zero. From case i) we derive immediately that now

$$p_2(x) = -b_4 z_2 x + b_2 + z_2 z_4,$$

hence $b_2 = -z_2 z_4$ and $b_4 = 0$.

case iii): Since $a_{2j} = a_{j2} = b_2 = 0$, for $1 \le j \le 4$, the zero order coefficient of p_2 reduces to $-z_3$. We conclude that $z_2 = 0$ implies $z_3 = 0$, so this case doesn't enter $dt \otimes d\mathbb{P}$-a.s.

case iv): In this case $a_{ij} = b_k = 0$, for all $(i,j) \neq (4,4)$ and $k \neq 4$. Also $\frac{\partial}{\partial z_4} F(x,z) = \frac{\partial}{\partial x} F(x,z) = 0$. Hence $p_2(x) = p_3(x) = 0$, independently of the choice of b_4 and a_{44}.

Summarizing the four cases we conclude that equation (4) implies

$$\begin{aligned} b_1 &= 0 & b_3 &= -z_3 z_4 \\ b_2 &= z_3 - z_2 z_4 & a_{ij} &= 0, \quad \text{for } (i,j) \neq (4,4). \end{aligned}$$

Whereas b_4 and a_{44} are arbitrary real, resp. nonnegative real, numbers whenever $z_2 = z_3 = 0$. Otherwise $b_4 = a_{44} = 0$.

This has to hold for $dt \otimes d\mathbb{P}$-a.e. (t, ω). So Z is uniquely (up to indistinguishability) determined and satisfies

$$\begin{aligned} Z_t^1 &= Z_0^1 \\ Z_t^2 &= Z_0^2 + \int_0^t \left(Z_s^3 - Z_s^2 Z_s^4 \right) ds \\ Z_t^3 &= Z_0^3 - \int_0^t Z_s^3 Z_s^4\, ds \\ Z_t^4 &= Z_0^4 + \int_0^t b_s^4\, ds + \sum_{j=1}^d \int_0^t \sigma_s^{4j}\, dW_s^j \end{aligned}$$

for some progressively measurable processes b^4 and σ^{4j}, $j = 1, \dots, d$, being compatible with (3) and vanishing outside the set $\{(t,\omega) \mid Z_t^2(\omega) = Z_t^3(\omega) = 0\}$.

Note that Z^2 and Z^3 satisfy (path-wise) a system of linear ODE's with continuous coefficients. Hence they are indistinguishable from zero on Ω_0. So the statement of the theorem is proved on Ω_0. It remains to prove it on $\Omega_1 := \Omega \setminus \Omega_0$.

Introduce the stopping time $\tau := \inf\{s > 0 \mid Z_s^2 = Z_s^3 = 0\}$. We have just argued that $\Omega_0 \subset \{\tau = 0\}$. By continuity of Z also the reverse inclusion holds, hence $\Omega_0 = \{\tau = 0\}$. The stopped process $Z^\tau =: Y$ satisfies (path-wise) the following system of linear stochastic integral equations

$$
\begin{aligned}
Y_t^1 &= Z_0^1 \\
Y_t^2 &= Z_0^2 + \int_0^{t\wedge\tau} \left(Y_s^3 - Y_s^2 Z_0^4\right) ds \\
Y_t^3 &= Z_0^3 + \int_0^{t\wedge\tau} Y_s^3 Z_0^4 \, ds \\
Y_t^4 &= Z_0^4, \quad \text{for } 0 \le t < \infty.
\end{aligned} \tag{9}
$$

We have used the fact that $b^4 = b^4 1_{[\tau,\infty]}$ and $\sigma^4 = \sigma^4 1_{[\tau,\infty]}$. Then the last equation follows from an elementary property of the stopped stochastic integral:

$$\sum_{j=1}^d \int_0^{t\wedge\tau} \sigma_s^{4j} \, dW_s^j = \sum_{j=1}^d \int_0^t \left(\sigma_s^{4j} 1_{[\tau,\infty]}\right) 1_{[0,\tau]} \, dW_s^j = \sum_{j=1}^d \int_0^t \sigma_s^{4j} 1_{[\tau]} \, dW_s^j = 0,$$

by continuity of W.

The system (9) has the unique solution for $0 \le t < \infty$

$$
\begin{aligned}
Y_t^1 &= Z_0^1 \\
Y_t^2 &= Z_0^2 e^{-Z_0^4 (t\wedge\tau)} + Z_0^3 (t\wedge\tau) e^{-Z_0^4 (t\wedge\tau)} \\
Y_t^3 &= Z_0^3 e^{-Z_0^4 (t\wedge\tau)} \\
Y_t^4 &= Z_0^4.
\end{aligned}
$$

Since $Z = Y$ on the stochastic interval $[0,\tau]$ and since $Y_t \neq 0$, $\forall t < \infty$, $\mathbb{P}$-a.s. on Ω_1, it follows by the continuity of Z, that $\Omega_1 = \{\tau > 0\} = \{\tau = \infty\}$. Inserting this in the above solution, the theorem is proved also on Ω_1. □

5. E-consistent Itô processes

Note that by definition Z is consistent if and only if $\mathbb{P}$ is a martingale measure for the discounted bond price processes. We could generalize this definition and call a state space process Z *e-consistent* if there exists an equivalent martingale measure $\mathbb{Q}$. Then obviously consistency implies e-consistency, and e-consistency implies the absence of arbitrage opportunities, as it is well known.

In case where the filtration is generated by the Brownian motion W, i.e. $(\mathcal{F}_t) = (\mathcal{F}_t^W)$, we can give the following stronger result:

Proposition 5.1. *If $(\mathcal{F}_t) = (\mathcal{F}_t^W)$, then any e-consistent Itô process Z is of the form as stated in Theorem 4.1. In particular the corresponding interest rate model is purely deterministic.*

Proof. Let Z be an e-consistent Itô process under $\mathbb{P}$, and let $\mathbb{Q}$ be an equivalent martingale measure. Since $(\mathcal{F}_t) = (\mathcal{F}_t^W)$, we know that all $\mathbb{P}$-martingales have the representation property relative to W. By Girsanov's theorem it follows therefore that Z remains an Itô process under $\mathbb{Q}$. In particular Z is a consistent state space Itô process with respect to the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{0\le t<\infty}, \mathbb{Q})$. Hence Z is of the form as stated in Theorem 4.1.

Since $\mathcal{F}_0^W$ consists of sets of measure 0 or 1, the processes Z^1, Z^2, Z^3 are (after changing the Z_0^i's on a set of measure 0) purely deterministic and therefore also $(r(t,.))_{0\le t<\infty}$, see Remark 4.2. □

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