

# Financial Policy in a Liquidity Trap

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## Abstract

A policy of recapitalizing leverage-constrained capital producers reduces the severity of a liquidity trap or avoids one altogether. Recapitalization policy sharply dominates fiscal stimulus. Both policies allow the savings and investment market to equilibrate with a smaller fall in output and inflation, but fiscal stimulus discourages savings while recapitalization boosts investment. So, recapitalization leads to a faster recovery in output.

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# 1 Introduction

While manipulation of the real rate of interest is the foremost tool of modern central banks, during the Japanese Lost Decade and the recent crisis in U.S. financial markets policymakers faced limits to interest-rate driven monetary policy. Near the lower bound on nominal interest rates, the monetary authority cannot reduce the nominal rate to the extent consistent with a Taylor rule when inflation or output are below-target. The classical explanation of a liquidity trap is that because the real interest rate is unable to fall to the level it would attain in the absence of a lower bound on the nominal interest rate, savings are too attractive relative to investment when the lower bound binds. The savings-investment market must equilibrate through a fall in output. The New Keynesian Phillips curve implies that a fall in output comes with deflation, which by raising the real interest rate exacerbates the misalignment of savings and investment and forces the equilibrating fall in output to be large. The classical explanation explains the source of the problem as a paradox of thrift, whereby excessive demand for savings lowers the interest rate and leads into the trap. The recipe for escape is to induce people to consume more and save less through expansionary fiscal policy.

In this paper, we instead focus on policies that raise investment demand to accommodate shifts in savings. We aim to understand whether the U.S. government's recent policy of recapitalizing financial institutions was an antidote to a liquidity trap, and how it differed from traditional fiscal policy. Specifically, we show that (1) an unexpected capital quality shock, by decreasing the net worth of capital producers, pushes down investment and the nominal interest rate, (2) a policy that recapitalizes leverage-constrained capital producers increases desired investment and consequently mitigates or avoids the deflationary investment and output decline characteristic of a liquidity trap, and (3) recapitalization policy is more effective at sustaining output than traditional fiscal policy.

First, we analyze a standard New Keynesian Dynamic Stochastic General Equilibrium (DSGE) model in the vein of Christiano, Eichenbaum, and Evans (henceforth CEE) [1]. We explore the effect of an increase during a liquidity trap of tax-financed investment subsidies. Investment subsidies lower the borrowing costs of capital producers. The resulting boost to desired investment permits equilibration at a lower rate of deflation and a much smaller decline in income.

To integrate financial frictions into the standard model, we assume capital producers, whom have the special ability to pursue profitable capital production projects, cannot commit to honor a contract that asks them to pay too high a fraction of project revenues. We choose to model this friction, originally developed by [2], for its straightforward intuition and tractability. Because our focus is on policy and liquidity traps, we do not attempt to model financial intermediation, especially since developing such a model is itself an ongoing research topic (see, e.g., [3]). The limited commitment problem constrains how much of a project's financing can come from outsiders and prevents the internal and external required rates of return on investment from being equal. The net worth of capital producers becomes a natural driver of investment, and in turn current investment drives future net worth. Because low investment today depresses net worth in future periods, and low future net worth depresses investment in the future, investment, output, and inflation recover more slowly than in the frictionless model. Also in contrast to a frictionless model, where an unexpected capital quality shock creates a scarcity which drives up investment and the real interest rate, now the shock's negative impact on net-worth reduces investment and forward consumption growth, which pushes down the real interest rate. A large enough

shock induces a liquidity trap with falls in investment, output, and inflation; however, tax-financed recapitalizations of capital producers boost current investment and raise the path of future investment, output, and inflation during a liquidity trap, as well as allowing the real interest rate to fall further.

A sufficiently large recapitalization outright evades a liquidity trap. Recapitalizations induce increases in inflation. Given a large enough intervention, the monetary authority's Taylor rule will prescribe a nominal interest rate above its lower bound.

Furthermore, recapitalizations ameliorate the liquidity trap far more than does an equal-sized increase in neutral government spending. A relatively modest recapitalization increases not only same-period investment and output, but also helps the economy to recover from the capital quality shock. The resulting increase in future inflation reduces the real interest rate in the liquidity trap, thus mitigating its severity. Traditional fiscal stimulus does not aid future recovery. We conclude that policy prescriptions in a liquidity trap should not focus on traditional Keynesian stimulus to the exclusion of other policies that more directly impinge on the market for savings and investment.

## 2 Monetary Policy Commitment, Traditional Fiscal Stimulus, and Investment Policies

The central bank's primary weapon against a liquidity trap should be commitment to downward deviations from Taylor-type policy rules sustained for some time after the economy reattains a positive nominal rate. Eggertsson and Woodford [4], for example, demonstrate that for a class of simple New-Keynesian DSGE models, a price level (rather than inflation rate) target can attain most of the gains from a fully optimal monetary policy that balances efficiency concerns outside liquidity traps with the need to reduce their impact. The price level target reduces the severity of a liquidity trap by raising inflation expectations, since the more deflationary is the liquidity trap, the more inflationary the public expects the aftermath to be. However, the central bank may have limited ability to credibly commit to a price level target, particularly after a prolonged period targeting an inflation rate. Doubts about central bank efficacy in a liquidity trap generate demand for complementary policy interventions that might alleviate one.

Interest in fiscal policy during a liquidity trap has enjoyed a revival from Keynes' original discussion of the Paradox of Thrift in his *General Theory* [5] to recent work by Krugman [6], Eggertsson and Woodford [4, 7], and Christiano, Eichenbaum, and Rebelo [8] (henceforth, CER). CER show that government fiscal stimulus can increase aggregate expenditures and boost marginal costs and aggregate prices. Like a decline in aggregate output, fiscal stimulus decreases households' discretionary income, which helps equilibrated desired savings and investment. But unlike a decline in aggregate output, fiscal stimulus is inflationary. Inflation reinforces an inward shift in the savings curve. The fall in income needed to equilibrate savings and investment is therefore much smaller with fiscal stimulus. We instead focus on interventions boosting demand for loanable funds.

Two recent papers that also emphasize the importance of investment-driven policies during liquidity traps are by Eggertsson [9] and Mankiw and Weinzierl [10]. However, neither consider an environment with an explicit financial friction. Additionally, [10] focuses on a two-period model that, although intuitive, cannot capture the non-trivial dynamic effects of policy intervention during liquidity traps which we describe here.

### 3 Standard Model

An infinitely-lasting economy is inhabited by a continuum of infinitely-lived identical households (measure  $\eta^h = 1$ ). There are three types of goods: capital, intermediate, and final. Households derive direct utility only from final goods. Capital depreciates gradually while the intermediate and final goods are perishable. Households may provide any amount of labor to intermediate good firms, and face direct disutility of labor.

First, we describe the time structure of the economy for clarity. We explain the structure of production and solve for the optimality conditions of the households. Finally, we complete the model by specifying government fiscal and monetary policy and imposing aggregate resource constraints.

#### 3.1 Timing

At the beginning of period  $t$ , capital owners rent the capital to Calvo-pricing monopolists, who combine the capital with household labor to build differentiated intermediate goods. The capital stock depreciates by  $1 - \delta_t$  during intermediate good production. The intermediate firms sell their goods to perfectly competitive, Dixit Stiglitz final good producers in exchange for final goods. Households receive labor, capital and profit income from intermediate good firms. Labor income, capital rents, and profit are paid in final goods. Then, final goods are either consumed or sold to competitive capital producers who pay in claims to new capital in period  $t + 1$ .

#### 3.2 Capital Producers

A capital building project yields 1 unit of capital at time  $t + 1$  for each unit of final good invested at time  $t$ . In the model without financial frictions, there is no conflict of interest between the capital producers and households. The contracting problem between them is therefore trivial. Let  $i_t$  denote the amount of final goods employed by a capital producer. If  $d_t$  denotes the amount of credit (in units of final good) households extend a capital producer, and  $x_t$  denotes an investment subsidy, then the flow of funds constraint is  $d_t = (1 - x_t) i_t$ . Let  $1 + r_{t+1}^d$  equal the market rate of return on household credit. The capital producer's problem is to choose project size  $i_t$  and household credit  $d_t$  that maximizes profits:

$$\max_{i_t, d_t} \text{payoff}_{t+1}^K i_t - \left(1 + r_{t+1}^d\right) d_t \text{ s.t.}$$

$$d_t = (1 - x_t) i_t$$

Here,  $\text{payoff}_t^K$  is the  $t + 1$  final goods income to a unit of capital. The price of a claim to a unit of future capital,  $q_t$ , will be shown to satisfy  $q_t = \frac{\text{payoff}_{t+1}^K}{1+r_{t+1}^d}$  in equilibrium. If we substitute the contract's constraint into the objective and set profits equal to zero, we obtain

$$\text{payoff}_{t+1}^K = \left(1 + r_{t+1}^d\right) (1 - x_t)$$

so that equilibrium  $q_t$  satisfies:

$$q_t = (1 - x_t) \tag{1}$$

### 3.3 Households

Household preferences are represented by the utility function

$$U \equiv \sum_{t=0}^{\infty} \beta^t \left[ \ln(c_t) - \frac{\psi}{2} \ell_t^2 \right]$$

Households choose consumption and labor effort each period  $\{c_t, \ell_t\}$  to maximize utility subject to their budget constraint (in units of final good)

$$c_t + d_t + \frac{1}{P_t} b_t + q_t k_{t+1}^h = (1 + r_t^d) d_{t-1} + \frac{1}{P_t} (1 + R_t^{rf}) b_{t-1} + \text{payoff}_t^K k_t^h + w_t \ell_t + o_t + x_t^h,$$

where

$$\text{payoff}_t^K \equiv r_t + q_t(1 - \delta_t).$$

The right-hand side of the budget constraint shows the interest income from previous-period loans  $d_{t-1}$  to capital producers; interest income (at nominal rate  $R_t^{rf}$ ) from previous-period nominal purchases  $b_{t-1}$  of government debt; the payoff to capital  $k_t^h$  delivered to the household at the beginning of the period, composed of rents  $r_t$  from intermediary goods producers and resale income after depreciation; wages  $w_t$  from the contemporaneous labor effort decision  $\ell_t$ ; the household's share of intermediaries' profits  $o_t$ ; and any lump sum government transfers  $x_t^h$ . Each household chooses between consuming  $c_t$  final goods; loaning  $d_t$  final goods to capital producers; buying nominally risk free one-period government debt  $b_t$ ; and purchasing claims  $k_{t+1}^h$  to future physical capital. The real values of nominal purchases equal the nominal purchase values divided by the nominal price of a final good,  $P_t$ .

The first-order conditions for household optimization are

$$\lambda_t = \frac{1}{c_t} \tag{2}$$

$$1 = \beta \frac{\lambda_{t+1}}{\lambda_t} (1 + r_{t+1}^d) \tag{3}$$

$$1 = \beta \frac{\lambda_{t+1}}{\lambda_t} \times \frac{1 + R_{t+1}^{rf}}{\Pi_{t+1}} \tag{4}$$

$$1 = \beta \frac{\lambda_{t+1}}{\lambda_t} \left( \frac{\text{payoff}_{t+1}^K}{q_t} \right) \tag{5}$$

$$\psi \ell_t = \lambda_t w_t^h \tag{6}$$

where  $\lambda_t$  is the household's Lagrangian multiplier on her time  $t$  budget constraint, interpreted as the individual household's marginal value of real income at that date, and  $\Pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$  denotes gross final good price inflation between periods  $t + 1$  and  $t$ . Conditions (3) and (4) imply that the return on household loans to capital producers equals the return on government bonds:

$$1 + r_{t+1}^d = \frac{1 + R_{t+1}^{rf}}{\Pi_{t+1}} = 1 + r_{t+1}^{rf}.$$

### 3.4 Production Sector

#### 3.4.1 Intermediate Producers

Intermediate firm  $j \in (0, 1)$  produces type- $j$  intermediate goods  $y_{jt}$  according to a Cobb-Douglas production utilizing capital  $k_{jt}$  and labor  $\ell_{jt}$  with a fixed cost  $\Theta$ :

$$y_{jt} = k_{jt}^{\theta_k} (\ell_{jt})^{1-\theta_k} - \Theta,$$

where  $\theta_k \in (0, 1)$ . Intermediate goods producers compete monopolistically and periodically choose an optimal nominal price  $\tilde{p}_{jt}$  in the manner of CEE to solve

$$\max_{\tilde{p}_{j,t}} \sum_{k=0}^{\infty} (\beta\mu)^k \lambda_{t+k} \left[ \frac{p_{j,t+k} y_{j,t+k}}{P_{t+k}} - (1-\nu) s_{t+k} y_{j,t+k} \right],$$

where  $s_{t+k}$  denotes time  $t+k$  real marginal costs,  $\mu$  is the probability to each intermediate goods producer of not reoptimizing at any given period, and nominal price  $p_{j,t+k}$  evolves according to a lagged indexation rule

$$p_{j,t+k} = \prod_{i=0}^{k-1} (\Pi_{t+i})^{\zeta} \tilde{p}_{jt},$$

where  $\zeta \in [0, 1]$  denotes the degree of indexation of not-reoptimizing monopolists' prices to lagged inflation.  $\nu = 1/\xi_p$  is a cost subsidy which corrects the monopolistic output distortion in steady state.

#### 3.4.2 Final Good Producers

Final goods firm  $j \in (0, 1)$  produces final goods  $Y_t$  via a Dixit-Stiglitz aggregator

$$Y_t = \left( \int_0^1 y_{jt}^{\frac{\xi_p-1}{\xi_p}} dj \right)^{\frac{\xi_p}{\xi_p-1}} \quad (7)$$

for elasticity of marginal rate of technical substitution  $\xi_p > 1$ .

The final good producer's problem is to choose quantities  $\{y_{jt}\}$  given prices  $\{p_{jt}\}$  for each intermediate good and final good price  $P_t$

$$\max_{y_{jt}} P_t \left( \int_0^1 y_{jt}^{\frac{\xi_p-1}{\xi_p}} dj \right)^{\frac{\xi_p}{\xi_p-1}} - \int_0^1 p_{jt} y_{jt} dj.$$

#### 3.4.3 New-Keynesian Phillips Curve

The log-linear approximation of the pricing rule for the intermediate firms about a zero inflation steady state reduces to the New Keynesian Phillips Curve. The New Keynesian Phillips Curve shows the relationship between nominal and real variables that results from the assumed price-setting rigidities of the intermediate firms.

$$\hat{\Pi}_t - \zeta \hat{\Pi}_{t-1} = \frac{(1-\mu\beta)(1-\mu)}{\mu} \hat{s}_t + \beta (\hat{\Pi}_{t+1} - \zeta \hat{\Pi}_t) \quad (8)$$

### 3.5 Monetary Policy

We abstract away from explicitly modeling open market operations and assume that money and bonds are in zero-supply. Monetary policy sets  $R_{t+1}^{rf}$ , the short-term nominal interest rate, according to the following rule:

$$\hat{R}_{t+1}^{rf} = \max \left\{ -1 + lb, \rho_\pi \hat{\Pi}_t \right\} \quad (9)$$

This is a Taylor rule that respects a lower bound ( $lb$ ) on the nominal short-term rate. We permit  $lb > 0$  in recognition that, in the real world, the central bank's target rate typically departs from its normal relation with macroeconomic variables at a small, but positive number. The assumption of a monetary policy of this form is not innocuous since Eggertsson and Woodford [7] have shown that commitment by the monetary authority to expansionary deviations from the Taylor rule after the nominal rate rises above its lower bound substantially alleviates the distress otherwise associated with a liquidity trap. We choose to focus on policy that complements the Taylor Rule. In order to solve the dynamics of the economy, we will rewrite the monetary policy rule in two parts:

$$\hat{Z}_{t+1} = \rho_\pi \hat{\Pi}_t \quad (10)$$

$$\hat{R}_{t+1}^{rf} = \max \left\{ -1 + lb, \hat{Z}_{t+1} \right\} \quad (11)$$

### 3.6 Government

Fiscal policy consists of neutral government spending and spending on investment subsidies. Spending is financed by lump sum taxes. We assume the government does not borrow. Thus, a given schedule of aggregate tax and expenditure flows  $\{X_t^h, x_t I_t, G_t\}$  must satisfy

$$x_t I_t + X_t^h + G_t = 0 \quad (12)$$

We assume government spending does not yield direct utility to any agent. Inclusion of a direct benefit in the household's momentary utility function would not alter the evolution of the economy as long as household utility from government spending was separable from household consumption and labor; the welfare calculation, of course, would be affected even under separability<sup>1</sup>.

### 3.7 Aggregation

Aggregate consumption and labor are simple scalings of individual consumption and labor:  $C_t^h = \eta^h c_t$ ,  $L_t^h = \eta^h \ell_t^h$ . The law of motion for the aggregate capital stock and the economy's resource constraint are

$$K_{t+1} = (1 - \delta_t) K_t + I_t \quad (13)$$

$$Y_t = C_t^h + I_t + G_t. \quad (14)$$

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<sup>1</sup>The assumption that government spending is wasteful (but not harmful) is not as unfair as it may seem. Because government spending eats up output, but households want to mitigate the decline in their private consumption, households work harder and build more output. By contrast, if private and public consumption were modeled as substitutes in the utility function, then a rise in government spending would lead households to substitute away from private consumption and overall output would not increase as much.

Last, the supply of aggregate capital and labor must equal the demand for capital and labor:  $K_t = \int_0^1 k_{jt} dj$ ,  $L_t^h = \eta^h \ell_t^h = \int_0^1 \ell_{jt} dj$ .

### 3.8 Equilibrium

A recursive “monetary” equilibrium of this economy is a vector of functions that map the endogenous state variables  $\{K_t, \Pi_{t-1}\}$  and exogenous sequences  $\{x_t, G_t, \delta_t\}$  to the choice variables and market prices of the economy, represented by the vector

$$z_t = [ \lambda_t \quad c_t \quad \ell_t^h \quad I_t \quad K_{t+1} \quad q_t \quad Z_{t+1} \quad R_{t+1}^{rf} \quad \dots \\ \dots \quad P_t \quad Y_t \quad w_t \quad r_t \quad \{p_{jt}\} \quad \{y_{jt}\} \quad \{\ell_{jt}\} \quad \{k_{jt}\} \quad X_t^h ]^T$$

such that the household and firm problems are satisfied, the government’s policy rules are satisfied, all markets clear, and the aggregate resource constraints are respected.

## 4 Model with a Financial Friction

This section explains how the economic environment changes when the economy’s capital producers must be provided incentives to not adversely affect the returns to their creditors in an attempt to achieve a higher private return. Capital producers now earn positive profits and accumulate wealth. The unit population of agents is split into a continuum of measure  $\eta^h \in (0, 1)$  for households and  $\eta^f = 1 - \eta^h$  for capital producers. While a household is infinitely lived, capital producers face a constant probability of death  $1 - \tau^f$ .<sup>2</sup>

### 4.1 Capital Producers

#### 4.1.1 Capital accumulation and the contracting problem

Capital producers are risk-neutral agents with the exclusive ability to pursue capital-building projects. They discount the future at rate  $\gamma$ ; we assume  $\gamma$  is high enough to guarantee that a capital producer invests her entire net worth until she dies, at which time she consumes her net-worth and exits the economy.

Since capital producers perfectly compete for household loans, in the absence of any contracting friction would compete away any rents. We force a wedge between the return on investment of capital producers and households by assuming that the capital producer can abscond a fraction  $\phi$  of project revenues.

The capital producer’s problem is similar to that in the unconstrained model, except that now the contract must satisfy an incentive compatibility constraint, and the resource constraint includes the net worth of the capital producer:<sup>3</sup>

$$\max_{h_t, d_t} \text{payoff}_{t+1}^K i_t - \left(1 + r_{t+1}^{rf}\right) d_t \text{ s.t.}$$

<sup>2</sup>The model requires that capital producers periodically die so that they do not accumulate over time enough net worth to render irrelevant the moral hazard problem described below.

<sup>3</sup>The assumption that the contract maximizes the capital producer’s payoff is without loss of generality because the financier will compete for funds until her own incentive constraint binds. If the contract maximized the payoff to households, the household would choose the lowest possible return for the financier such that the incentive constraint of the capital producer binds. Then, the households would expand the project size until the rate of return on their investment was equal to the rate of return on a riskless bond, i.e. until the household IR binds. Thus, the equilibrium contract would be identical.



$$\text{payoff}_{t+1}^K i_t - \left(1 + r_{t+1}^{rf}\right) d_t \geq \phi \text{payoff}_{t+1}^K i_t \quad (15)$$

$$\text{payoff}_{t+1}^K i_t - \left(1 + r_{t+1}^{rf}\right) d_t \geq \left(1 + r_{t+1}^{rf}\right) n_t \quad (16)$$

$$d_t + n_t = i_t \quad (17)$$

The incentive compatibility (IC) constraint (15) states that the profits of the capital producer must exceed what she could anyway obtain by running away. The capital producer wants to maximize the size of the project in order to earn the highest possible excess return over households.<sup>4</sup> To attract funds, the capital producer must raise the households' share; she will do this until one of the constraints binds. We assume and verify in our analysis that  $\phi$  is high enough such that the IC constraint binds and the individual rationality constraint (16) does not bind. Thus, the excess return to new capital over the risk free rate is positive.

Conditional on a binding IC constraint, we show the capital producer's minimum necessary share limits the project's size. The binding incentive compatibility constraint implies

$$(1 - \phi) \text{payoff}_{t+1}^K i_t = \left(1 + r_{t+1}^{rf}\right) d_t$$

Let  $\frac{1}{L_t} \equiv \frac{n_t}{i_t}$  denote the inverse of leverage. Substituting into the budget constraint (17) and recalling that  $q_t = \frac{\text{payoff}_{t+1}^K}{1+r_{t+1}^{rf}}$ , we obtain

$$\frac{1}{L_t} = 1 - (1 - \phi) q_t$$

Leverage, the ratio of total funds invested to net worth, is therefore increasing in  $q_t$ . This is not to say, however, that the financial friction implies that investment responds more strongly to changes in the price of capital. This relationship only explains why, in a model with financial frictions, movements in investment are tied to the net worth of capital producers. As will be clear soon, the financial friction induces a form of intertemporal investment adjustment costs.

Conveniently, the leverage ratio is invariant to idiosyncratic capital producer wealth; the aggregate leverage ratio equals each capital producer's leverage choice. So, we can easily aggregate across capital producers to obtain

$$I_t = L_t N_t \quad (18)$$

where  $N_t$  is aggregate capital producer net worth available to be invested at that date and  $I_t$  is aggregate investment.

#### 4.1.2 Capital producer net worth and consumption

Capital producers earn at time  $t$  income from their share  $K_t^f$  of new capital they produce at the beginning of the period. A random fraction  $1 - \tau^f$  of capital producers die, while  $\tau^f$  of those capital producers who entered the period continue on. Specifically, the amount of capital producer net worth at time  $t$  that will be available for investment at that date is

$$N_t = \tau^f \text{payoff}_t^K K_t^f + X_t, \quad (19)$$

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<sup>4</sup>In a perfect foresight equilibrium, agents have no incentive to trade contingent claims. In the non-stochastic steady state, the probability of any shock is zero. Once a shock is realized, agents know the state with certainty.

where  $X_t$  denotes recapitalization expenditures and the capital producers' share of new capital produced obeys

$$K_t^f = \phi I_{t-1}. \quad (20)$$

In words, capital producers' share of new capital equals the incentive-compatible share of revenues they extract from their projects.

Exiting capital producers consume an amount of final goods

$$C_t^f = (1 - \tau^f) \text{payoff}_t^K K_t^f. \quad (21)$$

Finally, we show that the friction induces a form of intertemporal investment adjustment costs. Combining the equations for capital producer net worth (19), capital producer capital stock (20), and the relationship between total investment and capital producer net worth (18), we obtain

$$I_{t+1} = L_{t+1} \left\{ \left[ \tau^f \phi \text{Payoff}_{t+1}^K \right] I_t + X_{t+1} \right\} \quad (22)$$

## 4.2 Government and Aggregation

The government budget constraint with recapitalization expenditures  $X_t$  is

$$X_t + X_t^h + G_t = 0$$

For aggregation, we take into account the consumption of capital producers. The economy's resource constraint is:

$$Y_t = C_t^h + C_t^f + I_t + G_t \quad (23)$$

## 4.3 Equilibrium

A recursive "monetary" equilibrium of this economy is a set of functions that map the state variables  $\{K_t, K_t^f, \Pi_{t-1}\}$  and exogenous sequences  $\{X_t, G_t, \delta_t\}$  to the choice variables and market prices of the economy, represented by the vector

$$z_t = [ \lambda_t \quad c_t \quad \ell_t^h \quad I_t \quad K_{t+1} \quad K_{t+1}^f \quad q_t \quad Z_{t+1} \quad R_{t+1}^f \quad \dots \\ \dots \quad P_t \quad Y_t \quad w_t \quad r_t \quad \{p_{jt}\} \quad \{y_{jt}\} \quad \{\ell_{jt}\} \quad \{k_{jt}\} \quad X_t^h ]^T$$

such that the household and firm problems are satisfied, the government's policy rules are satisfied, all markets clear, and the aggregate resource constraints are respected.

In the absence of switching in the monetary regime, there exists a unique nonexplosive equilibrium for the linearized economy, as in the standard model. In the model with financial frictions, there exist two liquidity trap equilibria. We select for analysis the liquidity trap equilibrium that converges to the unique non-trap equilibrium as the magnitude of the fundamental shock decreases. The other equilibrium can best be thought of as a sunspot. It exists independently of any fundamental volatility; that is, even in absence of a shock, an anticipated change in the monetary policy regime itself generates a deflationary contraction that validates the regime change. This equilibrium may be of interest to scholars studying self-fulfilling regime changes; however, we focus on regime changes caused by changes in fundamentals.

## 5 Parameter Definition and Calibration

We specify parameters for the financial friction model. The standard model parameters differ in the following way:  $\eta^f = 0$  ( $\eta^h = 1$ ) and  $\phi = 0$ , and  $\tau^f$  is irrelevant.

$lb$	0.25	$\theta_h$	0.64	$\zeta$	1
$\beta$	0.995	$\theta_k$	0.36	$\eta^f$	0.1
$\delta$	0.02	$\mu$	0.75	$\phi$	0.67
$\xi_p$	6	$\rho_\pi$	1.5	$\tau^f$	0.97

Table 1: Exogenous Parameter Specification

Most of the parameter values are within the range of standard values in the New Keynesian DSGE literature.  $\xi_p = 6$  gives a steady state markup over marginal cost of 20%. The probability of an intermediate firm not re-optimizing  $\mu = 0.75$  implies that on average an intermediate goods firm re-optimizes once every four quarters. Not-reoptimizing monopolists automatically update their prices according to lagged inflation ( $\zeta = 1$ ). We choose the Taylor rule parameter  $\rho_\pi$  to match simulations by Christiano, Eichenbaum, and Rebelo [8] and an annualized lower bound of 0.25 % to approximate the upper bound of the range of target interest rates set by the US Federal Reserve in 2008-2009, and the Bank of Japan during the mid-1990s and 2000s, when these central banks implemented near-zero interest rate policies.

We calculate a non-stochastic steady state with zero net inflation. The capital producer parameters  $\phi$  and  $\tau^f$  match a steady state annualized return on equity (ROE) of 12%, equal to the average ROE for US Banks between 1984 and 2010. Those parameters also match a steady state leverage of  $L = 4$  as in Gertler and Kiyotaki [3].

We calibrate the households' time discount  $\beta$  to support an annualized net interest rate of 1.925%, equal to the observed historical average US Treasury bill rate adjusted upwards for Krishnamurthy and Vissing-Jorgensen's [11] estimates of the effects of "liquidity" and "safety" premia on Treasury yields.

We calibrate monopolists' fixed costs  $\Theta$  to extinguish steady state monopolist profits. Steady state government expenditures  $G$ . The weight  $\psi$  on the quadratic disutility of labor matches steady state household effort equal to unity (as in [1]).

## 6 Shock Specification and Interpretation

The analysis begins with the standard model, which illustrates the stark differences between government spending and investment subsidies. In the standard model, we implement a shock to the investment tax (negative subsidy) in period 1. [12] and [1] make use of this shock as a proxy for the presence of financial frictions during recessions. The shock creates a wedge between the physical return to capital production and households' perceived return on capital production. The primary effect of the shock is to distort the households' consumption / savings decision, lowering investment. In equilibrium, lower investment implies negative forward consumption growth and so the real rate falls. Because the consequent decrease in output is deflationary, the nominal rate also falls.

In order to induce a liquidity trap and study the effect of government policy in the model with an *explicit* financial friction, we use a positive shock to the rate of depreciation in period 1. The direct effect of the shock (an increase in  $\delta_1$ ) is to reduce period 1 capital

producer net worth. This can be seen from equation (19) at time 1, reproduced here for convenience:

$$N_1 = \tau^f \left[ r_1^k + (1 - \delta_1) q_1 \right] K_1^f + X_1$$

Because investment expenditures are tied to net worth, the decline in net worth drags down investment, despite a rise in the value of new capital. As a result, the real interest rate falls. Again, because the consequent decrease in output is deflationary, the nominal rate falls as well.

One interpretation of the depreciation shock is that of an unexpected change in capital quality. Consider a model with multiple capital goods. In period 1, the agents find out that the capital mix existent at the start of period 1 becomes less effective at producing intermediate goods, starting in period 2 and beyond. Additionally, agents know in period 1 what is the “right” mix for new capital going forward. Thus, the returns to producing new capital become large. However, capital producers’ net worth comes from capital rents and the sale of the capital mix to households. The capital mix at period 1 is worth only a fraction of what it would have been worth in steady state, which depresses the net worth of capital producers and drags down investment.

In the standard model, an unexpected increase in  $\delta_1$  would induce increases in real and nominal interest rates. At period 1, capital that had accumulated previously would become less useful, but building new capital would be very attractive. Without the financial friction, there would be no tightening leverage constraint to offset the increased incentive to invest. Thus, initial desired investment would increase, and so would the equilibrium real and nominal rates<sup>5</sup>.

## 7 Standard Model Dynamics

### 7.1 Pure effect of a shock to the investment tax

In the analysis of the model without explicit financial frictions, we explore an increase in the tax on investment from zero to 1%. The direct effect is to depress investment and increase consumption. The drop in current investment reduces the future stock of capital. This increases the incentive to save in future periods. As a result, the forward rate of consumption growth declines. As the households’ optimal savings condition (4) shows, negative forward consumption growth is in turn consistent with initial declines in the real interest rate.

The nominal interest rate will also fall. This is because households’ desire to smooth consumption dampens the initial-period increase in consumption, so the drop in investment exceeds the rise in consumption, and output falls. The decline in labor reduces marginal costs and, under the New Keynesian Phillips Curve, (8), the rate of inflation. The Taylor rule, (9), then implies a more-than-proportional decline in the nominal rate.

In response to a large enough shock, the nominal rate violates its lower bound. We will use the shock to explore the efficacy of different policies in a liquidity trap.

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<sup>5</sup>Presumably, the nominal rate could still decrease in response to current deflation, even if the real rate increased. But the Taylor rule would prescribe a more-than-proportional interest rate decline in response to that deflation. For the real interest rate to still rise, then, would require an even deeper *future* deflation. But in the present model, the initial period shock must have its largest effect in the first period.

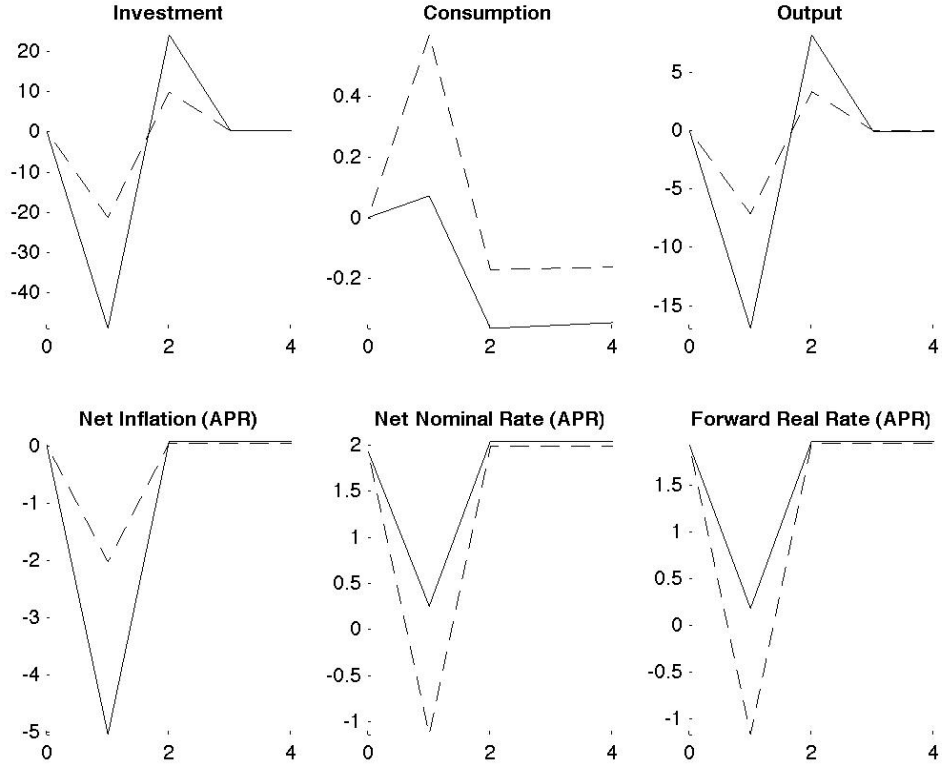


Figure 1: Baseline dynamics with and without a liquidity trap  
 Legend- Solid: Respects ZLB. Dashed: Ignores ZLB.

## 7.2 Interaction of the shock and a liquidity trap at period 1

The regime change from a Taylor-rule to a fixed rate significantly magnifies the early effects of the financial friction shock; investment, consumption, and inflation all fall further than they would when ignoring the lower bound. Relative to the dynamics of the shock in the absence of a liquidity trap, respecting the lower bound on the nominal rate prevents the real interest rate from falling as far as it would in the absence of a binding constraint.

The higher real rate has two effects. First, notice that the households' optimal savings condition, (4), reproduced below, implies that the higher real rate increases the forward rate of consumption growth.

$$1 = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-1} \times \frac{1 + R_{t+1}^{rf}}{\Pi_{t+1}}$$

Given the negative effect of lower investment on the future productive capacity of the economy, forward consumption does not increase relative to its steady state, so initial consumption falls. Second, consider the no arbitrage condition between bond purchases and real investment, displayed below, which emerges from (4) and (5):

$$\frac{1 + R_{t+1}^{rf}}{\Pi_{t+1}} = \frac{\text{payoff}_{t+1}^K}{q_t}$$

This no-arbitrage condition implies that the rate of return to investment must increase. In order for this to occur, future capital rents must rise. A deeper decline in initial investment exerts upward pressure on future rents.

Since investment and consumption decline relative to the baseline path, output also falls. The deeper decline in output magnifies the initial deflation.

Because investment falls more sharply in a liquidity trap, when the economy escapes the liquidity trap, the incentive to invest is very high. Additional savings are financed out of reduced consumption and increases in hours worked and labor income. Responses after the second period very-nearly coincide with those obtained when ignoring the lower bound on nominal interest rates. In both cases, investment remains slightly elevated, consumption remains mildly depressed, and the capital stock drifts up toward its steady state value.

### 7.3 Policy Experiments

Instead of considering the optimal implementable paths for consumption and labor, we consider the differential welfare effects and output multipliers of neutral government spending, as well as an investment subsidy that effectively reverses some of the financial friction shock. Because this model has capital, the equilibrium relationships between policy shocks and welfare or output are not analytically tractable. We instead build intuition by exploring simulations over a wide range of shock values.

Consider a [variable] shock that implies the household pays a tax rate between zero and 1.7% on each final good invested into a capital production project. We show that increasing government spending raises output but is not sufficiently effective to raise economic welfare, even during a liquidity trap. On the other hand, an investment subsidy that directly lowers the magnitude of the investment tax both raises output significantly more than government spending and is welfare increasing. Additionally, the differential output and welfare effects of the subsidy policy versus government spending are amplified by the presence of a liquidity trap.

Our exposition focuses on a model with linear capital production because it starkly demonstrates that there exist alternative fiscal policies that dominate traditional fiscal stimulus in a liquidity trap. However, the results are not sensitive to linear capital production. The qualitative results from this section obtain even under large adjustment costs in capital production and high curvature in momentary utility over consumption. Higher adjustment costs and higher curvature in preferences increase the government spending multiplier relative to the subsidy-spending multiplier, and the government spending multiplier can exceed one under non-separable preferences over consumption and leisure. But investment subsidies still dominate neutral government spending, over a broad range of specifications.

#### 7.3.1 Government spending

Suppose that the government increases its expenditures by 0.1% of steady state output. Figure 2 shows the initial period output multiplier of government spending, i.e. the rise in initial period output per additional unit of final good lost to government expenditures. Even though the spending multiplier is not large, and in particular very far from unity (the value above which private disposable income would not fall in response to an increase in government spending), it is nonetheless positive.

The presence of a liquidity trap at period 1 triples the multiplier. I.e., the boost to

output resulting from an increase in government spending is larger in a liquidity trap than outside a liquidity trap, which is consistent with [8]. In a liquidity trap, the monetary authority no longer responds to the higher rate of inflation with a more-than-proportionate increase in nominal interest rates. Thus, the real interest rate remains almost fixed, instead of rising and thereby eliciting further declines in private expenditures.

However, even in a liquidity trap, our model implies that government spending reduces private expenditures. To understand how this can occur, consider that government spending reduces the amount of final goods available for private expenditure, at any level of output. The household would like to smooth the temporary negative wealth effect of this intervention, but is only partially willing to do so. So, the household works harder in the initial period, but also invests and consumes less.

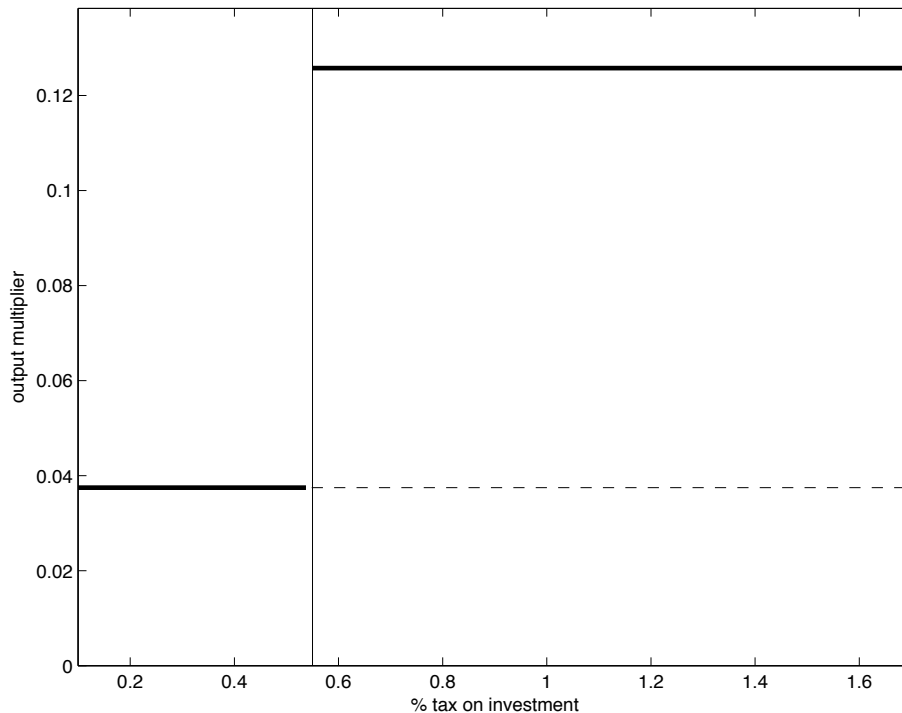


Figure 2: Fiscal Output Multiplier in the Standard Model

Legend- Solid: respects ZLB. Dashed: Ignores ZLB.

### 7.3.2 Undoing a Financial Friction Shock

Suppose that the government creates an investment subsidy worth 0.1% of steady state output. Figure 3 shows the initial period output multiplier of the subsidy, i.e. the rise in initial period output per additional unit of final good payed out in subsidies. The subsidy multiplier without even considering a liquidity trap to just above 20, which suggests the

investment subsidy is very effective. Even so, the presence of a liquidity trap at period 1 quadruples the multiplier to just above 80.

An investment subsidy substantially reduces the severity of the liquidity trap, because the intervention increases the households' effective rate of return to building new units of capital by lowering the cost to households of investment. Thus, the return to investment can equal the return on the nominally risk free asset, at a higher level of investment. As a result, equilibrium investment increases.

Why isn't higher investment financed out of decreased consumption? First, in a liquidity trap higher output and higher inflation do not elicit an increase in the nominal interest rate. Thus, the real interest rate falls despite the increase in initial investment and the second-period stock of capital; the interest rate places upward pressure on current consumption.

Furthermore, higher future capital will reduce the incentive to save in the future, and increase productive capacity. Higher future consumption is consistent only with higher initial consumption, if the real interest rate does not rise. Because both investment and consumption increase substantially, output and inflation also rise.

Investment subsidies overwhelmingly dominate government spending for two reasons: (1) the investment subsidy directly addresses the problem by literally reducing the magnitude of the the shock, and (2) the investment subsidy does not waste final goods. However, if government spending was not wasteful, and instead households substituted private consumption for government consumption, the mechanism through which government spending could increase output would be diminished. Thus, government spending would continue to be ineffective, even if less welfare-reducing.

### 7.3.3 Welfare

Subsidizing investment dominates increasing government spending in welfare terms, inside or outside a liquidity trap. Government spending is not welfare increasing either inside or outside a liquidity trap. That government spending is harmful outside a liquidity trap is hardly surprising, as the steady state is efficient and the destruction of final goods can achieve no good. That government spending is harmful even when a liquidity trap exists reflects that even with a liquidity trap, government spending is not sufficiently stimulative to overcome the loss of output.

## 8 Financial Friction Model Dynamics

### 8.1 Pure effect of a capital quality shock

The shock permanently degrades the future productivity of the capital stock entering period 1 by 11%. This damages capital producers' balance sheets by lowering the value of capital sold by capital producers to households in the initial period. Because moral hazard constrains capital producer leverage, falling net worth drags down overall investment, consistent with (18), reproduced below:

$$I_t = \frac{N_t}{1 - (1 - \phi) q_t}$$



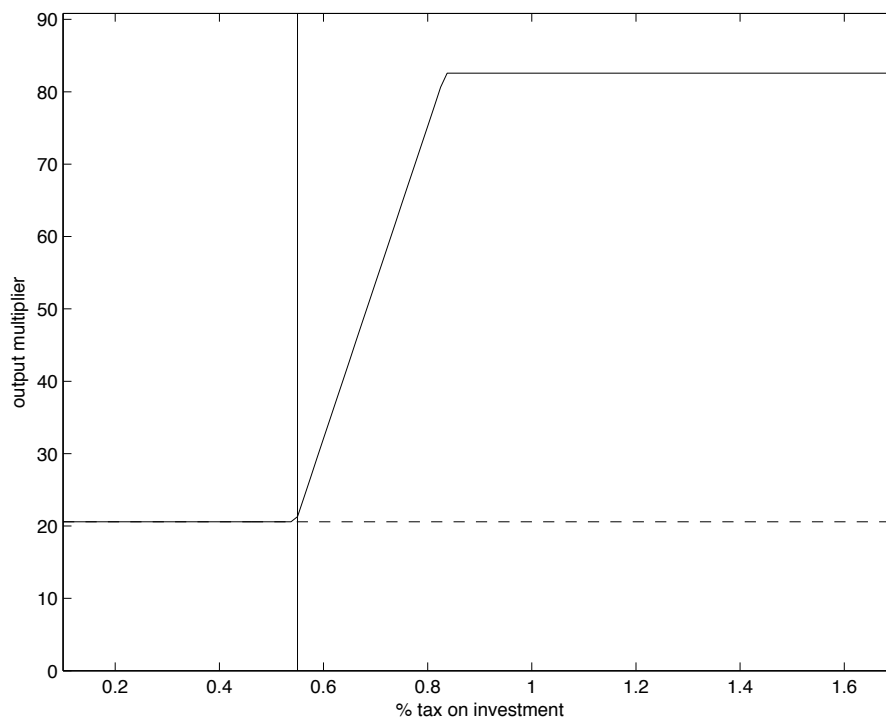


Figure 3: Subsidy Output Multiplier in the Standard Model  
 Legend- Solid: Respects ZLB. Dashed: Ignores ZLB.

Low investment today depresses net worth in future periods. Thus, investment, output, and inflation recover more slowly than in the model without financial frictions. As capital producer net worth recovers, so does investment. Investment overshoots, because the marginal value of new capital rises on impact. In anticipation of reduced future productive capacity and, later, a large demand for household savings as capital producer net worth recovers, consumption-smoothing households reduce consumption for several periods.

The persistence of low inflation after the shock is consistent with a slower recovery in the nominal interest rate, a feature that becomes important when the lower bound binds. When ignoring the lower bound on the nominal interest rate, the initial decline and later recovery in investment demand - and, hence, savings - result in negative forward consumption growth, which is consistent with an initial decline in the real interest rate.

As in the standard model analysis, we will use the shock to explore the efficacy of different policies in a liquidity trap.

## 8.2 Interaction of a capital quality shock and a liquidity trap at periods 1 and 2

If the shock to the capital stock is sufficiently large, the nominal interest rate will violate its lower bound. When the lower bound limits the downward movement of the nominal rate, the real interest rate is higher than it otherwise would be. The higher real

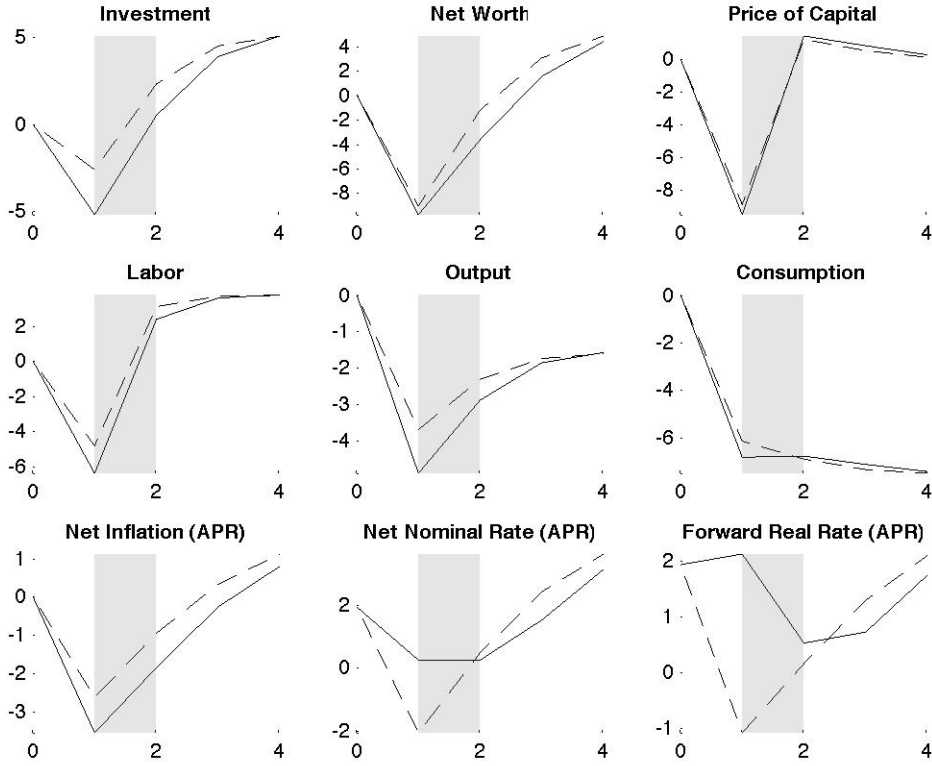


Figure 4: Baseline dynamics with and without a liquidity trap  
Legend- Solid: Respects ZLB. Dashed: Ignores ZLB.  
Shaded: Liquidity trap.

interest rate increases the costs of borrowing for leverage-constrained capital producers; this effect reduces investment. To see this, consider that we can use (18), (5), and (4) to write

$$I_t = \frac{N_t}{1 - (1 - \phi) \frac{\text{payoff}_{t+1}^K}{1+r_{t+1}^f}}$$

Additionally, the households' intertemporal optimality condition, (4), implies that the forward rate of consumption growth must rise, which is accomplished with a deeper decline in initial consumption. Thus, the initial declines in output and inflation are larger, when the model respects the lower bound on nominal interest rates.

The deeper fall in initial investment also suppresses future net worth; this in turn constrains future investment, and further slows the recovery in output and inflation. The lower rates of inflation in the future in turn further increase the initial real interest rate; thus, the contraction induced by the downwardly-immobile nominal interest rate reinforces itself.

Notice that the path of consumption when ignoring the lower bound on the nominal rate falls beneath the path of consumption after period 2, when the liquidity trap ends. By contrast, in the frictionless model, consumption when ignoring the lower bound everywhere exceeded consumption when respecting the lower bound. In the model with financial fric-

tions, the liquidity trap damages capital producer net worth. This constrains investment and, therefore, savings even after exiting the liquidity trap, despite the high return to investment. The weak response of investment respecting the lower bound allows consumption to recover relatively faster.

### 8.3 A capital quality shock with a liquidity trap at period 1 and recapitalization policy

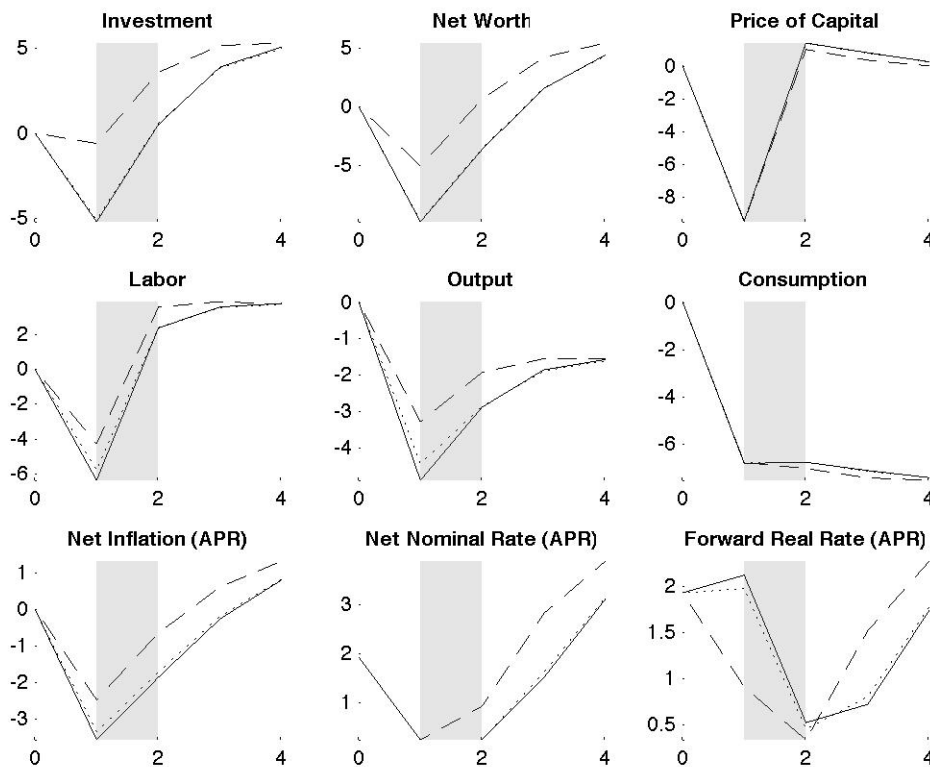


Figure 5: Baseline liquidity trap dynamics with and without intervention  
 Legend- Solid: No policy. Dotted: Fiscal. Dashed: Recapitalization.  
 Shaded: Liquidity trap.

We now explore the effects of a recapitalization of magnitude equal to 0.4% of steady state output. A one-period recapitalization of the capital producers substantially reduces the severity and duration of the liquidity trap. The recapitalization relaxes the constraint on investment arising from the moral hazard. As a result, the recapitalization offsets the decline in investment.

Importantly, the increase in investment from the intervention boosts future net worth, which in turn helps investment and output to recover more rapidly. The resulting higher future rate of inflation reduces the initial-period real interest rate. Thus, the expansionary effects of the recapitalization reinforce themselves.

As in the frictionless model, it is important that in the liquidity trap the monetary authority does not respond to higher inflation with a larger-than-proportional increase in the nominal interest rate. Because the real interest rate instead falls sharply relative to the no-intervention path, the forward growth rate of consumption must decline. This effect helps prevent initial consumption from falling further.

Because the recapitalization reduces the damage done by the liquidity trap to capital producer net worth, investment and savings recover more rapidly after the liquidity trap. As a result, household consumption under the intervention falls below household consumption without the intervention for a few periods after exiting the liquidity trap. This is an improvement over no-intervention, as the return to investment is high and low net worth therefore very costly. Recall, also, that consumption when ignoring the liquidity trap falls below consumption when respecting the liquidity trap, starting in period 2. In both the baseline and financial friction models, the investment boosting policy moves consumption closer to the path obtained when ignoring the lower bound.

#### **8.4 A capital quality shock with a liquidity trap at periods 1 and 2 and government spending.**

Suppose instead that the same magnitude of resources are used for fiscal stimulus. Neutral government spending elicits an almost one-to-one increase in output in the model with financial frictions. As in the standard model, higher government demand for final goods is satisfied by increased labor and output, which in turn reduces the extent of the deflation. However, the fiscal stimulus multiplier is larger in the model with the financial friction.

Why do we obtain this result? Since financiers are not taxed by the government to finance spending, the ratio of financier net worth to aggregate wealth increases in response to an increase in government spending. Thus, financiers will want to borrow a large proportion of household wealth for investment given some interest rate; or, put otherwise, financiers are willing to pay a higher rate to borrow a given fraction of household wealth. Consequently, households have a greater incentive to work more in order to compensate for the tax loss and thus be able to lend to financiers. Hence, investment remains almost fixed, and the fiscal stimulus multiplier is near unity in the model with financial frictions. In the model without financial frictions, by contrast, households reduce investment in order to avoid working a great deal more under higher government spending.

#### **8.5 Output multipliers**

In this section, we explore how the output multipliers for these policies vary with the magnitude of the shock to capital quality. Figure 6 plots initial-period output multipliers for different magnitudes of the capital shock. The dashed line and the line with circles correspond to multipliers for recapitalization, while respecting the zero lower bound and while ignoring the zero lower bound, respectively. The dotted line and the line with cross marks correspond to multipliers for an equivalent increase in government consumption, while respecting the zero lower bound and while ignoring the zero lower bound, respectively.

First, notice that both recapitalization and higher government spending boost output more inside of a liquidity trap. This additional benefit is due to the downward effect of the policies on the path of the real interest rate, i.e. the cost of borrowing, during a liquidity

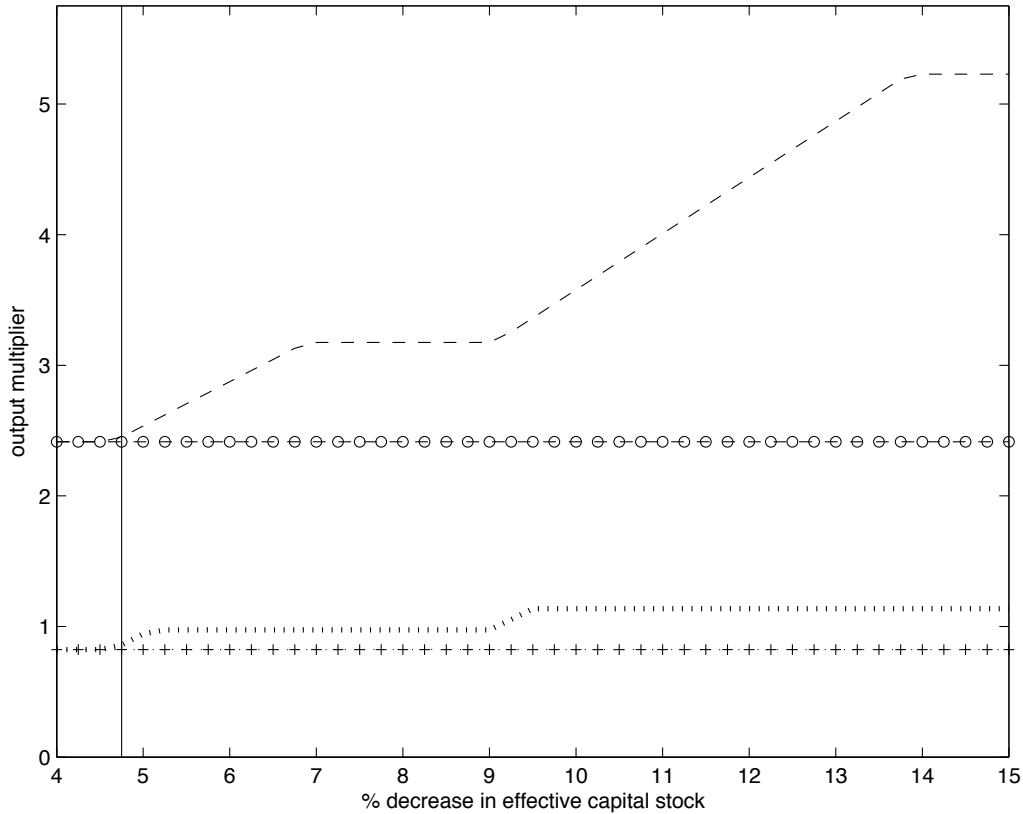


Figure 6: Output multipliers in the model with a financial friction

Legend- Dashed: Recap policy respects LB.    Circled/Dashed: Recap policy ignores LB.    Dotted: Fiscal policy respects LB.    Cross-marked/Dotted: Fiscal policy ignores LB.

trap. The real interest rate decreases because (1) during a liquidity trap an inflationary rise in output is consistent with a less-than-proportional response in the nominal interest rate and (2) future inflation rises in response to an increase in current inflation to be consistent with the New Keynesian Phillips curve.

Second, notice the multipliers that respect the lower bound increase in steps. For example, consider recapitalization. As the shock becomes large enough to trigger a liquidity trap, the multiplier rises linearly up to a plateau. To understand this, consider that the liquidity trap starts out very small. In this case, the recapitalization is more than large enough to cause the economy to exit the liquidity trap. As a result, some of the inflation caused by the policy triggers increases in the central bank's target rate. But as the shock and the liquidity trap grow in magnitude, the margin by which the policy avoids the liquidity trap shrinks. Less and less of the policy stimulus is offset by hikes in the target rate. And once the shock surpasses a certain threshold, the (fixed) recapitalization no longer permits an outright escape. At this point, the output multiplier plateaus, until the shock becomes just large enough to lengthen the zero lower bound episode by another period, and the multiplier ascends another step.

Finally, a liquidity trap expands the gap between the output multipliers for recapitalization and government consumption. The reason for this is that recapitalization accelerates the replenishment of investors' net worth, and strengthens the future recoveries in

investment and output. This inflationary future expansion in turn lowers the real cost of borrowing inside the liquidity trap. This strong intertemporal effect gives recapitalization an edge over traditional fiscal expansion. Furthermore, as the liquidity trap deepens, the multiplier for recapitalization increases by more than does the multiplier for government consumption. The longer is the liquidity trap, the longer the future expansion is unimpeded by rising nominal interest rates.

## 8.6 Welfare

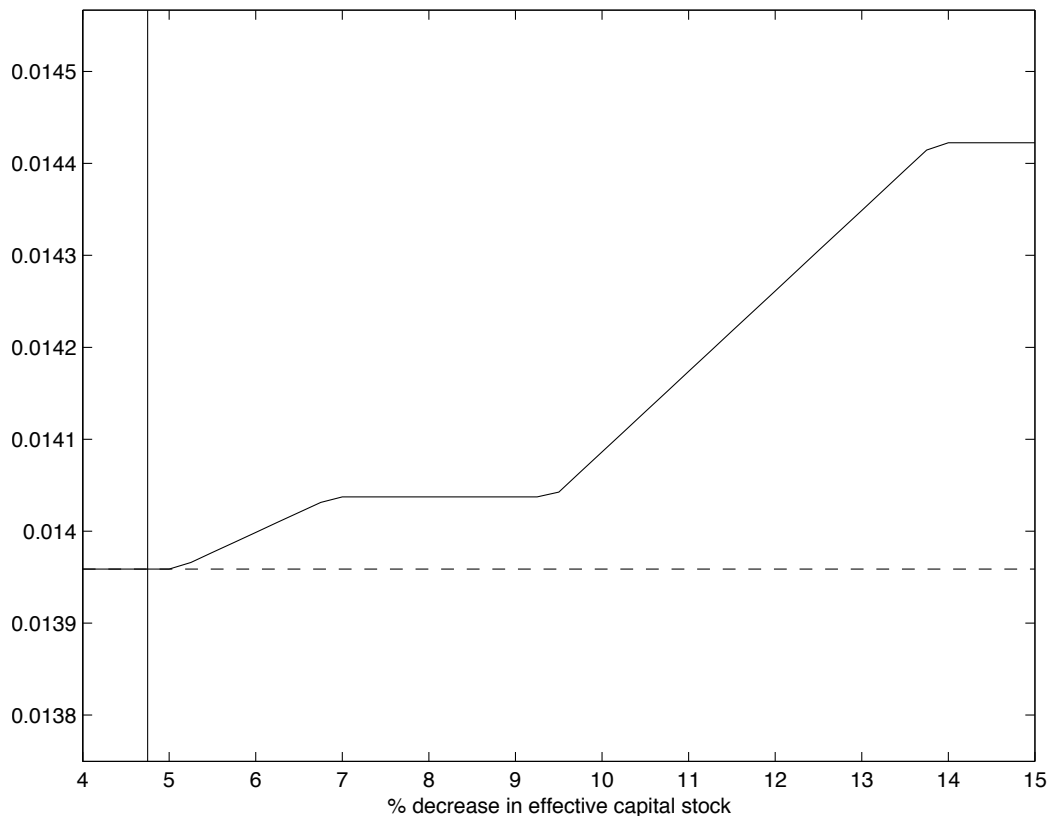


Figure 7: Difference in % welfare change due to policy

Legend- Solid: respects LB. Dashed: ignores LB

Finally, we illustrate the welfare<sup>6</sup> dominance of recapitalization policy. Figure (7) displays the difference between percent changes in welfare (relative to no intervention) under recapitalization, and percent changes in welfare under increased government consumption. The solid line displays this difference when the lower bound on nominal interest rates is respected. The dashed line ignores the lower bound.

<sup>6</sup>In this model, because there are two types of agent, we assume that the government's welfare function equals the population-weighted sum of utilities across agents, and that the government discounts the utilities of future generations of capital producers at the same discount rate as households discount future utility streams.

First, notice that the difference is always positive. This is consistent with the greater stimulative effect of recapitalization in accelerating recovery after the depreciation shock, inside and outside of a liquidity trap. The magnitude of the welfare advantage, of course, depends on how government spending enters into household preferences.

Second, and more important, the liquidity trap augments the welfare advantage of recapitalization over government spending. This is consistent with the above findings for the output multiplier. Recapitalization raises the future paths of investment, output, and inflation. As the liquidity trap becomes deeper and longer, recapitalization policy becomes comparatively even more welfare-increasing.

As in the standard model, only the investment-specific policy is welfare-improving. Government spending becomes relatively more beneficial in a liquidity trap, but household income still does not rise enough, and that rise is too transitory, to improve welfare in the long-run.

## 9 Limitations, Conclusion, and Further Thought

We showed that subsidizing investment and recapitalizing leverage constrained investors dampen the effects of a liquidity trap. Our results strongly suggest that researchers and policymakers who wish to mitigate the ill effects of a liquidity trap ought not limit their investigations to traditional fiscal stimulus. Different specifications can yield vastly different government spending multipliers in the liquidity trap, but for a very broad range of specifications, policies that support investment dominate government consumption.

We omit any sensible discussion of the role of money and other liquid securities such as Treasury bonds in the economy. Because the existence of a liquidity trap is inherently tied to corner conditions in money demand and monetary policy, it is an essential ingredient. We have followed the tradition of Woodford [13] in excluding money demand from our analysis, but we remain skeptical. For example, Alfred Pigou [14] argued that nocive effects of deflation during a liquidity trap may be undone by the consumption-boosting effect of rising real balances. Milton Friedman [15] argued that, because the central bank always has the option to buy other assets and thus increase the total circulation of cash and bonds (quantitative easing), a liquidity trap need never occur. We cannot discuss the validity of their arguments within our framework because they rely on an explicit model of liquidity demand where the equilibrium holding of liquid assets is strictly positive. Furthermore, under many standard models of liquidity demand, the lower bound on nominal interest rates is a limiting case; we are skeptical of the utility of a linear solution method for characterizing - even approximately - the asymptotic dynamics of liquidity demand near a liquidity trap. Due to computational limitations, we forsake the richer analysis of liquidity demand for the tractability of a linear approximation. We remain nonetheless confident that central banks face constraints on interest rate policy, and moreover, that our frameworks are sufficient to compare traditional fiscal policy to investment subsidies and recapitalizations.

Last, we do not take into account the effect of government recapitalizations on the ex-ante private incentives of investors. Tax-financed recapitalizations typically occur contingent on a sharp decrease in the ex-post aggregate return on investment in the economy. Investors, anticipating this, may be enticed to correlate their investments (as in [16]). Any complete analysis of the effectiveness of government support of financial firms must take this cost into account, since perhaps recapitalizing the financial sector may be ex-ante sub-optimal despite our results. Whether a central bank should subsidize the financial sector in

a liquidity trap will depend on whether such costs dominate the considerable ex post gains we have documented.



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## 10 Technical Appendix

### 10.1 Summary

We study the linearized economy described recursively by the evolution of a vector of variables  $\hat{z}_t$  (where  $\hat{z}_t = \frac{z_t - \bar{z}}{\bar{z}}$  with  $\bar{z}$  denoting the steady state value of  $z_t$ ) given by the linearized versions of each of the systems of equations, the sequence of which defines the conditions for the Walrasian sequence of markets equilibrium in the nonlinear economy. The vector of variables and the corresponding system of equations are:

$$z_t = \begin{bmatrix} q_t & \lambda_t & c_t & l_t & \Pi_t & I_t & K_{t+1} & Z_t & R_t^{rf} \end{bmatrix}^T$$

$$\begin{bmatrix} (5) & (2) & (4) & (14) & (8) & (1) & (13) & (10) & (11) \end{bmatrix}^T$$

When ignoring the zero lower bound or considering only small shocks, we are free to use the standard toolkit for finding a unique, nonexplosive “minimum state variable” (MSV) solution to the linearized version of the dynamic stochastic general equilibrium (DSGE) model. To conduct experiments while respecting the zero lower bound, we tell the following story. As of date zero, the agents are in a non-stochastic steady state. At date one, the agents wake up to find that one or more of the exogenously given variables (e.g.,  $\delta_t$ ) deviates from its steady-state value for several periods before returning at some future date to its steady-state value, where it will remain forevermore.

We use a shooting procedure to determine the evolution of the linearized economy consistent with agent observation at period one of the sequence of shocks. To accommodate the regime shift that occurs when the zero lower bound binds, we draw on techniques developed by Christiano, Eichenbaum, and Rebelo (2009). These techniques are documented in detail in the technical appendix. Briefly, we hypothesize that the economy enters the zero lower bound at date  $t_1 \geq 1$  and exits at some date  $t_2 + 1 > t_1$ . For periods  $t = t_1, \dots, t_2$ , the economy obeys the solution to the linearized model without the zero lower bound, with the exception that the nominal interest rate is fixed at zero, instead of fluctuating according to a Taylor-type rule. For all other periods, the economy obeys the system with the Taylor-type rule.

Denote by  $T$  the final period for which exogenously given quantities deviate from their steady state values. Then at date  $\max\{t_2 + 1, T + 1\}$  the economy has exited the zero lower bound and the shocks have concluded, and so the vector of normalized-deviations in endogenous variables  $\hat{z}_t$  follows the relation

$$\hat{z}_{t+1} = A\hat{z}_t$$

defining the stationary solution, where  $A$  is the policy matrix from the unique nonexplosive, minimum state variable (MSV) solution to the linearized DSGE version of the model. This relation plus the time  $\max\{t_2 + 1, T + 1\}$ -system of linearized equations gives a terminal condition that must be satisfied by the sequence of endogenous variables  $\{\hat{z}_t\}$ . The free components of  $\hat{z}_1$  are adjusted until the terminal condition just holds.

Last, we check that the Taylor-type interest rate that would hold at each date in the absence of the zero lower bound-constraint, but with the previous and contemporaneous realizations of endogenous quantities, would, in fact, be negative for precisely periods  $t = t_1, \dots, t_2$ . If not, we hypothesize an alternative  $t_1$  or  $t_2$ , and repeat the experiment.

## 10.2 Linearization

Our dynamic analysis explores the evolution of an economy when it is dislodged from a non-stochastic steady state. We approximate the true evolution of the economy within small deviations of steady state with first-order Taylor expansions. The linear solution method normally assumes that the system of equations defining an economic equilibrium is time-invariant. However, our analysis considers a constraint on monetary policy, the zero lower bound on the nominal interest rate, which only occasionally binds. The variable structure of the system of equations requires special care.

We consider the evolution of an economy whose agents ignore uncertainty. This is a strong assumption and relaxing it would likely reveal richer dynamics enhancing the risks and the effect of a liquidity trap; for example, in an economy “near” a liquidity trap, households may desire to engage in additional precautionary savings to guard against the possibility of a future discontinuous decline in income, a desire which itself could augment pressures driving an economy into a liquidity trap. The very difficult technical problems encountered when attempting such an analysis outside of highly stylized examples, however, gives the stronger assumption the benefit of much improved tractability, while still permitting us to obtain many insights that would likely remain qualitatively unchanged under a more general treatment.

Assume  $t = 0$  denotes a point in time where the economy is in steady state. Denote the vector of endogenous variables belonging to period  $t$  by  $z_t$ . We denote by  $\hat{z}_t$  the net normalization of  $z_t$  about the corresponding vector  $\bar{z}$  of steady-state values. We specify an exogenous sequence of shocks to technological and/or policy variables, where a “shock” denotes an exogenous quantity’s deviation from its steady state value. Denote by  $s_t$  exogenously given quantities at time  $t$ , and by  $\hat{s}_t$  the vector of shocks to said quantities normalized about the vector  $\bar{s}$  of the exogenous quantities’ steady state values.

Denote by  $\text{sys}_t$  the system of equations defining equilibrium conditions at date  $t$  when outside a liquidity trap. The system  $\text{sys}_t$  depends (possibly non-linearly) on  $z_{t-1}$ ,  $z_t$ ,  $z_{t+1}$  and  $s_t$ ,  $s_{t+1}$ . In a stochastic model,  $\text{sys}_t$  will include expectations. Since we consider shocks which are perfectly observable as of  $t = 1$ ,  $\text{sys}_t$  is deterministic in our analysis. Therefore,  $\text{sys}_t(\bar{z}, \bar{s}) = 0$  at each date outside the ZLB (we denote the zeroth term  $d_t$ ), and a first-order Taylor approximation about the non-stochastic steady state  $\bar{z}, \bar{s}$  yields

$$\begin{aligned} \text{sys}_t &\approx \text{sys}_t(\bar{z}, \bar{s}) + \nabla_{z_{t-1}} \text{sys}_t(\bar{z}, \bar{s}) \bar{z} \cdot \hat{z}_{t-1} + \nabla_{z_t} \text{sys}_t(\bar{z}, \bar{s}) \bar{z} \cdot \hat{z}_t \\ &\quad + \nabla_{z_{t+1}} \text{sys}_t(\bar{z}, \bar{s}) \bar{z} \cdot \hat{z}_{t+1} + \nabla_{s_t} \text{sys}_t(\bar{z}, \bar{s}) \bar{s} \cdot \hat{s}_t + \nabla_{s_{t+1}} \text{sys}_t(\bar{z}, \bar{s}) \bar{s} \cdot \hat{s}_{t+1} \\ &=: \alpha_2 \hat{z}_{t-1} + \alpha_1 \hat{z}_t + \alpha_0 \hat{z}_{t+1} + \beta_1 \hat{s}_t + \beta_0 \hat{s}_{t+1} + d_t \end{aligned} \quad (24)$$

Normally,  $\text{sys}_t$  would only include a monetary policy equation of the form  $r_t = f(r_{t-1}, y_t, \pi_t)$  summarizing a Taylor-type policy rule. However, because during a liquidity trap (i.e., when  $t$  lies between  $t_1$  and  $t_2$ )  $r_t = 0$  regardless of economic fundamentals, we make use of an auxiliary variable  $h_t$  such that  $h_t = f(r_{t-1}, y_t, \pi_t)$  and  $r_t = h_t$  hold outside the liquidity trap, while  $h_t = f(r_{t-1}, y_t, \pi_t)$  and  $r_t = 0$  during the liquidity trap.

Denote by  $\overline{\text{sys}}_t$  the system of equations when  $h_t = f(r_{t-1}, y_t, \pi_t)$  but  $r_t = 0$ . While it is not strictly necessary to enlarge the system and include  $h_t$  in addition to  $r_t$ , it makes the analysis more transparent and later facilitates checks that the ZLB in fact binds for those periods when it is hypothesized to bind. Suppose that the monetary policy equation giving  $r_t$  is the last equation in  $\text{sys}_t$ . Then the only difference in the log-linear approximations

$\text{sys}_t$  and  $\widetilde{\text{sys}}_t$  comes at the last row. That is,

$$\widetilde{\text{sys}}_t \approx \alpha_2 \hat{z}_{t-1} + \tilde{\alpha}_1 \hat{z}_t + \alpha_0 \hat{z}_{t+1} + \beta_1 \hat{s}_t + \beta_0 \hat{s}_{t+1} + d_t$$

where  $\tilde{\alpha}_1$  is just  $\alpha_1$  with the last row replaced by zeros until the last element, which is 1, and  $d_t$  is vector of rows except for the last element, which is 1. Thus, the last equation of  $\widetilde{\text{sys}}_t$  is  $\frac{r_t - r_{ss}}{r_{ss}} + 1 = 0$ , or  $r_t = 0$ , as desired.

### 10.3 Recursive formula when shocks and regime switches are foreseen

Our objective is to obtain a recursive form such that given variables up to time  $t + 1$  we may back out  $\hat{z}_{t+1}$ . The form we would like is

$$\hat{z}_{t+1} = A_1 \hat{z}_t + A_2 \hat{z}_{t-1} + B_1 \hat{s}_t + B_2 \hat{s}_{t+1} + F(d_{t+1}, d_t)$$

It turns out, however, that such a form is impossible to achieve, but adding a dependence on  $\hat{s}_{t+2}$  allows for

$$\hat{z}_{t+1} = A_1 \hat{z}_t + A_2 \hat{z}_{t-1} + B_1 \hat{s}_t + B_2 \hat{s}_{t+1} + B_3 \hat{s}_{t+2} + F(d_{t+1}, d_t) \quad (25)$$

If  $\alpha_0$  was invertible, backing out  $\hat{z}_{t+1}$  would be very easy. However,  $\alpha_0$  is not in general invertible. In order to derive a solution akin to the result in the invertible case, we use the QZ decomposition (or generalized Schur decomposition). Define  $\ell$  satisfying  $n - \ell \equiv \text{rank}(\alpha_0) < \text{length}(z) \equiv n$ . The QZ decomposition of matrices  $\alpha_0$  and  $\alpha_1$  yields a pair of orthonormal matrices  $Q$  and  $Z$ , and upper triangular matrices  $H_0$  and  $H_1$  that satisfy

$$Q\alpha_0 Z = H_0, \quad Q\alpha_1 Z = H_1$$

A similar result applies for  $\alpha_0$  and  $\tilde{\alpha}_1$ , yielding

$$\tilde{Q}\alpha_0 \tilde{Z} = \tilde{H}_0, \quad \tilde{Q}\tilde{\alpha}_1 \tilde{Z} = \tilde{H}_1$$

By construction  $ZZ' = I$  and  $\tilde{Z}\tilde{Z}' = I$ . Define  $\gamma_t \equiv Z'\hat{z}_t$  and  $\tilde{\gamma}_t \equiv \tilde{Z}'\hat{z}_t$  and  $X_{t+1} \equiv \beta_1 \hat{s}_t + \beta_0 \hat{s}_{t+1} + d_t$ , where  $d_t = d$  inside the liquidity trap and 0 otherwise. Premultiplication of (24) by  $Q$  gives

$$Q\alpha_0 Z Z' \tilde{Z} \tilde{Z}' \hat{z}_{t+1} + Q\alpha_1 Z Z' \hat{z}_t + Q\alpha_2 \hat{z}_{t-1} + QX_{t+1} = 0 \quad (26)$$

or

$$H_0 Z' \tilde{Z} \tilde{\gamma}_{t+1} + H_1 \gamma_t + Q\alpha_2 \hat{z}_{t-1} + QX_{t+1} = 0 \quad (27)$$

We develop a solution of the form (25) from (27) because the formulae resulting from development of (27) will permit us to easily summarize - via a few, simple substitutions - the solutions for each of the following cases: when both times  $t + 1$  and  $t$  are outside the ZLB; when  $t + 1$  is inside the ZLB while  $t$  is not; when both  $t + 1$  and  $t$  are inside the ZLB; and when time  $t + 1$  is outside the ZLB while time  $t$  is inside the ZLB.

Since  $Q$  and  $Z$  are invertible,  $H_0$  and  $\alpha_0$  must be of equal rank. Since  $H_0$  is a triangular matrix, at least  $\ell$  of its rows must have a zero on the main diagonal (otherwise,  $H_0$ 's rank would be higher than  $n - \ell$ ). Because we may reorder the QZ decomposition such that the spanned dimensions of  $H_0$  occupy the first  $\ell$  places in  $H_0$ , assume the first  $n - \ell$  diagonal elements are non-zero and the  $\ell$  zeros on the diagonal of  $H_0$  are located in its lower-right block. Moreover, note that the entire  $\ell$  bottom rows of  $H_0$  must equal 0. It is convenient to

split  $H_0$  and  $H_1$  about the element  $(m-l, m-l)$  to yield four submatrices for each original matrix, and  $Z$  and  $\gamma$  into two submatrices with height  $m-l$  and  $\ell$ .

$$H = \left[ \begin{array}{c|c} H^{11} & H^{12} \\ \hline H^{21} & H^{22} \end{array} \right] \quad Z' = \left[ \begin{array}{c} Z'_1 \\ Z'_2 \end{array} \right] \quad \gamma_t = \left[ \begin{array}{c} \gamma_t^1 \\ \gamma_t^2 \end{array} \right] \quad Q = \left[ \begin{array}{c} Q_1 \\ Q_2 \end{array} \right]$$

Thus, by construction  $H_0^{11}$  is invertible,  $H_0^{21} = 0$ , and  $H_0^{22} = 0$ . Also,  $H_1^{21} = 0$ , while  $H_1^{22}$  is upper-triangular and invertible<sup>7</sup>. Finally,  $Z_1$  and  $Z_2$  preserve the property  $Z'_i Z_i = I$  for  $i = 1, 2$ , although  $Z_i$  are not square.

The lower- $\ell$  equations of system (27) yield:

$$\begin{aligned} H_1^{22} \gamma_t^2 + Q_2 \alpha_2 \hat{z}_{t-1} &= -Q_2 X_{t+1} \quad \Rightarrow \\ \gamma_t^2 &= -(H_1^{22})^{-1} Q_2 [X_{t+1} + \alpha_2 \hat{z}_{t-1}] \end{aligned}$$

We are also interested in  $\tilde{\gamma}_{t+1}^2$ . When at time  $t+1$  the economy is in the ZLB,  $\tilde{\alpha}_1$  will replace  $\alpha_1$  in the system of equations at time  $t+1$ , and premultiplication by  $\tilde{Q}$  instead of  $Q$  will analogously yield<sup>8</sup>

$$\tilde{\gamma}_{t+1}^2 = -\left(\tilde{H}_1^{22}\right)^{-1} \tilde{Q}_2 [X_{t+2} + \alpha_2 \hat{z}_t]$$

The term  $H_0 Z' \tilde{Z}$  requires some care. Written out in blocks it looks like

$$\begin{aligned} &\left[ \begin{array}{c|c} H_0^{11} & H_0^{12} \\ \hline H_0^{21} & H_0^{22} \end{array} \right] \left[ \begin{array}{c} Z'_1 \\ Z'_2 \end{array} \right] \left[ \begin{array}{c|c} \tilde{Z}_1 & \tilde{Z}_2 \end{array} \right] = \left[ \begin{array}{c|c} H_0^{11} & H_0^{12} \\ \hline H_0^{21} & H_0^{22} \end{array} \right] \left[ \begin{array}{c|c} Z'_1 \tilde{Z}_1 & Z'_1 \tilde{Z}_2 \\ \hline Z'_2 \tilde{Z}_1 & Z'_2 \tilde{Z}_2 \end{array} \right] \\ &= \left[ \begin{array}{c|c} H_0^{11} Z'_1 \tilde{Z}_1 + H_0^{12} Z'_2 \tilde{Z}_1 & H_0^{11} Z'_1 \tilde{Z}_2 + H_0^{12} Z'_2 \tilde{Z}_2 \\ \hline H_0^{21} Z'_1 \tilde{Z}_1 + H_0^{22} Z'_2 \tilde{Z}_1 & H_0^{21} Z'_1 \tilde{Z}_2 + H_0^{22} Z'_2 \tilde{Z}_2 \end{array} \right] = \left[ \begin{array}{c|c} G_0^{11} & G_0^{12} \\ \hline G_0^{21} & G_0^{22} \end{array} \right] \end{aligned}$$

The upper  $(m-l)$  block of equations in (27) is

$$G_0^{11} \tilde{\gamma}_{t+1}^1 + G_0^{12} \tilde{\gamma}_{t+1}^2 + H_1^{11} \gamma_t^1 + H_1^{12} \gamma_t^2 + Q_1 \alpha_2 \hat{z}_{t-1} + Q_1 X_{t+1} = 0$$

or, after substitution for  $\gamma_t^2$  and  $\tilde{\gamma}_t^2$

$$\begin{aligned} &G_0^{11} \tilde{\gamma}_{t+1}^1 - G_0^{12} \left\{ \left(\tilde{H}_1^{22}\right)^{-1} \tilde{Q}_2 [X_{t+2} + \alpha_2 \hat{z}_t] \right\} \\ &\quad + H_1^{11} \gamma_t^1 - H_1^{12} \left\{ (H_1^{22})^{-1} Q_2 [X_{t+1} + \alpha_2 \hat{z}_{t-1}] \right\} + Q_1 \alpha_2 \hat{z}_{t-1} + Q_1 X_{t+1} = 0 \end{aligned}$$

Collecting terms, we have

$$\begin{aligned} &G_0^{11} \tilde{\gamma}_{t+1}^1 + \left[ H_1^{11} Z'_1 - G_0^{12} \left(\tilde{H}_1^{22}\right)^{-1} \tilde{Q}_2 \alpha_2 \right] \hat{z}_t + \left[ Q_1 \alpha_2 - H_1^{12} (H_1^{22})^{-1} Q_2 \alpha_2 \right] \hat{z}_{t-1} \\ &\quad - G_0^{12} \left(\tilde{H}_1^{22}\right)^{-1} \tilde{Q}_2 X_{t+2} - H_1^{12} (H_1^{22})^{-1} Q_2 X_{t+1} + Q_1 X_{t+1} = 0 \end{aligned}$$

<sup>7</sup>Golub and Van Loan [17] show in page 377, theorem 7.7.1, that  $H_0^{22} = 0$  implies that  $H_1^{22}$  is full-rank

<sup>8</sup>The presence of  $X_{t+2}^2$  in the equation below justifies the assertion that the formula for  $\hat{z}_{t+1}$  must depend on  $\hat{s}_{t+2}$ , as well as  $\hat{s}_t$  and  $\hat{s}_{t+1}$ .

Assume  $G_0^{11}$  is invertible. Then,

$$\begin{aligned}\tilde{\gamma}_{t+1}^1 = & - (G_0^{11})^{-1} \left[ H_1^{11} Z_1' - G_0^{12} \left( \tilde{H}_1^{22} \right)^{-1} \tilde{Q}_2 \alpha_2 \right] \hat{z}_t \\ & - (G_0^{11})^{-1} \left[ Q_1 \alpha_2 - H_1^{12} \left( H_1^{22} \right)^{-1} Q_2 \alpha_2 \right] \hat{z}_{t-1} \\ & + (G_0^{11})^{-1} \left[ G_0^{12} \left( \tilde{H}_1^{22} \right)^{-1} \tilde{Q}_2 X_{t+2} + \left\{ H_1^{12} \left( H_1^{22} \right)^{-1} Q_2 - Q_1 \right\} X_{t+1} \right]\end{aligned}$$

Now we can re-convert from  $\tilde{\gamma}_{t+1}$  to  $\hat{z}_{t+1}$ :

$$\begin{aligned}\hat{z}_{t+1} = & \begin{bmatrix} \tilde{Z}_1 & \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{t+1}^1 \\ \tilde{\gamma}_{t+1}^2 \end{bmatrix} = \tilde{Z}_1 \tilde{\gamma}_{t+1}^1 + \tilde{Z}_2 \tilde{\gamma}_{t+1}^2 \\ \hat{z}_{t+1} = & \tilde{Z}_1 \tilde{\gamma}_{t+1}^1 - \tilde{Z}_2 \left( \tilde{H}_1^{22} \right)^{-1} \tilde{Q}_2 [\alpha_2 z_t + X_{t+2}]\end{aligned}$$

Finally, we have

$$\hat{z}_{t+1} = A_1 \hat{z}_t + A_2 \hat{z}_{t-1} + C_1 X_{t+2} + C_2 X_{t+1}$$

or, exactly as in the motivation

$$\hat{z}_{t+1} = A_1 \hat{z}_t + A_2 \hat{z}_{t-1} + B_1 \hat{s}_{t+2} + B_2 \hat{s}_{t+1} + B_3 \hat{s}_t + B_4 d_{t+1} + B_5 d_t$$

where

$$\begin{aligned}A_1 = & - \tilde{Z}_1 (G_0^{11})^{-1} \left[ H_1^{11} Z_1' - G_0^{12} \left( \tilde{H}_1^{22} \right)^{-1} \tilde{Q}_2 \alpha_2 \right] - \tilde{Z}_2 \left( \tilde{H}_1^{22} \right)^{-1} \tilde{Q}_2 \alpha_2 \\ A_2 = & - \tilde{Z}_1 (G_0^{11})^{-1} \left[ Q_1 \alpha_2 - H_1^{12} \left( H_1^{22} \right)^{-1} Q_2 \alpha_2 \right] \\ B_1 = & C_1 \beta_0 \\ B_2 = & C_1 \beta_1 + C_2 \beta_0 \\ B_3 = & C_2 \beta_1 \\ B_4 = & C_1 \\ B_5 = & C_2 \\ C_1 = & \left[ \tilde{Z}_1 (G_0^{11})^{-1} G_0^{12} \left( \tilde{H}_1^{22} \right)^{-1} - \tilde{Z}_2 \left( \tilde{H}_1^{22} \right)^{-1} \right] \tilde{Q}_2 \\ C_2 = & \tilde{Z}_1 (G_0^{11})^{-1} \left\{ H_1^{12} \left( H_1^{22} \right)^{-1} Q_2 - Q_1 \right\}\end{aligned}$$

Now, it's simple to analyze what happens when the system changes from lying outside the ZLB to entering a liquidity trap, and back again. Before  $t_1 - 1$ , the period  $t + 1$  belongs to to the same system as period  $t$ . Therefore, we substitute  $\{Z, Q, H_1, H_0\}$  for every  $\{\tilde{Z}, \tilde{Q}, \tilde{H}_1, \tilde{H}_0\}$  in the above expressions, which gives the same formula for  $\hat{z}_{t+1}$  as would obtain if we solved after replacing (27) with

$$H_0 \gamma_{t+1} + H_1 \gamma_t + Q \alpha_2 \hat{z}_{t-1} + Q X_{t+1} = 0$$

reflecting that both the time  $t$  and time  $t + 1$  systems are outside the ZLB. At  $t_1 - 1$ , the notation applies perfectly and each regime should be respected. More clearly, when  $t = t_1 - 1$  we can only obtain  $\tilde{\gamma}_{t+1}^2$ , not  $\gamma_{t+1}^2$ , because the time  $t + 1$  system is in the ZLB,

and yet the time  $t$  system from which  $\tilde{\gamma}_{t+1}^1$  is derived is outside of the ZLB; therefore, we must incorporate  $\tilde{\gamma}_{t+1}$  into the time  $t$  system transformed using the QZ decomposition for  $\alpha_0$  and  $\alpha_1$ , not  $\tilde{\alpha}_1$ . From  $t_1$  until  $t_2 - 1$   $\{\tilde{Z}, \tilde{Q}, \tilde{H}_1, \tilde{H}_0\}$  should replace  $\{Z, Q, H_1, H_0\}$  since only  $\tilde{\text{sys}}_t$  applies, obtaining the same formula for  $\hat{z}_{t+1}$  as would be obtained if we replaced (27) with

$$\tilde{H}_0 \tilde{\gamma}_{t+1} + \tilde{H}_1 \tilde{\gamma}_t + \tilde{Q} \alpha_2 \hat{z}_{t-1} + \tilde{Q} X_{t+1} = 0$$

At  $t_2$ , all notation should be reversed - i.e.  $\{Z, Q, H_1, H_0\}$  should be exchanged with  $\{\tilde{Z}, \tilde{Q}, \tilde{H}_1, \tilde{H}_0\}$  and vice versa; this is the opposite case to that encountered at  $t = t_1 - 1$ . Finally, at  $t_2 + 1$  and beyond we again substitute  $\{Z, Q, H_1, H_0\}$  for  $\{\tilde{Z}, \tilde{Q}, \tilde{H}_1, \tilde{H}_0\}$ , as we did before  $t_1 - 1$ .

## 10.4 Computing the policy matrix for the stationary economy

Now we describe the derivation of  $A$  satisfying

$$\hat{z}_{t+1} = A \hat{z}_t$$

for all  $t \geq \max\{t_2, T\}$ .

Ignoring shocks - which will not enter into the MSV solution for  $Y_{t+1}$  we are seeking for  $t \geq \max\{t_2, T\}$  - we have

$$\begin{bmatrix} \alpha_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{z}_{t+1} \\ \hat{z}_t \end{bmatrix} + \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I & 0 \end{bmatrix} \begin{bmatrix} \hat{z}_t \\ \hat{z}_{t-1} \end{bmatrix} = 0$$

or

$$aY_{t+1} + bY_t = 0 \tag{28}$$

We will attempt to find a linear one-stage Markov solution of the form  $\hat{z}_{t+1} = A \hat{z}_t$  or  $DY_{t+1} = 0$ , also called a minimum state variable (MSV) solution. Define  $\ell$  such that  $m - \ell \equiv \text{rank}(a) < \text{length}(Y) \equiv m \equiv 2n$ . Note that matrices  $a$  and  $b$  are square. A QZ decomposition of  $a$  and  $b$  allows us to rewrite (28) as

$$\begin{aligned} QaZZ'Y_{t+1} + QbZZ'Y_t &= 0 \\ H_0\gamma_{t+1} + H_1\gamma_t &= 0 \end{aligned} \tag{29}$$

The bottom  $\ell$ -equations of this system are

$$H_0^{22}\gamma_{t+1}^2 + H_1^{22}\gamma_t^2 = 0$$

Remember, both  $H_0^{22}$  and  $H_1^{22}$  are upper triangular matrices.  $H_0^{22}$  is a matrix of zeros (since the rank of  $H_0$  must be  $m - \ell$  and we can reorder the rows appropriately while preserving the decomposition), while  $H_1^{22}$  must be invertible. Therefore,  $\gamma_t^2 = 0 \forall t$ , or  $Z_2'Y_t = 0$ . This condition provides  $\ell$  equations relating  $\hat{z}_{t+1}$  to  $\hat{z}_t$ , which means we have found part of the MSV solution. Recall the condition for an MSV solution is the existence of  $n \times 2n$  matrix  $D$  s.t.  $DY_{t+1} = 0$ . With the condition  $Z_2'Y_t = 0$  in hand, we may now write  $D = \begin{bmatrix} \tilde{D} \\ Z_2' \end{bmatrix}$ ,

where  $\tilde{D}$  is an  $n - \ell \times 2n$  matrix that we will now find.

The top  $m - \ell$  equations of (29) may be written

$$H_0^{11}\gamma_{t+1}^1 + H_1^{11}\gamma_t^1 = 0$$

Since  $H_0^{11}$  is invertible we have

$$\begin{aligned}\gamma_{t+1}^1 &= - (H_0^{11})^{-1} H_1^{11} \gamma_t^1 \\ \gamma_{t+1}^1 &= \Pi \gamma_t^1 \text{ where} \\ \Pi &\equiv - (H_0^{11})^{-1} H_1^{11}\end{aligned}$$

Therefore,

$$\begin{aligned}Z'Y_{t+1} &= \begin{bmatrix} \gamma_{t+1}^1 \\ \gamma_{t+1}^2 \end{bmatrix} = \begin{bmatrix} \Pi \gamma_t^1 \\ 0 \end{bmatrix} = \begin{bmatrix} \Pi \gamma_t^1 \\ 0 \end{bmatrix} \\ Y_{t+1} &= [ Z_1 \quad Z_2 ] \begin{bmatrix} \Pi \gamma_t^1 \\ 0 \end{bmatrix} = Z_1 \Pi Z_1' Y_t\end{aligned}$$

Next we use the diagonal decomposition of  $\Pi = P\Lambda P^{-1}$ , where  $\Lambda$  is a diagonal matrix containing the eigenvalues of matrix  $\Pi$  on the main diagonal, and  $P$  is the matrix of eigenvectors, to write

$$Y_{t+1} = Z_1 P^{-1} \Lambda P Z_1' Y_t \quad (30)$$

Recursively substituting forward, we have

$$Y_{t+k} = Z_1 P^{-1} \Lambda^k P Z_1' Y_t$$

For economic plausibility, we only consider “nonexplosive” solutions where  $\lim_{t \rightarrow \infty} Y_t = 0$ , i.e., a solution which eventually returns to steady state. In order to extinguish explosive paths, we require that for every eigenvalue  $j$  of  $\Pi$  greater than one we have  $P_j Z_1' Y_{t^*} = 0$  where  $P_j$  is the row of  $P$  corresponding to the eigenvector for eigenvalue  $j$ , and  $t^* = \max\{t_2 + 1, T + 1\}$ , the date at which the economy becomes stationary. We write  $(P, \Lambda)$  such that the first  $q$  rows of  $\Lambda$  correspond to explosive eigenvalues. Separating  $P$  into  $\begin{bmatrix} P_e \\ P_s \end{bmatrix}$ , where  $P_e$  equals the first  $q$  rows of  $P$ , we have the condition  $P_e Z_1' Y_{t^*} = 0$ . This condition in turn implies (via backwards recursion on (30) from any  $t + 1 \geq t^*$ ) that  $P_e Z_1' Y_{t+1} = 0$ . Also, in our example there are exactly  $q = n - l$  explosive eigenvalues. Thus,  $\tilde{D} = P_e Z_1'$  and

$$D = \begin{bmatrix} P_e Z_1' \\ Z_2' \end{bmatrix}$$

satisfies  $DY_{t+1} = 0$  at every date  $t + 1 \geq t^*$ .

Finally, separate  $D$  horizontally into two  $n \times n$  matrices,  $D = [ D_1 \quad D_2 ]$ , so that  $DY_{t+1} = D_1 z_{t+1} + D_2 z_t = 0$ . Then  $z_{t+1} = Az_t$  for  $A = -D_1^{-1} D_2$ , where we have assumed and verified that  $D_1$  is an invertible matrix.

## 10.5 Restriction on $z_1$

If a liquidity trap begins at time  $t > 2$ :

$$\begin{aligned}\gamma_1^2 &= - (H_1^{22})^{-1} Q_2 [X_2 + \alpha_2 \hat{z}_0] = - (H_1^{22})^{-1} Q_2 X_2 \\ Z_2' \hat{z}_1 &= - (H_1^{22})^{-1} Q_2 X_2\end{aligned}$$

Otherwise:

$$\tilde{Z}_2' \hat{z}_1 = - (\tilde{H}_1^{22})^{-1} \tilde{Q}_2 X_2$$