Student Seminar in Pure Mathematics

Homological Algebra

William Roberts Ballard
Céline Berthaud
Milo Nicolas Jacques Blum
Bryan Briod
Eric Yen-Yo Chen
Virgile Simon Constantin
Benoît Juan Cuenot
Fabien Donnet-Monay
Mathis Duguin
Haotian Lyu
Héloïse Camille Charlotte Mansat
Jorge Martín Álvarez
Claudio Pfammatter
Alexandre Anselme Antonio Pons
Deborah Righi
Runchi Tan
Dév Raj Joe Vorburger
Dimitri Wyss
Qian Yao
Zichun Zhou

$$\begin{tikzcd}
 & & & C \arrow[dl, "x"] \arrow[ddr, "y"] & \\
 & & A' \arrow[dll, "i"'] \arrow[ddd, "(Tj)i"] & & \\
B' \arrow[uurrr, "v"] \arrow[ddrr, "j"'] & & & & B' \arrow[ull, "\exists g"'] \arrow[dl, dashed, "\delta"'] \\
 & & & A \arrow[ulll, dashed, "u"] \arrow[uuu, dashed, "vu"] & \\
 & & C' \arrow[ur, dashed, "\partial"] \arrow[uurr, "\exists f"'] & &
\end{tikzcd}$$

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Chapter 0. Preface

The present manuscript grew out of the student seminar in pure mathematics at EPFL, organized by Dimitri Wyss as the main lecturer and Eric Yen-Yo Chen as the teaching assistant, which took place in the fall semester of 2024. The goal was to get familiar with the basics of homological algebra following the Weibel's book[1].

The seminar consisted of 2 hours of lectures and 2 hours of exercises per week for 14 weeks. The lectures, notes, and solutions were given and written by the students participating in the seminar course. Each chapter ends with a few exercises, the solutions of which can be found at the end of the notes.

The goal of the course was to introduce the derived category of an abelian category and to give an idea of why this might be a useful concept. To this end, we discuss the composition of derived functors and the total tensor product. In order to get there in time, we made the choice of discussing only a few applications along the way, mostly related to the functors Hom and Ext.

We finish by taking full responsibility for the possible typos and errors that you may find in these notes.

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[1] Weibel CA. *An Introduction to Homological Algebra.* Cambridge University Press; 1994.

Chapter 1. Abelian Categories

Abelian categories generalize some very useful features of the category of abelian groups. In particular, morphisms in abelian categories can be added and possess the notions of kernel and cokernel with the desired properties. Thanks to these properties, the discipline of homological algebra becomes a powerful tool, with direct applications in areas such as topology or algebraic geometry.
Intuitively, abelian categories resemble the category of abelian groups **Ab**. Apart from **Ab** itself, some examples of abelian categories include the category R-**Mod** of left (equivalently right) modules over a given ring R, the category $\Bbbk$-**Vect** of vector spaces over a field $\Bbbk$ or the category $\mathbf{Ab}_{\text{fin}}$ of finitely generated abelian groups.
We will need some preliminary definitions before getting to the the concept of abelian category.

1.1 Definitions and basic properties

Definition 1.1. A category $\mathcal{A}$ is called a *preadditive category* or **Ab**-*category* if, for every pair of objects A and B in $\mathcal{A}$, the hom-set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ has the structure of an abelian group where composition distributes over addition. This means, given a diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

then $f(g + g')h = fgh + fg'h$.

Notice that, in an **Ab**-category, the hom-set $\mathrm{Hom}_{\mathcal{A}}(A, A)$ has the structure of an associative ring.

Definition 1.2. Given two **Ab**-categories $\mathcal{A}, \mathcal{B}$, a functor $F : \mathcal{B} \to \mathcal{A}$ is called an *additive functor* if, for each pair of objects B, B' in $\mathcal{B}$, the map

$$\mathrm{Hom}_{\mathcal{B}}(B, B') \longrightarrow \mathrm{Hom}_{\mathcal{A}}(F(B), F(B'))$$

is a group homomorphism, i.e. $F(f + f') = F(f) + F(f')$ for $f, f' : B \to B'$.

Definition 1.3. An **Ab**-category $\mathcal{A}$ is called an *additive category* if it has an initial object 0 and a product $A \times B$ for each pair of objects A, B in $\mathcal{A}$.

Proposition 1.4. *In an additive category, the coproduct of any two objects exists and is isomorphic to their product.*

Proof. Let $\mathcal{A}$ be an additive category and A, B two objects in $\mathcal{A}$. Consider the product $A \times B$ and the maps

$$\begin{tikzcd} A \arrow[r, "\alpha"] & A \times B & B \arrow[l, "\beta"'] \end{tikzcd}$$

given by $\alpha = \mathrm{id}_A \times 0$ and $\beta = 0 \times \mathrm{id}_B$. Let us see that this defines a coproduct. Given an object C and maps $f : A \to C$ and $g : B \to C$, we seek for a unique $h : A \times B \to C$ such that $h\alpha = f$ and $h\beta = g$. Set $h = f \circ \pi_A + g \circ \pi_B$. Then, one checks that $h\alpha = (f\pi_A + g\pi_B)(\mathrm{id}_A \times 0) = f\mathrm{id}_A + g0 = f$, and equivalently $h\beta = g$. Uniqueness follows by observing that if $f = \mathrm{id}_A$, one needs $h\alpha = \mathrm{id}_A$, and similarly with $g = \mathrm{id}_B$. Thus, $A \times B$ satisfies the universal property of coproducts. □

Using induction, the statement follows for any finite coproduct.

Definition 1.5. An *abelian category* is an additive category $\mathcal{A}$ satisfying the following properties:

(i) Every map in $\mathcal{A}$ has a kernel and a cokernel.

(ii) Every monic in $\mathcal{A}$ is the kernel of its cokernel.

(iii) Every epi in $\mathcal{A}$ is the cokernel of its kernel.

Remark 1.6. In the abelian categories mentioned at the start of the section, the kernel and cokernel of a given map are the usual ones in group, module or vector space homomorphisms, respectively. However, we notice that **Groups** is not an abelian category: it suffices to take the inclusion map $H \hookrightarrow G$ of a non-normal subgroup H of a group G. This is a monic map, but it can never be a kernel, as kernels are normal subgroups.

In particular, in an abelian category monics are kernels and epis are cokernels. Next, we can also define the notion of image of a morphism, by replicating the construction in abelian groups. In this way, the resulting object will match the image set in the case that the corresponding objects contain underlying sets.
Given a morphism $f : A \to B$, consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, dashed] & B \arrow[r] & \operatorname{coker} f \\
& \ker(\operatorname{coker} f) \arrow[u] &
\end{tikzcd}$$

where the arrows $B \to \operatorname{coker} f$ and $\ker(\operatorname{coker} f) \to B$ are given by definition of kernel and cokernel. By the universal property of the kernel, there exists a unique $A \to \ker(\operatorname{coker} f)$. In abelian groups, given that $\operatorname{coker} f \cong B/\operatorname{im} f$, this is exactly the codomain restriction of f to its image $\operatorname{im} f$. Hence, we define the object

$$\operatorname{im} f := \ker(\operatorname{coker} f).$$

The next result has been proved in Exercise 1(c).

Proposition 1.7. *The map $A \to \operatorname{im} f$ is epi.*

Once we have defined the image, the notion of exactness arises.

Definition 1.8. A sequence of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ is called *exact* (at B) if $g \circ f = 0$ and the canonical morphism $\operatorname{im} f \to \ker g$ is an isomorphism.

Definition 1.9. A subcategory $\mathcal{B} \subseteq \mathcal{A}$ is called an *abelian subcategory* if

(i) $\mathcal{B}$ is abelian.

(ii) Every exact sequence in $\mathcal{B}$ is also exact in $\mathcal{A}$.

Definition 1.10. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$ and $\mathcal{B}$ is said to be *left-exact* (resp. *right-exact*) if, for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{A}$, the sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

(resp. the sequence $F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$) is exact in $\mathcal{B}$.
A functor is called *exact* if it is simultaneously left- and right-exact.

Definition 1.11. A contravariant additive functor $F : \mathcal{A} \to \mathcal{B}$ is *left-exact* (resp. *right-exact*) if $F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ is left-exact (resp. right).

Let us now see a couple of non-trivial examples of abelian categories.

Example 1.12.

(i) Let R be a commutative ring, and consider the subcategory R-$\mathbf{Mod}_{\mathrm{fin}}$ of finitely generated R-modules. This category is abelian if and only if R is a Noetherian ring. Indeed, if R is Noetherian, the kernel and image of a morphism are finitely generated submodules of the corresponding modules, and thus lie in the category. This need not hold if R is not Noetherian, e.g. $R = K[X_1, X_2, \dots]$ for some field K. For instance, the kernel of the morphism $K[X_1, X_2, \dots] \to K$ mapping each $X_i \mapsto 0$ is the submodule $(X_1, X_2, \dots)$, which is not finitely generated.

(ii) Given a category I and an abelian category $\mathcal{A}$, the functor category $\mathcal{A}^I$ is also an abelian category.

The next result presents characterizations of additive and abelian categories. Its proof is left to the reader

Proposition 1.13. *Let $\mathcal{C}$ be a full subcategory of an abelian category $\mathcal{A}$. Then:*

(i) $\mathcal{C}$ is additive if and only if 0 is in $\mathcal{C}$ and $\mathcal{C}$ is closed under products.

(ii) $\mathcal{C}$ is abelian and $\mathcal{C} \hookrightarrow A$ is exact if and only if $\mathcal{C}$ is additive and closed under kernels and cokernels.

Finally, we consider the following important example of a left-exact functor.

Proposition 1.14. *Let $\mathcal{A}$ be an abelian category and M an object in $\mathcal{A}$. Then $\operatorname{Hom}_{\mathcal{A}}(M,-) : \mathcal{A} \to \mathbf{Ab}$ is a left-exact functor.*

Proof. Consider an exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ in $\mathcal{A}$. Let us check that

$$0 \longrightarrow \operatorname{Hom}(M,A) \xrightarrow{\alpha_*} \operatorname{Hom}(M,B) \xrightarrow{\beta_*} \operatorname{Hom}(M,C)$$

is also exact. First, at $\operatorname{Hom}(M,A)$, consider $\gamma \in \operatorname{Hom}(M,A)$ such that $\alpha_*\gamma = \alpha \circ \gamma = 0$. Then, as α is monic by exactness, $\gamma = 0$. Thus, α_* is also monic.
Then, at $\operatorname{Hom}(M,B)$, it is clear that $(\beta_* \circ \alpha_*)(\gamma) = \beta \circ \alpha \circ \gamma$ for any $\gamma \in \operatorname{Hom}(M,A)$. It remains to check that $\ker \beta_* \subseteq \operatorname{im} \alpha_*$ as subgroups of $\operatorname{Hom}(M,B)$. For that, take $\delta \in \operatorname{Hom}(M,B)$ such that $\beta_*\delta = \beta \circ \delta = 0$. Then, since $A \cong \ker(\beta)$ by exactness of the original sequence, the universal property of kernels implies that there exists a (unique) $\gamma \in \operatorname{Hom}(M,A)$ such that $\alpha \circ \gamma = \alpha_*\gamma = \delta$, as in the diagram.

$$\begin{tikzcd}
 & A \cong \ker\beta \arrow[d, "\alpha"] & \\
M \arrow[ru, dashed, "\gamma"] \arrow[r, "\delta"] & B \arrow[r, "\beta"] & C
\end{tikzcd}$$

Thus, $\delta \in \operatorname{im} \alpha_*$ as we wished. ☐

Corollary 1.15. *The functor $\operatorname{Hom}_{\mathcal{A}}(-,M)$ is a left-exact contravariant functor.*

Proof. It suffices to notice that $\operatorname{Hom}_{\mathcal{A}}(A,M) = \operatorname{Hom}_{\mathcal{A}^{\mathrm{op}}}(M,A)$ for any A in $\mathcal{A}$, as arrows are reversed in $\mathcal{A}^{\mathrm{op}}$. The statement then follows from the previous proposition. ☐

1.2 Freyd-Mitchell embedding theorem

The following result provides an inclusion of any abelian category into a category of modules over a certain ring, preserving the underlying categorical structure. This is a very powerful result, as it allows to regard objects in abelian categories as modules, thus possessing the numerous well-known algebraic properties of these.
For its statement, recall that a category is called *small* if the class of all its objects is a set. Furthermore, in the next pages we will work with the *Yoneda embedding* $h : \mathcal{A} \to \mathbf{Ab}^{\mathcal{A}}$, defined on any abelian category $\mathcal{A}$. This is given by $h(A) = \operatorname{Hom}_A(-,A)$ for each object A in $\mathcal{A}$.

Theorem 1.16. *Let $\mathcal{A}$ be a small abelian category. Then, there exists a ring R and an exactly fully faithful functor $i : \mathcal{A} \hookrightarrow R$-**Mod**.*

Proof (sketch). The Yoneda embedding $h : \mathcal{A} \to \mathbf{Ab}^{\mathcal{A}}$ is a left-exact functor, but not right-exact in general. It factors through the category $\mathcal{L}$ of left-exact functors $\mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$. This is an abelian category, and the functor $\mathcal{A} \to \mathcal{L}$ is actually exact.
It can be shown that $\mathcal{L}$ is equivalent to R-**Mod** for $R := \operatorname{Hom}_{\mathcal{L}}(P,P)$, where P is some special object. ☐

Example 1.17. Consider $\mathcal{A}$ to be the category of $\mathbb{Z}$-graded R-modules for a ring R. Then A injects into the category of $\prod_{i\in\mathbb{Z}} R$-modules of the form $\bigoplus_{i\in\mathbb{Z}} M_i$.

1.3 The Yoneda lemma

The *Yoneda lemma* is one fundamental result in category theory, providing a representation of any small category $\mathcal{C}$ via functors from $\mathcal{C}$ to **Set**. Here, we state a special instance of the main result.

Theorem 1.18. *The Yoneda embedding* $h : \mathcal{A} \to \mathbf{Ab}^{\mathcal{A}}$ *reflects exactness, i.e. a sequence* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ *is exact in* $\mathcal{A}$ *if for all* M *in* $\mathcal{A}$*, the sequence*

$$\operatorname{Hom}_{\mathcal{A}}(M, A) \xrightarrow{\alpha_*} \operatorname{Hom}_{\mathcal{A}}(M, B) \xrightarrow{\beta_*} \operatorname{Hom}_{\mathcal{A}}(M, C)$$

is exact.

Proof. Consider a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathcal{A}$. First, we may take $M = A$ in the hypothesis. By exactness of the sequence in $\mathbf{Ab}^{\mathcal{A}}$, we have

$$0 = (\beta_* \circ \alpha_*)(\mathrm{id}_A) = \beta \circ \alpha \circ \mathrm{id}_A = \beta \circ \alpha.$$

It remains to show that the universal map between $\operatorname{im} \alpha$ and $\ker \beta$ is an isomorphism. In this case, pick $M = \ker \beta$ with $i : \ker \beta \to B$. Since $\beta \circ i = \beta_* i = 0$, then $i \in \operatorname{im} \alpha_*$, i.e. there exists $\gamma : \ker \beta \to A$ such that $i = \alpha_* \gamma = \alpha \circ \gamma$. Next, let us consider the construction of $\operatorname{im} \alpha$ as in the definition, giving rise to arrows $A \xrightarrow{\alpha'} \operatorname{im} \alpha \xrightarrow{j} B$. Given that $\beta \circ j \circ \alpha' = \beta \circ \alpha = 0$ and α' is epi, then $\beta \circ j = 0$ as well. Hence, by the universal property of the kernel, there exists $\delta : \operatorname{im} \alpha \to \ker \beta$ such that $j = i \circ \delta$. This is displayed in the following diagram.

$$\begin{tikzcd}
& \ker \beta \arrow[dl, dashed, "\gamma"'] \arrow[d, "i"] & \\
A \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & B \arrow[r, "\beta"] & C \\
& \operatorname{im} \alpha \arrow[u, "j"'] \arrow[uu, dashed, bend right=80, "\delta"'] &
\end{tikzcd}$$

Finally, notice that $j\alpha'\gamma\delta = \alpha\gamma\delta = i\delta = j$. Since j is monic, this implies $\alpha'\gamma\delta = \mathrm{id}_{\operatorname{im} \alpha}$. A parallel argument leads to $\delta\alpha'\gamma = \mathrm{id}_{\ker \beta}$, and we deduce that δ is an isomorphism, as we wanted to show. ☐

Remark 1.19. The converse of the Yoneda lemma is not true in general. For example, consider the exact sequence $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ in the category of abelian groups. Taking $M = \mathbb{Z}/2\mathbb{Z}$, one gets the following sequence between hom-sets

$$\operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi_*} \operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

However, π_* cannot be surjective because $\operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ is the trivial group while $\operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Thus note that, in Proposition 1.14, we need the injectivity of α for the resulting sequence to be exact at the middle position.

This allows to state the following result between adjoint functors, proved in Exercise 3. Recall the definition of an adjoint pair of functors: given categories $\mathcal{A}$ and $\mathcal{B}$, a pair of functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ is said to be adjoint if there is a natural isomorphism $\operatorname{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{A}}(A, R(B))$.

Corollary 1.20. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be abelian categories, and let* $L : \mathcal{A} \to \mathcal{B}$ *and* $R : \mathcal{B} \to \mathcal{A}$ *be an adjoint pair of functors. Then* L *is right-exact and* L *is left-exact.*

1.4 Warm up Exercises Week 1

The reader may want to consult the Appendix of Weibel's book for a reminder on general category theory.

(i) (a) Show that $\mathbb{Z} \subset \mathbb{Q}$ is epi in the category of rings and that $\mathbb{Q} \subset \mathbb{R}$ is epi in the category of Hausdorff topological spaces.

    (b) Show that in the category of groups, monics are just injective set maps and kernels are monics whose image is a normal subgroup.

(ii) Show that the functor category $\mathcal{A}^I$ is a category whenever I is small. In that case, show that the *Yoneda embedding*

$$h : I \to \mathbf{Sets}^{I^{op}}, \qquad h_i(j) = \mathrm{Hom}_I(j, i)$$

is fully faithful.

(iii) Let I and $\mathcal{A}$ be categories and assume that every functor $F : I \to \mathcal{A}$ has a limit. Assuming that I is small, show that

$$\lim : \mathcal{A}^I \to \mathcal{A}, \quad F \mapsto \lim_{i \in I} F_i$$

is a functor and that the universal property of $\lim_{i \in I} F_i$ for all F is equivalent to the assertion that lim is right adjoint to the diagonal functor $\Delta : \mathcal{A} \to \mathcal{A}^I$. Dually, show that colim is left adjoint to Δ.

(iv) Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be two functors and assume that there are natural transformations $\eta : \mathrm{id}_{\mathcal{A}} \Rightarrow RL$ and $\epsilon : LR \Rightarrow \mathrm{id}_{\mathcal{B}}$ such that the composites

$$L(A) \xrightarrow{L(\eta_A)} LRL(A) \xrightarrow{\epsilon_{L(A)}} L(A) \text{ and } R(B) \xrightarrow{\eta_{R(B)}} RLR(B) \xrightarrow{R(\epsilon_B)} R(B),$$

are the identity for all objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Show that (L, R) is an adjoint pair of functors.

1.5 Exercises Week 2

(i) Prove the following statements in an abelian category without using the Freyd-Mitchell embedding theorem:

    (a) Assume $0 \to A$ is a kernel for $f : A \to B$. Then f is monic. Dually if $B \to 0$ is a cokernel of f, then f is epi.

    (b) A morphism $f : A \to B$ that is both monic and epi is an isomorphism.

    (c) For any morphism $f : A \to B$ the induced morphism $A \to \mathrm{im}(f)$ is epi.

(ii) Given a category I and an abelian category $\mathcal{A}$, show that the functor category $\mathcal{A}^I$ is also an abelian category and that the kernel of $\eta : B \to C$ is the functor $i \mapsto \ker(\eta(i))$.

(iii) Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be an adjoint pair of additive functors. That is, there is a natural isomorphism

$$\tau : \mathrm{Hom}_{\mathcal{A}}(L(A), B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(A, R(B)).$$

Then L is right exact, and R is left exact.

(iv) (5-Lemma) Consider the following commutative diagram in an abelian category $\mathcal{A}$:

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "a"] & B' \arrow[r] \arrow[d, "b"] & C' \arrow[r] \arrow[d, "c"] & D' \arrow[r] \arrow[d, "d"] & E' \arrow[d, "e"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E.
\end{tikzcd}$$

Show that if b and d are monic and a is an epi, then c is monic. Dually, show that if b and d are epis and e is monic, then c is an epi. In particular, if a, b, d and e are isomorphisms, then so is c.

Hint: You may use the Freyd-Mitchell embedding theorem

Chapter 2. Chain Complexes

We present below the formalism of chain complexes in a general abelian category, similar to the usual theory of chain complexes for R-modules. We fix throughout an abelian category $\mathcal{A}$.

Definition 2.1. A chain complex $C_\bullet$ in $\mathcal{A}$ is a family of objects $\{C_n\}_{n\in\mathbb{Z}}$ equipped with morphisms $d_n : C_n \to C_{n-1}$ (called differentials) such that $d_n \circ d_{n+1} = 0 \ \forall n \in \mathbb{Z}$.
We define the n-cycles and n-boundaries respectively by $Z_n(C_\bullet) = \ker(d_n)$, $B_n(C_\bullet) = \operatorname{im}(d_{n+1})$.

By abuse of notation we will often write the d_n in the above definition as d, so that the condition $d_n \circ d_{n+1} = 0$ becomes $d^2 = 0$.

Definition 2.2. We say $C_\bullet$ is bounded above if $\exists N \in \mathbb{Z}$ such that $C_n = 0 \ \forall n > N$. One similarly defines boundedness below. If $C_\bullet$ is bounded below by a and above by b, we say it has amplitude in $[a, b]$

Definition 2.3. A morphism of chain complexes $u_\bullet : C_\bullet, D_\bullet$ is a collection of morphisms $\{u_n : C_n \to D_n\}_{n\in\mathbb{Z}}$ satisfying appropriate compatibility conditions, namely such that the following diagram commutes $\forall n \in \mathbb{Z}$

$$\begin{tikzcd}
C_n \arrow[r, "d_n"] \arrow[d, "u_n"'] & C_{n-1} \arrow[d, "u_{n-1}"] \\
D_n \arrow[r, "\delta_n"'] & D_{n-1}
\end{tikzcd}$$

here we have noted the differentials of $C_\bullet$ by d, and those of $D_\bullet$ by δ.

One can easily verify that this defines a category of chain complexes over $\mathcal{A}$, which we denote by $\mathrm{Ch}(\mathcal{A})$.

2.1 Homology

Having defined chain complexes, the next natural step is to define the most important operation on these: homology functors.
Given a chain complex $C_\bullet$ with differentials d, note that $d_{n+1} : C_{n+1} \to C_n$ factors as $C_{n+1} \xrightarrow{e} \operatorname{im}(d_{n+1}) \xrightarrow{m} C_n$ for e, m epi and monic respectively. We thus obtain

$$0 = d_n \circ d_{n+1} = d_n \circ m \circ e \implies d_n \circ m = 0$$

By the universal property of kernels, m thus factors through $\ker(d_n)$.

Definition 2.4. Let $C_\bullet \in \mathrm{Ch}(\mathcal{A})$. The n-th homology of $C_\bullet$ is

$$H_n(C_\bullet) = \operatorname{coker}(\operatorname{im}(d_{n+1}) \to \ker(d_n))$$

with the implicit morphism a factor of m as above.

Remark 2.5. $H_n(C_\bullet) = 0$ iff C is exact at n (this is easy for R-modules, and one can then use Freyd-Mitchell).

We have yet to show that homology defines a collection of functors.

Proposition 2.6. $H_n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ *is a functor*

Proof. Consider a morphism $u_\bullet : C_\bullet \to D_\bullet$. Denote by d, δ the differentials of $C_\bullet, D_\bullet$ respectively. Let ι_n denote the natural morphisms $\iota_n : \ker(d_n) \to D_n \ \forall n \in \mathbb{Z}$. Now we have $\forall n \in \mathbb{Z}$, $\delta_n \circ u_n \circ \iota_n = u_{n-1} \circ d_n \circ \iota_n = 0$ so by the universal property of kernels we get a diagram

$$\begin{tikzcd}
\ker(d_n) \arrow[r] \arrow[d] & \ker(\delta_n) \arrow[d] \\
H_n(C_\bullet) & H_n(D_\bullet)
\end{tikzcd}$$

In R-**mod**, the top map is $u_n|_{\ker(d_n)}$ and the diagram can be completed with a unique induced morphism $\theta_n : H_n(C_\bullet) \to H_n(D_\bullet)$ down below making the diagram commute. By Freyd-Mitchell we obtain such a morphism in the general case. If we thus define $H_n(C_\bullet)$ on morphisms by $H_n(u_\bullet) = \theta_n$, it is then easy to verify functoriality by the uniqueness property of θ_n. ☐

Since we will usually be interested in maps preserving homology, it is worth defining a new notion of isomorphism.

Definition 2.7. A quasi-isomorphism of chain complexes over $\mathcal{A}$ is a morphism of chain complexes over $\mathcal{A}$, $u_\bullet : C_\bullet \to D_\bullet$, such that $H_n(u_\bullet)$ is an isomorphism $\forall n \in \mathbb{Z}$

Example 2.8. (i) An isomorphism is clearly a quasi-isomorphism

(ii) The converse is not true. A typical counterexample comes from the following morphism in Ch($\mathbb{Z}$-**mod**)

$$\begin{tikzcd}
C_2 = 0 \arrow[r] & C_1 = \mathbb{Z} \arrow[r] \arrow[d] & C_0 = \mathbb{Z} \arrow[r] \arrow[d] & C_{-1} = 0 \\
D_2 = 0 \arrow[r] & D_1 = 0 \arrow[r] & D_0 = \mathbb{Z}/2\mathbb{Z} \arrow[r] & D_{-1} = 0
\end{tikzcd}$$

Here the map $C_0 \to D_0$ is just the usual quotient. One easily shows that this defines a quasi-isomorphism.

Having defined homology, the next natural step is to define cohomology.

Definition 2.9. A cochain complex in $\mathcal{A}$ is a family of objects $\{C^n\}_{n\in\mathbb{Z}}$ with a collection of morphisms $d^n : C^n \to C^{n+1}$ satisfying $d^{n+1} \circ d^n = 0 \ \forall n \in \mathbb{Z}$. One defines cocycles, coboundaries, and cohomology in a dual way to all the definitions built on chain complexes.

2.2 Ch($\mathcal{A}$) as an abelian category

It is natural to expect that Ch($\mathcal{A}$) should have the structure of an abelian category. To show this let us first define kernels and cokernels appropriately.

Lemma 2.10. *In* Ch($\mathcal{A}$)*, the kernel of* $f_\bullet : C_\bullet \to D_\bullet$ *is given by the chain complex* $\ker(f_\bullet)$ *with differentials induced by the universal properties of the kernels* $\ker(f_n)$ *(this will be elaborated on in the proof)*

Proof. Consider $f_\bullet : C_\bullet \to D_\bullet$ and the kernels $k_n : \ker(f_n) \to C_n$. Consider the following diagram

$$\begin{tikzcd}
 & & C_n \arrow[dd, "f_n"] \\
\ker(f_{n+1}) \arrow[urr, bend left, "d_{n+1}\circ k_{n+1}"] \arrow[drr, bend right] \arrow[r, dashed, "\exists! e_{n+1}"] & \ker(f_n) \arrow[ur] \arrow[dr, "0"] & \\
 & & D_n
\end{tikzcd}$$

where the bottom arrow is simply defined by the composition $f_n \circ d_{n+1} \circ k_{n+1}$, so as to make the diagram without e_{n+1} commute. Since

$$f_n \circ d_{n+1} \circ k_{n+1} = \delta_{n+1} \circ f_{n+1} \circ k_{n+1} = 0,$$

we are guaranteed the existence and uniqueness of e_{n+1} by the universal property of kernels. By the diagram, we have $k_n \circ e_{n+1} = d_n \circ k_{n+1}$ so $k_\bullet : \ker(f_\bullet) \to C_\bullet$ defines a morphism. It remains to show it satisfies the universal property of kernels.
Let let $K'_\bullet \in$ Ch($\mathcal{A}$) and $j_\bullet : K'_\bullet \to C_\bullet$ be such that $f_\bullet \circ j_\bullet = 0$. At level n this gives $f_n \circ j_n = 0$ and thus by the universal property of kernels we have morphisms $u_n : K'_n \to \ker(f_n)$ with $k_n \circ u_n = j_n$. We just need to show these define a morphism of chain complexes. We have the diagram

$$\begin{tikzcd}
K'_{n+1} \arrow[rrr, "g_{n+1}"] \arrow[dd, "j_{n+1}"'] \arrow[dr, "u_{n+1}"] & & & K'_n \arrow[dd, "j_n"] \arrow[dl, "u_n"'] \\
& \ker(f_{n+1}) \arrow[r, "e_{n+1}"] \arrow[dl, "k_{n+1}"'] & \ker(f_n) \arrow[dr, "k_n"] & \\
C_{n+1} \arrow[rrr, "d_{n+1}"] & & & C_n
\end{tikzcd}$$

It commutes along the bottom trapezoid and the two triangles. We wish to show the top trapezoid commutes. We have

$$\begin{aligned} d_{n+1} \circ j_{n+1} = d_{n+1} \circ k_{n+1} \circ u_{n+1} &= k_n \circ e_{n+1} \circ u_{n+1} \\ = j_n \circ g_{n+1} &= k_n \circ u_n \circ g_{n+1} \end{aligned}$$

Since k_n is monic (it is a kernel), we get $e_{n+1} \circ u_{n+1} = u_n \circ g_{n+1}$ as required ☐

Example 2.11. (i) If $\iota_\bullet : C_\bullet \to D_\bullet$ is injective (ie $\ker \iota_\bullet = 0$) we define

$$D_\bullet / C_\bullet = \operatorname{coker}(C_\bullet \to D_\bullet)$$

(ii) In R-**mod**, ie $C_\bullet \subset D_\bullet$ is a subcomplex, then D_n/C_n agrees with the usual quotient.

The next step in showing that $\mathrm{Ch}(\mathcal{A})$ is an abelian category is to show that it is additive and that it is Ab. We take as fact the following (it is easily shown):

Remark 2.12. In $\mathrm{Ch}(\mathcal{A})$ there is a zero object $\{0_\bullet : C_\bullet \to D_\bullet\}_{n\in\mathbb{Z}}$ and finite products $\{\prod_\alpha A_{\alpha,n}\}_{n\in\mathbb{Z}}$ with differentials

$$\prod_\alpha d_{\alpha,n} : \prod_\alpha A_{\alpha,n} \to \prod_\alpha A_{\alpha,n-1}.$$

Additionally, each hom set has the structure of an abelian group given by $f_\bullet + g_\bullet = \{f_n + g_n\}_{n\in\mathbb{Z}}$ for $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ two morphisms, and inversion similarly componentwise defined.

Theorem 2.13. *The category* $\mathrm{Ch}(\mathcal{A})$ *is an abelian category*

Proof. We have already seen that $\mathrm{Ch}(\mathcal{A})$ is additive and Ab. We will show that that every monic is the ker of its coker. Showing every epic is the coker of its ker is similar.

Let $f_\bullet : B_\bullet \to C_\bullet$ be monic. We have a sequence $\ker(f_\bullet) \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{f_\bullet} C_\bullet$. By definition $f_\bullet \circ \iota_\bullet = 0$, so we obtain $\iota_\bullet = 0$. Thus $\iota_\bullet$ factors through the 0 map $0_\bullet \to B_\bullet$. We also have that the 0 map factors through $\iota_\bullet$, and thus by uniqueness of kernels we deduce that $\ker(f_\bullet) = 0_\bullet$. Now in abelian categories, $\ker(f) = 0 \iff f$ is monic. We deduce that f_n is monic $\forall n \in \mathbb{Z}$. Now let $q_\bullet : C_\bullet \to \operatorname{coker}(f_\bullet)$ be the natural morphism and $g_\bullet : D_\bullet \to C_\bullet$ be a morphism such that $q_\bullet \circ g_\bullet$ is zero. Then this also holds at level n and f_n is the ker of its coker so we get a diagram

$$\begin{tikzcd}
B_n \arrow[r, "f_n"] & C_n \arrow[r, "q_n"] & \operatorname{coker}(f_n) \\
D_n \arrow[u, dashed, "\exists h_n"] \arrow[ur, "g_n"'] & &
\end{tikzcd}$$

and h_n defines a morphism of chain complexes, so that $f_\bullet$ satisfies the universal property of $\ker(\operatorname{coker}(f_\bullet))$ as required. ☐

2.3 Double complexes

The following objects will be of use when studying spectral sequences.

Definition 2.14. A double complex in $\mathcal{A}$ is a family $\{C_{p,q}\}_{(p,q)\in\mathbb{Z}^2}$ of objects in $\mathcal{A}$ with maps $d^h_{p,q} : C_{p,q} \to C_{p-1,q}$ and $d^v_{p,q} : C_{p,q} \to C_{p,q-1}$ satisfying

i) $d^h \circ d^h = 0$

ii) $d^v \circ d^v = 0$

iii) $d^v \circ d^h + d^h \circ d^v = 0$

Note that by the last condition, the squares in the following diagram usually don't commute

$$\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} & C_{p-1,q+1} \arrow[l] \arrow[d, "d^v"] & C_{p,q+1} \arrow[l, "d^h"] \arrow[d, "d^v"] & C_{p+1,q+1} \arrow[l, "d^h"] \arrow[d, "d^v"] & {} \arrow[l] \\
{} & C_{p-1,q} \arrow[l] \arrow[d, "d^v"] & C_{p,q} \arrow[l, "d^h"] \arrow[d, "d^v"] & C_{p+1,q} \arrow[l, "d^h"] \arrow[d, "d^v"] & {} \arrow[l] \\
{} & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "d^h"] \arrow[d] & C_{p+1,q-1} \arrow[l, "d^h"] \arrow[d] & {} \arrow[l] \\
& {} & {} & {} &
\end{tikzcd}$$

We say the double complex is bounded if it only has finitely many non-zero objects

Remark 2.15. Although we usually write chain complexes from left to right, we have noted them from right to left here to label a double complex with $\mathbb{Z}^2$ in the obvious way

Remark 2.16. Although a double chain complex generally is not a chain complex of chain complexes, we would like to identify it with one. An object of the category $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$ is a complex $\to C_{\bullet,q} \to C_{\bullet,q-1} \to$. Given a double complex as above, define maps $f_{p,q} = (-1)^p d^v_{p,q}$: these define chain morphisms between the horizontal complexes, and it is easy to see that it defines an object of the category $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$.

One can build a chain complex from a double complex by "collapsing along the diagonal" in two different ways

Definition 2.17. Given a double complex $C_{\bullet,\bullet}$, its total complexes are the collections of objects

$$T_n^{\prod} = \prod_{p+q=n} C_{p,q}, \quad T_n^{\oplus} = \bigoplus_{p+q=n} C_{p,q}$$

Note this may not exist if $C_{\bullet,\bullet}$ is not bounded

We have not yet defined a chain complex as we need to specify differentials. We basically do this componentwise and apply the appropriate universal properties. Note that from a component $C_{p,q}$, there are two maps to objects of lower degree, d^h and d^v.

Proposition 2.18. *$T_\bullet^{\prod}$ and $T_\bullet^{\oplus}$, when they exist, naturally form chain complexes with differentials $d^h + d^v$*

Proof. We show how to properly construct these differentials for $T_\bullet^{\prod}$, the proof for $T_\bullet^{\oplus}$ is analogous (but using coproduct properties instead).
Let $n \in \mathbb{Z}$. For each pair $(p, q) \in \mathbb{Z}^2$ with $p+q = n$, we have a projection map $\pi_{p,q} : T_n^{\prod} \to C_{p,q}$. Composing with $d^h_{p,q}$ and $d^v_{p,q}$ gives us maps

$$\tilde{d}^h_{p,q} : T_n \to C_{p-1,q}, \quad \tilde{d}^v_{p,q} : T_n \to C_{p,q-1}$$

By the universal property of the product we thus get maps $d^h_n, d^v_n : T_n \to T_{n-1}$. We define the differential $\partial_n : T_n \to T_{n-1}$ by $\partial_n = d^h_n + d^v_n$ (addition taken in $\mathrm{Hom}_{\mathcal{A}}(T_n, T_{n-1})$). To check that this defines a chain complex, we have (since composition is bilinear in Ab categories)

$$\partial_{n-1} \circ \partial_n = d^h_{n-1} \circ d^h_n + d^v_{n-1} \circ d^h_n + d^h_{n-1} \circ d^v_n + d^v_{n-1} \circ d^v_n$$

One quickly verifies that $d^h_{n-1} \circ d^h_n$ is induced (through the universal property of the product) by the family $\{d^h_{p-1,q} \circ d^h_{p,q} \circ \pi_{p,q}\}_{p+q=n}$, that $d^v_{n-1} \circ d^v_n$ is induced by $\{d^v_{p,q-1} \circ d^v_{p,q} \circ \pi_{p,q}\}_{p+q=n}$, and that $d^v_{n-1} \circ d^h_n + d^h_{n-1} \circ d^v_n$ is induced by $\{(d^v_{p-1,q} \circ d^h_{p,q} + d^h_{p,q-1} \circ d^v_{p,q}) \circ \pi_{p,q}\}_{p+q=n}$. Since all of these are families of 0 objects by the definition of a double complex, we deduce that $\partial_{n-1} \circ \partial_n = 0$ as required. □

2.4 Exercises Week 3

Throughout the whole sheet, $\mathcal{A}$ will denote an abelian category.

Remember that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in an abelian category is said to be *exact* at B if $g \circ f = 0$ and the induced map $\mathrm{im}(f) \to \ker(g)$ is an isomorphism.

(i) Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a sequence of chain complexes in $\mathcal{A}$. Show, without using Freyd-Mitchell embedding theorem, that if

$$0 \to A_n \to B_n \to C_n \to 0$$

is exact in $\mathcal{A}$ for all $n \in \mathbb{Z}$, then

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is exact in $\mathbf{Ch}(\mathcal{A})$.

(ii) Let $f_\bullet : B_\bullet \to C_\bullet$ and $g_\bullet : D_\bullet \to C_\bullet$ be morphisms of chain complexes. Assume that $f_\bullet$ is monic and that, for each $n \in \mathbb{Z}$, there exists a map $h_n : D_n \to B_n$ such that $f_n \circ h_n = g_n$. Show that it defines a morphism $h_\bullet$, such that $f_\bullet \circ h_\bullet = g_\bullet$ in the category of chain complexes. In other words, show that if the following diagram commutes in $\mathcal{A}$ for every $n \in \mathbb{Z}$,

$$\begin{tikzcd}
B_n \arrow[rr, "f_n"] & & C_n \\
& D_n \arrow[ul, dashed, "\exists h_n"] \arrow[ur, "g_n"'] &
\end{tikzcd}$$

then $h_\bullet$ is a morphism and the following commutes in $\mathbf{Ch}(\mathcal{A})$:

$$\begin{tikzcd}
B_\bullet \arrow[rr, "f_\bullet"] & & C_\bullet \\
& D_\bullet \arrow[ul, dashed, "h_\bullet"] \arrow[ur, "g_\bullet"'] &
\end{tikzcd}$$

(iii) Let $D_{\bullet,\bullet} = \{D_{p,q}\}$ be a bounded double complex with maps

$$d^h_{p,q} = d^h : C_{p,q} \to C_{p-1,q} \quad \text{and} \quad d^v_{p,q} = d^v : C_{p,q} \to C_{p,q-1}.$$

Show that, if each row (or each column) $D_{\bullet,q}$ is an exact sequence, then the total complex $\mathrm{Tot}(D_{\bullet,\bullet})$ is also exact.
Hint: You may need Freyd-Mitchell Embedding Theorem.

(iv) (Mapping Cone)

  (a) Let $f_\bullet : B_\bullet \to C_\bullet$ be a morphism of chain complexes. Show that $f_\bullet$ induces a natural double complex $D_{\bullet,\bullet}$ which contains exactly two non-trivial rows: $B_\bullet$ and $C_\bullet$.

  (b) We have seen that for every double complex $D_{\bullet,\bullet}$, by adding signs to all vertical maps $d^h_{p,q}$, one gets the maps $f_{p,q} := (-1)^p d^v_{p,q}$. This defines a morphism of chain complexes : $f_{\bullet,q} : C_{\bullet,q} \to C_{\bullet,q-1}$. Therefore, we may view $\{C_{\bullet,q}\}_{q\in\mathbb{Z}}$ as a chain complex over the category $\mathbf{Ch}(\mathcal{A})$ and $f_{\bullet,q}$ as the boundary map at the q-th position. Show that this sign trick gives us a 1-1 correspondence between the objects of $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$ and the collection of all double complexes over $\mathcal{A}$.

  (c) Because of (b), we would like to define the category of double complexes in such that it is isomorphic to $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$. Thus, we define a *morphism of double complexes* between $C_{\bullet,\bullet}$ and $D_{\bullet,\bullet}$ as a collection of maps $\{f_{p,q} : C_{p,q} \to D_{p,q}\}$, such that after turning $C_{\bullet,\bullet}, D_{\bullet,\bullet}$ into objects in $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$ and using the sign trick, $f_{\bullet,\bullet}$ is a morphism of $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$.

  Now let $B[-1]_\bullet$ be the chain complex defined by $B[-1]_n = B_{n-1}$ with differentials $-d$, where d are the differentials of B (see **Translation 1.2.8** in Weibel's book for a more general definition).

Show that one can consider $C_\bullet$ and $B[-1]_\bullet$ as double complexes in a natural way. Moreover, prove that we have the following exact sequence in the category of double complexes:

$$0 \to C_\bullet \to D_{\bullet,\bullet} \to B[-1]_\bullet \to 0.$$

Remark: The total complex of $D_{\bullet,\bullet}$ is the *mapping cone* of f', denoted by $\mathrm{cone}(f')$, where f' is a map which differs from f by some signs. We will encounter the mapping cone later (see section 1.5 in Weibel's book).

2.5 Long exact sequences

Theorem 2.19. *Let $\mathcal{A}$ be an abelian category, and suppose that*

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \arrow[r, "g_\bullet"] & C_\bullet \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence in $\mathrm{Ch}(\mathcal{A})$. *Then there is a collection of* natural[2] *maps* $\{\partial_n : H_n(C) \to H_{n-1}(A)\}_{n\in\mathbb{Z}}$, *which we call* connecting homomorphisms, *such that*

$$\cdots \xrightarrow{\tilde{g}_{n+1}} H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{\tilde{f}_n} H_n(B) \xrightarrow{\tilde{g}_n} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{\tilde{f}_{n-1}} \cdots$$

is an exact sequence in $\mathcal{A}$, where for each $n \in \mathbb{Z}$, $\tilde{f}_n$ (respectively, $\tilde{g}_n$) is the image of the chain map $f_\bullet$ (respectively, $g_\bullet$) under the functor $H_n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$.

Before proving this theorem, we state the *Snake Lemma*, which will help in our construction of the connecting homomorphisms $\partial_n : H_n(C) \to H_{n-1}(A)$.

Lemma 2.20 (The Snake Lemma)**.** *Let $\mathcal{A} = R$–**mod** for some ring R, and suppose that we have a commutative diagram in $\mathcal{A}$ of the form*

$$\begin{tikzcd} & A' \arrow[r, "p_1"] \arrow[d, "f"] & B' \arrow[r, "p_2"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i_1"] & B \arrow[r, "i_2"] & C & \end{tikzcd} .$$

Then, if the rows of this diagram are exact, there is an exact sequence

$$\ker(f) \xrightarrow{p_1} \ker(g) \xrightarrow{p_2} \ker(h) \xrightarrow{\partial} \mathrm{coker}(f) \xrightarrow{\tilde{\imath}_1} \mathrm{coker}(g) \xrightarrow{\tilde{\imath}_2} \mathrm{coker}(h) \ ,$$

where for $a \in A$, $\tilde{\imath}_1$ maps $a + f(A')$ to $i_1(a) + g(B')$, and for $b \in B$, $\tilde{\imath}_2$ maps $b + g(B')$ to $i_2(b) + h(C')$. Also, we can compute $\partial(c')$ for any $c' \in \ker(h)$. First, we find $b' \in B'$ such that $p_2(b') = c'$, then we find the unique $a \in A$ such that $i(a) = g(b')$. Then ∂ satisfies

$$\partial(c') = a + f(A'). \tag{1}$$

Proof. Please see Exercise 1 of this week's exercise sheet. ☐

Proof of Theorem 1.1. By the Freyd-Mitchell embedding Theorem, it suffices to consider the case when $\mathcal{A} = R$–**mod** for some ring R. Fix an integer n, and consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r, "f"] \arrow[d, "d^A"] & B_n \arrow[r, "g"] \arrow[d, "d^B"] & C_n \arrow[r] \arrow[d, "d^C"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "f"] & B_{n-1} \arrow[r, "g"] & C_{n-1} \arrow[r] & 0 \end{tikzcd} ,$$

the rows of which are exact by Week 3, Exercise 1. Note that each f_n is injective and each g_n is surjective. Hence we obtain via the Snake Lemma an exact sequence

$$Z_n(A) \xrightarrow{f} Z_n(B) \xrightarrow{g} Z_n(C) \longrightarrow A_{n-1}/dA_n \xrightarrow{\tilde{f}} B_{n-1}/dB_n \xrightarrow{\tilde{g}} C_{n-1}/dC_n \ ,$$

where the morphism $\tilde{f} : A_{n-1}/dA_n \to B_{n-1}/dB_n$ satisfies

$$\tilde{f}(a + dA_n) = f_{n-1}(a) + dB_n$$

---

[2] The sense in which the connecting homomorphisms are *natural* is explained in Remark 2.24.

for all $a \in A_{n-1}$. The morphism $\tilde{g} : B_{n-1}/dB_n \to C_{n-1}/dC_n$ satisfies an analogous definition and is surjective because g_{n-1} is surjective. Since $n \in \mathbb{Z}$ was arbitrary, we may construct a new commutative diagram

$$\begin{tikzcd} & A_n/dA_{n+1} \arrow[r, "\tilde{f}"] \arrow[d, "d_*^A"] & B_n/dB_{n+1} \arrow[r, "\tilde{g}"] \arrow[d, "d_*^B"] & C_n/dC_{n+1} \arrow[r] \arrow[d, "d_*^C"] & 0 \\ 0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f"] & Z_{n-1}(B) \arrow[r, "g"] & Z_{n-1}(C) & \end{tikzcd} \tag{2}$$

with exact rows. The morphism $d_*^A : A_n/dA_{n+1} \to Z_{n-1}(A)$ satisfies

$$d_*^A(a + dA_{n+1}) = d_n^A(a)$$

for all $a \in A_n$, and the morphisms d_*^B, d_*^C satisfy analogous relations. To see why the left square in (2) commutes, let $a \in A_n$ and observe that

$$\begin{aligned} d_*^B \circ \tilde{f}(a + dA_{n+1}) &= d_*^B(f_n(a) + dB_{n+1}) \\ &= d_n^B(f_n(a)) \\ &\overset{!}{=} f_{n-1}(d_n^A(a)) \\ &= f_{n-1} \circ d_*^A(a + dA_{n+1}), \end{aligned}$$

as needed, where the marked equality holds because $f_\bullet$ is a chain map from $A_\bullet$ to $B_\bullet$. An identical argument shows that the right square commutes. Note that the kernel of the morphism $d_*^A : A_n/dA_{n+1} \to Z_{n-1}(A)$ is

$$\begin{aligned} \ker(d_*^A) &= \{a + dA_{n+1} : a \in A_n,\ d_n^A(a) = 0\} \\ &= Z_n(A)/dA_{n+1} \\ &= H_n(A), \end{aligned}$$

and its cokernel is

$$\begin{aligned} \operatorname{coker}(d_*^A) &= Z_{n-1}(A)/\operatorname{im}(d_*^A) \\ &= Z_{n-1}(A)/\{d_n^A(a) : a \in A_n\} \\ &= Z_{n-1}/dA_n \\ &= H_{n-1}(A). \end{aligned}$$

Hence, a second application of the Snake Lemma to (2) yields a connecting homomorphism $\partial_n : H_n(C) \to H_{n-1}(A)$ such that the sequence

$$H_n(A) \xrightarrow{\tilde{f}_n} H_n(B) \xrightarrow{\tilde{g}_n} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{\tilde{f}_{n-1}} H_{n-1}(B) \xrightarrow{\tilde{g}_{n-1}} H_{n-1}(C).$$

is exact. We note that $\tilde{f}_k = H_k(f_\bullet)$ and $\tilde{g}_k = H_k(g_\bullet)$ for $k \in \{n, n-1\}$. Since this sequence is exact for all $n \in \mathbb{Z}$, we may paste them together to obtain the desired long exact sequence. ☐

Remark 2.21. Before continuing, we note that in the above proof we can actually calculate the image of $[c] \in H_n(C)$ under ∂_n using Equation (1). Let $c \in Z_n(C)$. Let $b \in B_n$ be such that $\tilde{g}_n$ maps $[b] \in B_n/dB_{n+1}$ to $[c]$, i.e., $[g_n(b)] = [c]$ in C_n/dC_{n+1}. Then there is $a \in Z_{n-1}(A)$ such that

$$f_{n-1}(a) = d_*^B([b]) = d_n^B(b).$$

It follows that the connecting homomorphism $\partial_n : H_n(C) \to H_{n-1}(A)$ satisfies $\partial_n([c]) = [a]$.

In our statement of Theorem 1.1, we mentioned that the connecting homomorphisms $\{\partial_n : H_n(C) \to H_{n-1}(A)\}_{n\in\mathbb{Z}}$ are *natural*. We now give the precise meaning of this statement. To do so, we introduce two new categories.

Remark 2.22. Let $\mathcal{C}$ be an abelian category. Then there is category $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ whose objects are the short exact sequences of chain complexes in $\mathcal{S}$. A morphism in $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ from $0 \to A \to B \to C \to 0$ to $0 \to A' \to B' \to C' \to 0$ is a commutative diagram of the shape

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0.
\end{tikzcd}$$

The category $\mathcal{L}(\mathcal{C})$ of long exact sequences in $\mathcal{C}$ is defined analogously.

Proposition 2.23. *Let $\mathcal{C}$ be an abelian category. Then we may define a functor from $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ to $\mathcal{L}(\mathcal{C})$ by mapping a short exact sequence of chain complexes in $\mathcal{C}$*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

to the long exact sequence in $\mathcal{C}$

$$\cdots \xrightarrow{\tilde{g}_{n+1}} H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{\tilde{f}_n} H_n(B) \xrightarrow{\tilde{g}_n} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{\tilde{f}_{n-1}} \cdots ,$$

and by mapping a morphism in $\mathcal{S}(\mathrm{Ch}(C))$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "u"] & B' \arrow[r, "v"] & C' \arrow[r] & 0
\end{tikzcd} \tag{3}$$

to the following morphism in $\mathcal{L}(\mathcal{C})$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(A) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\alpha}"] & H_n(B) \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{\beta}"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "\tilde{\gamma}"] & H_{n-1}(A) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\alpha}"] & \cdots \\
\cdots \arrow[r, "\partial'"] & H_n(A') \arrow[r, "\tilde{u}"] & H_n(B') \arrow[r, "\tilde{v}"] & H_n(C') \arrow[r, "\partial'"] & H_{n-1}(A') \arrow[r, "\tilde{g}"] & \cdots .
\end{tikzcd} \tag{4}$$

Proof. The only non-trivial part of this statement is that the diagram in (4) defines a morphism in $\mathcal{L}(C)$. This holds if each square in the ladder diagram commutes. $\tilde{\beta}_n\tilde{f}_n = \tilde{u}_n\tilde{\alpha}_n$ and $\tilde{\gamma}_n\tilde{g}_n = \tilde{v}_n\tilde{\beta}_n$ follow immediately from the commutativity of the diagram in (3) and the functoriality of H_n. Hence it suffices to show that

$$\begin{tikzcd}
H_n(C) \arrow[r, "\partial_n"] \arrow[d, "\tilde{\gamma}_n"] & H_{n-1}(A) \arrow[d, "\tilde{\alpha}_{n-1}"] \\
H_n(C') \arrow[r, "\partial'_n"] & H_{n-1}(A')
\end{tikzcd} \tag{5}$$

commutes. Let $[c]$ be an arbitrary element of $H_n(C)$, where $c \in Z_n(C)$. By Remark (2.21), if $b \in B_n$ is such that $\tilde{g}_n[b] = [c]$, and $a \in Z_{n-1}(A)$ is such that $f_{n-1}(a) = d_n^B(b)$, then $\partial_n[c] = [a]$. We want to find the image of $\tilde{\gamma}_n[c] \in H_n(C')$ under ∂'_n. Note that $\beta_n(b) \in B'_n$ is such that

$$\tilde{v}_n[\beta_n(b)] = \tilde{v}_n\tilde{\beta}_n[b] = \tilde{\gamma}_n\tilde{g}_n[b] = \tilde{\gamma}_n[c],$$

and $\alpha_{n-1}(a) \in Z_{n-1}(A')$ is such that

$$u_{n-1}(\alpha_{n-1}(a)) = \beta_{n-1}(f_{n-1}(a)) = \beta_{n-1}(d_n^B(b)) = d_n^{B'}(\beta_n(b)).$$

Thus ∂'_n maps $\tilde{\gamma}_n[c]$ to $[\alpha_{n-1}(a)] = \tilde{\alpha}_{n-1}[a] = \tilde{\alpha}_{n-1}\partial_n[c]$, i.e.,

$$\partial'_n\tilde{\gamma}_n[c] = \tilde{\alpha}_{n-1}\partial_n[c],$$

as needed. Hence we have mapped each morphism in $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ to a well-defined morphism in $\mathcal{L}(\mathcal{C})$. It is immediately seen that this mapping preserves identity morphisms and compositions. We thus have a functor from $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ to $\mathcal{L}(\mathcal{C})$. ☐

Remark 2.24. For each integer k, define functors L_k, R_k from $\mathcal{S}(\mathrm{Ch}(\mathcal{C}))$ to $\mathcal{C}$ such that

$$L_k\Big(0 \to A \to B \to C \to 0\Big) = H_k(A),$$

and

$$R_k\Big(0 \to A \to B \to C \to 0\Big) = H_k(C).$$

Then the commutativity of the diagram in (5) shows that for each integer n, we may define a natural transformation $\partial_n : R_n \Longrightarrow L_{n-1}$ by assigning to each short exact sequence $0 \to A \to B \to C \to 0$ of chain complexes in $\mathcal{C}$ the connecting homomorphism $H_n(C) \to H_{n-1}(A)$. It is for this reason that we say the connecting homomorphisms in Theorem 1.1 are natural.

We now give a two brief examples to demonstrate the usefulness of long exact sequences in algebraic topology.

Example 2.25. If X is a topological space, the singular complex $S(X)$ is the chain complex $C_\bullet(X)$, where for $n \geq 0$, $C_n(X)$ is the free abelian group with all singular n-simplices on X as its basis. Elements of $C_n(X)$ are called n-chains on X, and $H_n(X)$ is defined to be the nth homology group of $S(X)$. If A is a subspace of X, then there is a short exact sequence

$$0 \to S(A) \to S(X) \to S(X, A) \to 0, \tag{6}$$

where $S(X, A)$ is the quotient chain complex $S(X)/S(A)$. Let $H_n(X, A)$ be the nth homology group of $S(X, A)$. Each element of $H_n(X, A)$ is represented a relative n-cycle: an n-chain on X whose boundary, i.e., its image under the differential $C_n(X) \to C_{n-1}(X)$, actually lies in $C_{n-1}(A)$. Via Theorem 1.1, we obtain from (6) a long exact sequence

$$\cdots \to H_n(X) \to H_n(X, A) \overset{\partial}{\to} H_{n-1}(A) \to H_{n-1}(X) \to \cdots . \tag{7}$$

It turns out that the connecting homomorphism ∂ takes a relative cycle representing a class of $H_n(X, A)$ to the class in $H_{n-1}(A)$ represented by its boundary.

Example 2.26. Suppose that A, B are subspaces of a topological space X such that $X = \mathrm{int}(A) \cup \mathrm{int}(B)$. Then there is a short exact sequence of chain complexes

$$0 \to S(A \cap B) \to S(A) \oplus S(B) \to S(X) \to 0,$$

which is sent by the functor in Proposition 2.23 to the long exact sequence

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots .$$

This long exact sequence is known as the *Mayer-Vietoris sequence.*

2.6 Chain homotopies

Definition 2.27. A chain complex $C_\bullet$ is called *split* if there are morphisms $\{s_n : C_n \to C_{n+1}\}$, called *splitting maps*, such that $d_{n+1} = d_{n+1} s_n d_{n+1}$ for every integer n. A chain complex $C_\bullet$ is *split exact* if it is split and acyclic.

Example 2.28. We show that every chain complex $C_\bullet$ of vector spaces over a field k is split. For each integer n, pick $B'_n \leq C_n$ such that $C_n = Z_n \oplus B'_n$. Note that

$$B'_n \cong C_n/Z_n \cong \mathrm{im}(d_n) = B_{n-1}. \tag{8}$$

Let $\pi_n : Z_n \oplus B'_n \to Z_n$ and $\pi'_n : Z_n \oplus B'_n \to B'_n$ be projections. Then the isomorphism $C_n/Z_n \to B'_n$ is the the map taking $[c] \in C_n/Z_n$ to $\pi'_n(c)$. Also, the isomorphism $C_n/Z_n \to B_{n-1}$ maps $[c] \in C_n/Z_n$ to $d_n(c)$. Similarly, since $B_n = \mathrm{im}(d_{n+1})$ is a subspace of Z_n, we may pick $H'_n \leq Z_n$ such that $Z_n = B_n \oplus H'_n$. Note that

$$H'_n \cong Z_n/B_n = H_n. \tag{9}$$

Let $\rho_n : B_n \oplus H'_n \to B_n$ and $\rho'_n : B_n \oplus H'_n \to H'_n$ be the projections corresponding to this decomposition. Now we define the splitting map $s_n : C_n \to C_{n+1}$ to be the composition

$$C_n \xrightarrow{\pi_n} Z_n \xrightarrow{\rho_n} B_n \xrightarrow{q_n} B'_{n+1} \subseteq C_{n+1},$$

where q_n is from the same family of isomorphisms as in (8). What is the composition $d_{n+1}s_nd_{n+1}$? For an element $x \in C_{n+1}$, we have $d_{n+1}(x) \in B_n \subseteq Z_n$, thus

$$\begin{aligned} d_{n+1}s_n(d_{n+1}(x)) &= d_{n+1}q_n(d_{n+1}(x)) \\ &= d_{n+1}(\pi'_{n+1}(x)) \\ &= d_{n+1}(\pi_{n+1}(x)) + d_{n+1}(\pi'_{n+1}(x)) \\ &= d_{n+1}(x), \end{aligned}$$

thus $d_{n+1}s_nd_{n+1} = d_{n+1}$. Hence C is a split chain complex. Next, we determine the conditions under which C is not only split, but split exact. Note that for $x \in C_n$, we have $\rho_n\pi_n(x) = d_{n+1}(y)$ for some $y \in C_{n+1}$, thus

$$\begin{aligned} d_{n+1}s_n(x) &= d_{n+1}q_n(d_{n+1}(y)) \\ &= d_{n+1}(y) \\ &= \rho_n\pi_n(x) \end{aligned}$$

thus $d_{n+1}s_n$ is projection $C_n \to B_n$. Similarly, one may show that $s_{n-1}d_n$ is projection $C_n \to B'_n$. Since we have the decomposition $C_n = B_n \oplus H'_n \oplus H_n$, it follows from (9) that both the kernel and cokernel of the chain map $ds + sd : C_\bullet \to C_\bullet$ are the trivial homology complex $H_\bullet(C)$, that is, the complex with zero differentials whose nth object is $H_n(C)$. One may argue from here that $C_\bullet$ is exact if and only if $ds + sd$ is the identity chain map.

Remark 2.29. Given chain complexes $C_\bullet$ and $D_\bullet$, and *any* collection of morphisms $\{s_n : C_n \to D_{n+1}\}_{n\in\mathbb{Z}}$, let $f_n : C_n \to D_n$ be the morphism

$$f_n = d^D_{n+1}s_n + s_{n-1}d^C_n.$$

Then $f_\bullet : C_\bullet \to D_\bullet$ is in fact a chain map, since for any $n \in \mathbb{Z}$:

$$\begin{aligned} d^D_n f_n &= d^D_n d^D_{n+1}s_n + d^D_n s_{n-1}d^C_n \\ &= d^D_n s_{n-1}d^C_n \\ &= s_{n-2}d^C_{n-1}d^C_n + d^D_n s_{n-1}d^C_n \\ &= (s_{n-2}d^C_{n-1} + d^D_n s_{n-1})d^C_n \\ &= f_{n-1}d^C_n. \end{aligned}$$

We give a special name to chain maps that are of this form.

Definition 2.30. A chain map $f_\bullet : C_\bullet \to D_\bullet$ is *null homotopic* if there are morphisms $s_n : C_n \to D_{n+1}$, $n \in \mathbb{Z}$, such that

$$f_n = d^D_{n+1}s_n + s_{n-1}d^C_n$$

for every integer n. The maps $\{s_n\}_{n\in\mathbb{Z}}$ are called a *chain contraction* of f. If the identity morphism $C_\bullet \to C_\bullet$ of a chain complex $C_\bullet$ is null homotopic, we say that $C_\bullet$ is *contractible.*

Returning to Example 2.28, we see that a chain complex $C_\bullet$ of vector spaces over a field k is split exact if and only if $C_\bullet$ is contractible. It turns out that this is true in the general case.

Exercise 2.1. Show that a chain complex $C_\bullet$ is split exact if and only if it is contractible.

Definition 2.31. Let $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ morphisms of chain complexes. Then $f_\bullet, g_\bullet$ are *chain homotopic* if their difference $f - g$ is null homotopic. In this case, the corresponding family of maps $\{s_n\}_{n\in\mathbb{Z}}$ is called a *chain homotopy.* We may define an equivalence relation on morphisms of chain complexes by identifying chain maps that are chain homotopic.

The map $H_n(f) : H_n(C) \to H_n(D)$ induced by a chain map $f_\bullet : C_\bullet \to D_\bullet$ depends only on the homotopy class of $f_\bullet$:

Proposition 2.32. *If $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are homotopic maps of chain complexes, then these maps induce the same maps $H_n(C_\bullet) \to H_n(D_\bullet)$ on homology.*

Proof. It suffices to show that $H_n(f)$ is the zero map $H_n(C) \to H_n(D)$ for all $n \in \mathbb{Z}$ if $f : C_\bullet \to D_\bullet$ is null homotopic. Let $\{s_n\}_{n\in\mathbb{Z}}$ be a chain contraction of f, and let $[x] \in H_n(C)$, where $x \in Z_n(C)$. Then

$$f_n(x) = d^D_{n+1}s_n(x) + s_{n-1}d^C_n(x) = d^D_{n+1}s_n(x)$$

lies in $B_n(D)$, thus $[f(x)]$ is the zero element of $H_n(D)$. We conclude that $H_n(f)$ is the zero map $H_n(C) \to H_n(D)$. ☐

Remark 2.33. It can be shown that there is a homotopy category of chain complexes on $\mathcal{A}$, in which the objects are chain complexes on $\mathcal{A}$, and the morphisms are homotopy classes of chain morphisms. We have a special name for isomorphisms in this new category.

Definition 2.34. A chain map $f_\bullet : C_\bullet \to D_\bullet$ defines a *chain homotopy equivalence* if it is an isomorphism in the homotopy category of chain complexes, i.e., if there is a chain map $g_\bullet : D_\bullet \to C_\bullet$ such that $f_\bullet g_\bullet$ is chain homotopic to the identity on $D_\bullet$ and $g_\bullet f_\bullet$ is chain homotopic to the identity on $C_\bullet$. If there exists a chain homotopy equivalence between two chain complexes, we say that they are *homotopy equivalent.*

2.7 Mapping cones

The notion of homotopy in $\mathrm{Ch}(\mathcal{A})$ is closely related to the notion of homotopy in **Top**. Recall that continuous maps $f, g : X \to Y$ of topological spaces are homotopic in **Top** if there is a continuous map $H : [0,1]\times X \to Y$, called a homotopy, such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for all $x \in X$. Equivalently, if $\iota_0 : X \to \{0\}\times X$ and $\iota_1 : X \to \{1\} \times X$ are the natural inclusions, then a continuous map $H : [0,1] \times X \to Y$ defines a homotopy between f and g if $H \circ \iota_0 = f$ and $H \circ \iota_1 = g$. Our aim is to find a counterpart for this definition in $\mathrm{Ch}(\mathcal{A})$. We first find a chain complex $I_\bullet$ to act as an interval object. We assume that $\mathcal{A} = R$—mod for a ring R.

Definition 2.35. Let $I_\bullet$ be the simplicial chain complex of an interval, consisting of two vertices v_0, v_1 and one edge $e = [v_0, v_1]$. That is, $I_0 = R\langle v_0, v_1\rangle$, $I_1 = R\langle e\rangle$, and $I_k = 0$ otherwise. Also, we have $\partial_1[v_0, v_1] = v_1 - v_0$ and $\partial_k = 0$ otherwise.

In **Top**, the domain of the homotopy H is the product of the interval $[0,1]$ with X. In $\mathrm{Ch}(\mathcal{A})$, the appropriate notion of a product is the tensor product.

Definition 2.36. Given chain complexes $C_\bullet, D_\bullet$, let $C_\bullet \otimes D_\bullet$ be the chain complex such that

$$(C_\bullet \otimes D_\bullet)_n = \bigoplus_{i+j=n} C_i \otimes D_j,$$

with the nth differential on $C_\bullet \otimes D_\bullet$ defined as follows: if i, j are integers such that $i + j = n$, and $(x, y) \in C_i \times D_j$, then

$$d_n(x \otimes y) = d^C_i x \otimes y + (-1)^i(x \otimes d^D_j y).$$

Remark 2.37. Suppose that $C_\bullet$ is a chain complex. Then $I_\bullet \otimes C_\bullet$ is the chain complex such that at level n:

$$\begin{aligned}(I_\bullet \otimes C_\bullet)_n &= (I_0 \otimes C_n) \oplus (I_1 \otimes C_{n-1})\\ &= (\langle v_0\rangle \otimes C_n) \oplus (\langle v_1\rangle \otimes C_n) \oplus (\langle e\rangle \otimes C_{n-1})\\ &= C^{v_0}_n \oplus C^{v_1}_n \oplus C^e_{n-1},\end{aligned}$$

where $C^{v_i}_n = \langle v_i\rangle \otimes C_n$, and $C^e_{n-1} = \langle e\rangle \otimes C_{n-1}$. If $x \in C_n$, then

$$d_n(v_i \otimes x) = v_i \otimes d^C_n x,$$

and if $x \in C_{n-1}$, then

$$d_n(e \otimes x) = v_1 \otimes x - v_0 \otimes x - e \otimes d^C_{n-1}x.$$

There are also chain maps $\iota_0, \iota_1 : C_\bullet \to I_\bullet \otimes C_\bullet$ such that $\iota_0(x) = v_0 \otimes x$ and $\iota_1(x) = v_1 \otimes x$ for $x \in C_n$.

We give a new definition of homotopy in $\mathrm{Ch}(\mathcal{A})$ that mirrors the definition of homotopy in **Top** we saw earlier.

Definition 2.38. Chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic if there is a chain map $H : I_\bullet \otimes C_\bullet \to D_\bullet$ such that $H \circ \iota_0 = f_\bullet$ and $H \circ \iota_1 = g_\bullet$.

Proposition 2.39. *Chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic in the sense of Definition 2.31 if and only if they are chain homotopic in the sense of Definition 2.38.*

Proof. If $f_\bullet, g_\bullet$ are chain homotopic in the sense of Definition 2.38, then there is a chain map $H : I_\bullet \otimes C_\bullet \to D_\bullet$ such that $H \circ \iota_0 = f_\bullet$ and $H \circ \iota_1 = g_\bullet$. For $x \in C_n$, we have $e \otimes x \in (I \otimes C)_{n+1}$. Thus we may define $s_n : C_n \to D_{n+1}$ by letting $s_n(x) = H_{n+1}(e \otimes x)$. It follows that

$$\begin{aligned}
(d^D_{n+1}s_n + s_{n-1}d^C_n)(x) &= d^D_{n+1}H_{n+1}(e \otimes x) + H_n(e \otimes d^C_n x) \\
&\overset{!}{=} H_n d^{I\otimes C}_{n+1}(e \otimes x) + H_n(e \otimes d^C_n x) \\
&= H_n(v_1 \otimes x - v_0 \otimes x - e \otimes d^C_n x) + H_n(e \otimes d^C_n x) \\
&= H_n(v_1 \otimes x) - H_n(v_0 \otimes x) \\
&= H_n \circ \iota_1(x) - H_n \circ \iota_0(x) \\
&= g(x) - f(x),
\end{aligned}$$

where the marked equality follows because H is a chain map. Thus the maps $\{s_n\}_{n\in\mathbb{Z}}$ are a chain contraction of $f - g$, and we conclude that the chain maps $f_\bullet, g_\bullet$ are chain homotopic in the sense of Definition 2.31. Conversely, suppose that $\{s_n\}_{n\in\mathbb{Z}}$ is a chain contraction of $f - g$. Recall that $(I \otimes C)_n = C^{v_0}_n \oplus C^{v_1}_n \oplus C^e_{n-1}$. We define $H_n : (I \otimes C)_n \to D_n$ by requiring

$$H_n(v_0 \otimes x) = f_n(x) \quad \text{and} \quad H_n(v_1 \otimes x) = g_n(x)$$

for $x \in C_n$, and

$$H_n(e \otimes x) = s_{n-1}(x)$$

for $x \in C_{n-1}$. We leave it to the reader to show that $H : I_\bullet \otimes C_\bullet \to D_\bullet$ is a chain map such that $H \circ \iota_0 = f_\bullet$ and $H \circ \iota_1 = g_\bullet$, and thus that $f_\bullet, g_\bullet$ are chain homotopic in the sense of Definition 2.38. ☐

Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, we wish to define a new chain complex $\mathrm{cone}(f_\bullet)$, called the *mapping cone* of $f_\bullet$. This construction takes inspiration from the mapping cone C_f of a continuous map $f : X \to Y$ of topological spaces. Recall that to form C_f, we first take the cone CX of the space X (i.e., we take the quotient of the cylinder $I \times X$ by the equivalence relation that collapses $\{1\} \times X$ to a point) . Next, we glue CX to Y by taking the quotient of $CX \sqcup Y$ by the relation that glues $\{0\} \times X \subseteq CX$ to Y via $(0, x) \sim f(x)$. These two steps result in the mapping cone C_f. In $\mathrm{Ch}(\mathcal{A})$ we perform analogous maneuvers. First, we quotient the cylinder $I_\bullet \otimes C_\bullet$ by $\langle v_1\rangle \otimes C_\bullet$. The object at the nth degree of this quotient is

$$\frac{(I_\bullet \otimes C_\bullet)_n}{\langle v_1\rangle \otimes C_n} = \frac{C^{v_0}_n \oplus C^{v_1}_n \oplus C^e_{n-1}}{C^{v_1}_n} \cong C^{v_0}_n \oplus C^e_{n-1}.$$

The resulting chain complex is called the cone of $C_\bullet$ and is denoted by $\mathrm{cone}(C_\bullet)$ (cf. Definition 2.42). Next, we mimic the step of gluing CX to Y via f by taking the quotient of $\mathrm{cone}(C_\bullet) \oplus D_\bullet$ by a relation that identifies $C^{v_0}_n$ and D_n via $x \sim f_n(x)$ for $x \in C_n$. This results in a chain complex $\mathrm{cone}(f_\bullet)$, whose object at the nth degree is

$$\mathrm{cone}(f_\bullet)_n = C_{n-1} \oplus D_n.$$

The differentials for this chain complex are induced by the differential of the cylinder $I_\bullet \otimes C_\bullet$.

Definition 2.40. Given a chain morphism $f_\bullet : C_\bullet \to D_\bullet$, define the *mapping cone* of $f_\bullet$ to be the chain complex cone($f_\bullet$) whose object in the nth degree is $C_{n-1} \oplus D_n$, and whose nth differential $C_{n-1} \oplus D_n \to C_{n-2} \oplus D_{n-1}$ satisfies

$$\begin{aligned} d_n(x, y) &= (-d^C_{n-1}(x), d^D_n(y) - f_{n-1}(x)) \\ &= \begin{bmatrix} -d^C_{n-1} & 0 \\ -f_{n-1} & d^D_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

for $(x, y) \in C_{n-1} \oplus D_n$.

Remark 2.41. We see that cone($f_\bullet$) is indeed a chain complex by noting that the matrix representation of $d_n d_{n+1}$ is

$$\begin{bmatrix} -d^C_{n-1} & 0 \\ -f_{n-1} & d^D_n \end{bmatrix} \begin{bmatrix} -d^C_n & 0 \\ -f_n & d^D_{n+1} \end{bmatrix} = \begin{bmatrix} d^C_{n-1} d^C_n & 0 \\ f_{n-1} d^C_n - d^D_n f_n & d^D_n d^D_{n+1} \end{bmatrix} = 0,$$

since $f_\bullet$ is a chain map.

Definition 2.42. Given a chain complex $C_\bullet$, define the cone of $C_\bullet$ to be the mapping cone of the identity chain map $\mathrm{id}^C_\bullet : C_\bullet \to C_\bullet$. The cone of $C_\bullet$ is denoted by cone($C_\bullet$).

Proposition 2.43. *For any chain complex $C_\bullet$, the cone of $C_\bullet$ is split exact.*

Proof. We first find splitting maps $\{s_n : \mathrm{cone}(C_\bullet)_n \to \mathrm{cone}(C_\bullet)_{n+1}\}$. Let $s_n : C_{n-1} \oplus C_n \to C_n \oplus C_{n+1}$ be the map with matrix representation

$$s_n = \begin{bmatrix} 0 & -\mathrm{id}_n \\ 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} d_{n+1} s_n d_{n+1} &= \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \begin{bmatrix} 0 & -\mathrm{id}_n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & d_n \\ 0 & \mathrm{id}_n \end{bmatrix} \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \\ &= \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \\ &= d_{n+1}, \end{aligned}$$

which confirms that cone($C_\bullet$) is split. To see that cone($C_\bullet$) is acyclic, note that for each n:

$$\begin{aligned} \ker(d_n) &= \ker \begin{bmatrix} -d_{n-1} & 0 \\ -\mathrm{id}_{n-1} & d_n \end{bmatrix} \\ &= \{(x, y) \in C_{n-1} \oplus C_n : d_{n-1}x = 0, x = d_n y\} \\ &= \{(d_n y, y) : y \in C_n\} \\ &= \mathrm{im} \begin{bmatrix} -d_n & 0 \\ -\mathrm{id}_n & d_{n+1} \end{bmatrix} \\ &= \mathrm{im}(d_{n+1}). \end{aligned}$$

This confirms that cone($C_\bullet$) is split exact. □

Remark 2.44. Proposition 2.43 is the homological realization of the fact that the cone CX of any topological space X is contractible.

We demonstrate the value of mapping cones of chain maps in our final result, which reduces questions about quasi-isomorphisms to the study of exact complexes. We begin by recalling the definition of a quasi-isomorphism:

Definition 2.45. A chain map $f_\bullet : C_\bullet \to D_\bullet$ is called a *quasi-isomorphism* if its image $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$ under the homology functor H_n is an isomorphism for every integer n.

Proposition 2.46. *A chain map $f_\bullet : C_\bullet \to D_\bullet$ is a quasi-isomorphism if and only if the mapping cone complex of $f_\bullet$ is exact.*

Proof. Given a chain map $f_\bullet : C_\bullet \to D_\bullet$, we claim that there is a short exact sequence of chain complexes

$$0 \longrightarrow D_\bullet \xrightarrow{\alpha_\bullet} \mathrm{cone}(f_\bullet) \xrightarrow{\beta_\bullet} C_\bullet[-1] \longrightarrow 0, \tag{10}$$

where $C_\bullet[-1]$ is the (-1)th translate of $C_\bullet$, the map $\alpha_\bullet$ satisfies $\alpha_n(x) = (0, x)$ for $x \in D_n$, and the map $\beta_\bullet$ satisfies $\beta_n(x, y) = -x$ for $(x, y) \in C_{n-1} \oplus D_n$. So as long as one remembers that the nth differential for $C_\bullet[-1]$ is $-d^C_n$ (cf. Translation 1.2.8 in Weibel's book), it is easy to show that $\alpha_\bullet, \beta_\bullet$ are chain maps, and that the sequence in (10) is exact. By Theorem 1.1, there is a long exact sequence

$$\cdots \to H_{n+1}(\mathrm{cone}_f) \to H_n(C) \overset{\partial_n}{\to} H_n(D) \to H_n(\mathrm{cone}_f) \to H_{n-1}(C) \to \cdots, \tag{11}$$

where we have recalled that $H_n(C[-1]) \cong H_{n-1}(C)$. Let $x \in Z_n(C)$ to compute the image of $[x] \in H_n(C)$ under the connecting homomorphism ∂_n. β_{n+1} maps $(-x, 0)$ to x. Next, we have

$$d^{\mathrm{cone}}_{n+1}(-x, 0) = \begin{bmatrix} -d^C_n & 0 \\ -f_n & d^D_{n+1} \end{bmatrix} \begin{bmatrix} -x \\ 0 \end{bmatrix} = \begin{bmatrix} d^C_n(x) \\ f_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ f_n(x) \end{bmatrix},$$

since $x \in Z_n(C)$. Lastly, we note that α_n maps $f_n(x)$ to $(0, f_n(x))$. Thus the image of $[x]$ under the connecting homomorphism ∂_n is

$$\partial_n[x] = [f_n(x)] = \tilde{f}_n[x].$$

In particular, the connecting homomorphism ∂_n is precisely the map $H_n(f) : H_n(C) \to H_n(D)$. Since the sequence in (11) is exact, we conclude that $f_\bullet : C_\bullet \to D_\bullet$ is a quasi-isomorphism if and only if $H_n(\mathrm{cone}_f) = 0$ for all n, that is, if and only if the mapping cone complex of $f_\bullet$ is exact. ☐

2.8 Exercises Week 4

(i) Prove the following lemma stated during the lecture.

Lemma 2.47 (Snake lemma)**.** *Consider a commutative diagram of R-modules of the form*

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"'] & B' \arrow[r, "p"] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C. & \end{tikzcd}$$

If the rows are exact, there is an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \overset{\partial}{\to} \mathrm{coker}(f) \to \mathrm{coker}(g) \to \mathrm{coker}(h)$$

with ∂ defined by the formula

$$\partial(c') = i^{-1} g p^{-1}(c'), \quad c' \in \ker(h).$$

Moreover, if $A' \to B'$ is monic, then so is $\ker(f) \to \ker(g)$*, and if $B \to C$ is onto, then so is* $\mathrm{coker}(g) \to \mathrm{coker}(h)$*.*

(ii) Let $\mathcal{A}$ be an abelian category.

(a) Let

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

be a short exact sequence in $\mathbf{Ch}(\mathcal{A})$. Show that if two of the three complexes are exact, then so is the third.

(b) Let $f_\bullet : C_\bullet \to D_\bullet$ be a morphism in $\mathbf{Ch}(\mathcal{A})$. Show that if $\ker(f_\bullet)$ and $\mathrm{coker}(f_\bullet)$ are acyclic, then $f_\bullet$ is a quasi-isomomorphism. Is the converse true?

(iii) Consider the following chain complex of abelian groups

$$\cdots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \to \cdots .$$

Show that it is acyclic, but not split exact.

(iv) Let $\mathcal{A}$ be an abelian category and consider a morphism $f_\bullet : C_\bullet \to D_\bullet$ in $\mathbf{Ch}(\mathcal{A})$.
Show that $f_\bullet$ is null homotopic if and only if it extends to a map $(-s, f) : \mathrm{cone}(C_\bullet) \to D_\bullet$.

Chapter 3. Derived Functors

Throughout this whole chapter, $\mathcal{A}$ and $\mathcal{B}$ will denote two arbitrary abelian categories.

3.1 δ-functors

Definition 3.1. A *homological* (resp. *cohomological*) *δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $\{T_n : \mathcal{A} \to \mathcal{B}\}_{n\geq 0}$ (resp. $\{T^n : \mathcal{A} \to \mathcal{B}\}_{n\geq 0}$) together with a collection of morphisms $\{\delta_n : T_n(C) \to T_{n-1}(A)\}$ (resp. $\{\delta^n : T^n(C) \to T^{n+1}(A)\}$) defined for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, such that the two following conditions hold:

(i) For any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we have a long exact sequence

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\delta_{n+1}} T_n(A) \longrightarrow T_n(B) \longrightarrow T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \longrightarrow \cdots$$

$$\text{(resp.} \qquad \cdots \longrightarrow T^{n-1}(C) \xrightarrow{\delta^{n-1}} T^n(A) \longrightarrow T^n(B) \longrightarrow T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \longrightarrow \cdots)$$

(ii) For every morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

we have a commutative diagram for every $n \in \mathbb{Z}$:

$$\begin{tikzcd}
T_n(C') \arrow[r, "\delta_n"] \arrow[d] & T_{n-1}(A') \arrow[d] & \text{resp.} & T^n(C') \arrow[r, "\delta^n"] \arrow[d] & T^{n+1}(A') \arrow[d] \\
T_n(C) \arrow[r, "\delta_n"] & T_{n-1}(A) & & T^n(C) \arrow[r, "\delta^n"'] & T^{n+1}(A)
\end{tikzcd}$$

Remark 3.2. We make the convention that T_n (resp. T^n) is 0 for any $n < 0$. In particular by Definition 3.1.1, T_0 is right-exact (resp. T^0 is left-exact).

Example 3.3. (i) Homology gives a homological δ-functor $\mathrm{Ch}_{\geq 0}\mathcal{A} \to \mathcal{A}$, and cohomology gives a cohomological one $\mathrm{Ch}^{\geq 0}\mathcal{A} \to \mathcal{A}$.

(ii) Let A be an abelian group, and $p \in \mathbb{Z}$. We define

$$T_0(A) := A/pA, T_1(A) := {}_pA := \{a \in A | pa = 0\}, T_n = 0, \forall n \geq 2$$

and we want then to define δ_1 to get a homological δ-functor $\mathbf{Ab} \to \mathbf{Ab}$ (or a cohomological one by $T^0 := T_1, T^1 := T_0, \delta^0 := \delta_1$).

For this, let $0 \to A \to B \to C \to 0$ be a short exact sequence, and consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\cdot p"'] & B \arrow[r] \arrow[d, "\cdot p"'] & C \arrow[r] \arrow[d, "\cdot p"'] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma we have an exact sequence

$$0 \longrightarrow {}_pA \longrightarrow {}_pB \longrightarrow {}_pC \xrightarrow{\delta} A/pA \longrightarrow B/pB \longrightarrow C/pC \longrightarrow 0$$

and this gives us $\delta_1 := \delta$.

(iii) We can generalize the previous example to the category of R-modules for some ring R. To do that, let $r \in R, M \in R\text{-}\mathbf{mod}$, and define

$$T_0(M) := M/rM, T_1(M) := {}_rM$$

to get a homological δ-functor $R\text{-}\mathbf{mod} \to \mathbf{Ab}$.

(iv) In the same setting as in point 3., we can also define

$$T_n(M) := \operatorname{Tor}_n^R(R/(r), M), n \geq 0$$

and we will see later in the chapter why this is a homological δ-functor.

Definition 3.4. Let $S_\bullet, T_\bullet$ be two homological δ-functors (resp. $S^\bullet, T^\bullet$ two cohomological δ-functors). A *morphism* $S_\bullet \to T_\bullet$ (resp. $S^\bullet \to T^\bullet$) is a collection of natural transformations $\alpha_n : S_n \to T_n$ (resp. $\alpha^n : S^n \to T^n$) that commutes with δ, i.e. such that for any short exact sequence $0 \to A \to B \to C \to 0$, the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+1}(C) \arrow[r, "\delta^S_{n+1}"] \arrow[d, "(\alpha_{n+1})_C"'] & S_n(A) \arrow[r] \arrow[d, "(\alpha_n)_A"'] & S_n(B) \arrow[r] \arrow[d, "(\alpha_n)_B"'] & S_n(C) \arrow[r, "\delta^S_n"] \arrow[d, "(\alpha_n)_C"'] & S_{n-1}(A) \arrow[r] \arrow[d, "(\alpha_{n-1})_A"'] & \cdots \\
\cdots \arrow[r] & T_{n+1}(C) \arrow[r, "\delta^T_{n+1}"'] & T_n(A) \arrow[r] & T_n(B) \arrow[r] & T_n(C) \arrow[r, "\delta^T_n"'] & T_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

resp.

$$\begin{tikzcd}
\cdots \arrow[r] & S^{n-1}(C) \arrow[r, "\delta^{n-1}_S"] \arrow[d, "(\alpha^{n-1})_C"'] & S^n(A) \arrow[r] \arrow[d, "(\alpha^n)_A"'] & S^n(B) \arrow[r] \arrow[d, "(\alpha^n)_B"'] & S^n(C) \arrow[r, "\delta^n_S"] \arrow[d, "(\alpha^n)_C"'] & S^{n+1}(A) \arrow[r] \arrow[d, "(\alpha^{n+1})_A"'] & \cdots \\
\cdots \arrow[r] & T^{n-1}(C) \arrow[r, "\delta^{n+1}_T"'] & T^n(A) \arrow[r] & T^n(B) \arrow[r] & T^n(C) \arrow[r, "\delta^n_T"'] & T^{n+1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Definition 3.5. A homological δ-functor $T_\bullet$ (resp. cohomological δ-functor $T^\bullet$) is *universal* if for any other δ-functor $S_\bullet$ and any natural transformation $f_0 : S_0 \to T_0$ (resp. any $S^\bullet$ and any $f^0 : T^0 \to S^0$), there exists a unique morphism $\{f_n : S_n \to T_n\}$ extending f_0 (resp. there exists a unique morphism $\{f^n : T^n \to S^n\}$ extending f^0).

Example 3.6. We will see later that homology $H_* : \mathrm{Ch}_{\geq 0}\mathcal{A} \to \mathcal{A}$ and cohomology $H^* : \mathrm{Ch}^{\geq 0}\mathcal{A} \to \mathcal{A}$ are universal.

3.2 Projective resolutions

Definition 3.7. An object P in an abelian category $\mathcal{A}$ is *projective* if it satisfies the following universal property:
$\forall B \overset{g}{\twoheadrightarrow} C$ epimorphism, $P \overset{\gamma}{\to} C, \exists P \overset{\beta}{\to} B$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[d, "\gamma"] \arrow[dl, dashed, "\exists\beta"'] \\
B \arrow[r, two heads, "g"] & C
\end{tikzcd}$$

In other words, the morphism $\mathrm{Hom}_{\mathcal{A}}(P, B) \to \mathrm{Hom}_{\mathcal{A}}(P, C)$ induced by g is surjective.

Example 3.8. Free R-modules are projective, as you can lift the image by γ of a basis of P by g to get β.

Proposition 3.9. *An R-module is projective if and only if it's a direct summand of a free module.*

Proof. "$\Leftarrow$" This is clear by the universal property of coproduct and the fact that free R-modules are projective.

"$\Rightarrow$" Let A be a projective module, and $F(A)$ be the free R-module with basis $\{e_a\}_{a \in A}$. Note that $F(A)$ is equipped with a projection $\pi : F(A) \to A$.
Now, by the universal property of projective modules, we have a morphism $i : A \to F(A)$ such that $\pi i = id_A$, so that

$$0 \to A \xrightarrow{i} F(A) \to F(A)/A \to 0$$

is split, and that A is a direct summand of $F(A)$. □

Example 3.10. (i) Let $R := R_1 \times R_2$ a product of two rings, and $P = R_1 \times 0$ an R-module. P is projective as it is a direct summand of R, but it is not free as $(0, 1) \cdot P = 0$.

(ii) Let F be a field, $R = M_n(F), n > 1$. $V := F^n$ is a projective R-module, as $R = V^{\oplus n}$, but V is not free; indeed, if it was, its dimension as a F-vector space would be dn^2 for some $d \geq 0$, but $\dim_F V = n \neq dn^2$.

Definition 3.11. We say that an abelian category $\mathcal{A}$ *has enough projectives* if for every object A of $\mathcal{A}$, there is a projective module P and an epimorphism $P \twoheadrightarrow A$.

Remark 3.12. The category of finite abelian groups is an abelian category with no non-zero projective object.

Lemma 3.13. *Let $M \in \mathcal{A}$. M is projective if and only if* $\operatorname{Hom}_{\mathcal{A}}(M, -)$ *is exact.*

Proof. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$.
We already know that $\operatorname{Hom}_{\mathcal{A}}(M, -)$ is left exact, so saying that it is exact is equivalent to saying that

$$\operatorname{Hom}_{\mathcal{A}}(M, B) \to \operatorname{Hom}_{\mathcal{A}}(M, C)$$

is surjective. But this is exactly the definition that we gave (Definition 3.7), so we are done. □

Definition 3.14. A *left resolution* of $M \in \mathcal{A}$ is a chain complex $P_\bullet$ bounded below by 0, such that there is a map $P_0 \to M$ making the following sequence exact:

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

If every P_i is projective, then we say that $P_\bullet$ is a projective resolution.

Lemma 3.15. *Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object M of $\mathcal{A}$ has a projective resolution.*

Proof. We will construct P_i by induction. First, as $\mathcal{A}$ has enough projectives, there is a projective module and an epimorphism $P_0 \twoheadrightarrow M \to 0$. We moreover define $M_0 := \ker(P_0 \twoheadrightarrow M)$.

Inductively, having defined $P_k, M_k, \forall k \leq i-1$, we define P_i to be the projective module with an epimorphism $P_i \twoheadrightarrow M_{i-1} \to 0$, and M_i to be the kernel of this morphism, namely $M_i := \ker(P_i \twoheadrightarrow M_{i-1})$.
Writing d_i for the composite $P_i \to M_{i-1} \to P_{i-1}$, we have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & M_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[rr, "d_3"] \arrow[dr] & & P_2 \arrow[rr, "d_2"'] \arrow[ur] & & P_1 \arrow[rr, "d_1"] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
 & M_2 \arrow[ur] \arrow[dr] & & & & M_0 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where every $0 \to M_i \to P_i \to M_{i-1} \to 0$ is exact by definition of M_i and P_i.
But this exactness precisely gives us that

$$d_i(P_i) = M_{i-1} = \ker(d_{i-1}),$$

which shows that $P_\bullet$ is a left resolution of M. ☐

Theorem 3.16 (Comparison Theorem)**.** *Let $f : M \to N$ be a map in $\mathcal{A}$. Moreover let $P_\bullet \to M$ be a projective resolution, and $Q_\bullet \to N$ any left resolution.*
Then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ extending f, i.e. such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] & P_0 \arrow[r] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

This map is unique up to homotopy.

Proof. We do the proof by constructing f_n inductively, where $n \geq -1$.
For the base case, we define $f_{-1} := f, P_{-1} := M, Q_{-1} := N$, and moreover we denote by d_0 the maps $P_0 \to P_{-1}$ and $Q_0 \to Q_{-1}$, so that when we talk about $P_\bullet$ or $Q_\bullet$, we really mean the exact sequences extended by the term P_{-1} or Q_{-1}.

Now suppose we have constructed $f_k, \forall k \leq n$. By the equality $f_{n-1}d = df_n$, we have an induced map $f'_n : Z_n(P) \to Z_n(Q)$.
But $Z_n(P) = B_n(P)$ by exactness of $P_\bullet$, so $d : P_{n+1} \twoheadrightarrow P_n$ factorizes by $d : P_{n+1} \to Z_n(P)$ (and similarly for Q). We therefore have, by projectivity of P_{n+1}, the existence of a map f_{n+1} such that the following diagram commutes:

$$\begin{tikzcd}
 & P_{n+1} \arrow[d, two heads, "d"] \arrow[ddl, dashed, "\exists f_{n+1}"'] \\
 & Z_n(P) \arrow[d, "f'_n"] \\
Q_{n+1} \arrow[r, two heads, "d"] & Z_n(Q)
\end{tikzcd}$$

Moreover, $df_{n+1} = f'_n d = f_n d$ so this indeed is a chain map.

To see uniqueness up to homotopy, we let $g_\bullet$ be another candidate to extend f. We want to construct $\{s_n : P_n \to Q_{n+1}\}_{n \geq -1}$ such that $h := f - g = sd + ds$.
First define $s_{-1} = 0$. Note that $d_0 h_0 = h_{-1} d_0 = (f - f)d_0 = 0$, so that h_0 factors by $h_0 : P_0 \to Z_0(Q) = B_0(Q)$. By projectivity of P_0, h_0 lifts to a map $s_0 : P_0 \to Q_1$ such that $h_0 = ds_0 = s_{-1}d + ds_0$, and we have the base case.

Assume now by induction that we are given maps $s_i, \forall i < n$ such that $ds_i = h_i - s_{i-1}d$. In particular, the map $h_n - s_{n-1}d : P_n \to Q_n$ satisfies

$$d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - ds_{n-2})d = dh - hd + sdd = 0$$

and this maps factors through $Z_n(Q)$. As before, that means it lifts to a map $s_n : P_n \to Q_{n+1}$ with $ds_n = h_n - s_{n-1}d$, and we have our homotopy by induction. ☐

Lemma 3.17 (Horseshoe lemma)**.** *Given a commutative diagram*

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the column is exact and the rows are projective resolutions, and defining $P_n := P_n' \oplus P_n''$*, we have that* $P_\bullet$ *is a projective resolution of* A*, and that the right hand column of the diagram lifts to an exact sequence of chain complexes*

$$0 \to P' \xrightarrow{\iota} P \xrightarrow{\pi} P'' \to 0$$

where $\iota_n : P_n' \to P_n, \pi_n : P_n \to P_n''$ *are the natural inclusion and projection.*

Proof. See exercise sheet 5. ☐

3.3 Exercises Week 5

(i) Let $\mathcal{S}$ be the category of short exact sequences

$$0 \to A \to B \to C \to 0 \tag{12}$$

in an abelian category $\mathcal{A}$ and δ a homological δ-functor. Show that δ_i is a natural transformation from the functor sending (1) to $T_i(C)$ to the functor sending (1) to $T_{i-1}(A)$.

(ii) Show that a chain complex P is a projective object in **Ch** if and only if it is a split exact complex of projectives. *Hint*: To see that P must be split exact, consider the surjection from $cone(id_p)$ to $P[-1]$. To see that split exact complexes are projective objects, consider the special case $0 \to P_1 \cong P_0 \to 0$.

(iii) Use the previous exercise to show that if $\mathcal{A}$ has enough projectives, then so does the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$.

(iv) Prove the Horseshoe Lemma. More concretely, given a commutaive diagram

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
... \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
... \arrow[r] & P_2^{''} \arrow[r] & P_1^{''} \arrow[r] & P_0^{''} \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the column is exact and the rows are projective resolutions. Define $P_n = P'_n \oplus P''_n$. Prove $P_\bullet$ assemble to form a projective resolution P of A, and the right-hand column lifts to an exact sequence of complexes

$$0 \to P'_\bullet \xrightarrow{\iota} P_\bullet \xrightarrow{\pi} P''_\bullet \to 0.$$

Here $\iota_n : P'_n \to P_n$ denotes the natural inclusion and $\pi_n : P_n \to P''_n$ denotes the natural projection.

3.4 Left derived functors

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $\mathbf{R\text{-}Fun}(\mathcal{A}, \mathcal{B})$ and $\delta\text{-}\mathbf{Hom}(\mathcal{A}, \mathcal{B})$ be the category of right exact additive functors and the category of homological δ-functors between $\mathcal{A}$ and $\mathcal{B}$ respectively. There is a functor

$$\begin{aligned} \delta\text{-}\mathbf{Hom}(\mathcal{A}, \mathcal{B}) &\to \mathbf{R\text{-}Fun}(\mathcal{A}, \mathcal{B}) \\ (T_i)_{i\geq 0} &\mapsto T_0. \end{aligned}$$

We now want to construct a functor that goes in the opposite direction

$$\mathbf{R\text{-}Fun}(\mathcal{A}, \mathcal{B}) \to \delta\text{-}\mathbf{Hom}(\mathcal{A}, \mathcal{B}),$$

that sends right exact functors to universal homological δ-functors. To this end, we will assume that $\mathcal{A}$ has enough projectives.

Definition 3.18. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and assume that $\mathcal{A}$ has enough projectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive right exact functor and let A be an object in $\mathcal{A}$. Fix a projective resolution $P \to A$ and define

$$L_iF(A) := H_i(FP).$$

For any morphism $\delta : A \to A'$ in $\mathcal{A}$, take projective resolutions $P \to A$ and $P' \to A'$. By the comparison Lemma, there is a lift of δ, that is unique up to homotopy

$$\begin{tikzcd} \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\delta"] & 0 \\ \cdots \arrow[r] & P_0' \arrow[r] & A' \arrow[r] & 0. \end{tikzcd}$$

Hence, by applying F to the whole diagram, we get a lift of $F\delta$ that is also unique up to homotopy. This induces a morphism

$$L_i(\delta) : L_iF(A) \to L_iF(A').$$

This defines a functor called the *i-th left derived functor* of F.

Our next goal is to show that L_iF is indeed a well-defined additive functor, under the same hypotheses.

Lemma 3.19. *Let A be an object in $\mathcal{A}$. The object $L_iF(A)$ is well-defined, up to a canonical isomorphism.*

Proof. We will show that $L_iF(A)$ does not depend on the choice of projective resolution. Let $P \to A$ and $Q \to A$ be projective resolutions. Denote by $L_iF(-)_P$ and $L_iF(-)_Q$ the i-th left derived functor constructed using projective resolutions P and Q respectively. By the comparison Lemma, there are morphisms $f : P \to Q$ and $g : Q \to P$ lifting the identity of A. Moreover, the identity $\mathrm{id}_P : P \to P$ is also a lift of the id_A. The situation is illustrated in the following diagram, where the two squares are commutative:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"] \arrow[dd, bend right=60, "\mathrm{id}_P"'] & A \arrow[d, "\mathrm{id}_A"] \\ Q \arrow[r] \arrow[d, "g"] & A \arrow[d, "\mathrm{id}_A"] \\ P \arrow[r] & A. \end{tikzcd}$$

The comparison Lemma implies that any two lifts must be homotopic. Hence, we get

$$g \circ f \simeq \mathrm{id}_P,$$

since $g \circ f$ is also a lift of the identity. This implies

$$Fg \circ Ff \simeq F\mathrm{id}_P = \mathrm{id}_{FP}.$$

Thus,

$$L_iF(\mathrm{id}_A)_P \circ L_iF(\mathrm{id}_A)_Q = \mathrm{id}_{L_iF(A)_Q}.$$

A symmetrical argument shows that $L_iF(\mathrm{id}_A)_Q \circ L_iF(\mathrm{id}_A)_P = \mathrm{id}_{L_iF(A)_P}$, which concludes the proof. $\square$

Lemma 3.20. *Let $f : A \to A'$ be a morphism in $\mathcal{A}$. The morphism*

$$L_iF(f) : L_iF(A) \to L_iF(A')$$

is well-defined.

Proof. Take projective resolutions P and P' of A and A' respectively. Since two lifts of f are homotopic they must induce the same morphism on homology. □

Proposition 3.21. *Each L_iF is an additive functor.*

Proof. Let A be an object in $\mathcal{A}$ and let $P \to A$ be a projective resolution. Since id_P is a lift of id_A, we must have

$$L_iF(\mathrm{id}_A) = \mathrm{id}_{L_iF(A)}.$$

Now consider morphisms $f : A \to A'$ and $g : A' \to A''$ and chain maps $\tilde{f}, \tilde{g}$ lifting f and g respectively (for some chosen projective resolutions). Since $\tilde{g} \circ \tilde{f}$ is a lift of $g \circ f$ we can compute

$$L_iF(g \circ f) \cong H_iF(\tilde{g} \circ \tilde{f}) \cong H_iF(\tilde{g}) \circ H_iF(\tilde{f}) \cong L_iF(g) \circ L_iF(f).$$

Now let $f^1, f^2 : A \to A'$ be two morphisms with lifts $\tilde{f^1}$ and $\tilde{f^2}$ respectively. As the sum $\tilde{f^1} + \tilde{f^2}$ is a lift of $f^1 + f^2$, we deduce similarly that

$$L_iF(f^1 + f^2) = L_iF(f^1) + L_iF(f^2).$$ □

Theorem 3.22. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, where $\mathcal{A}$ has enough projectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive right exact functor. The left derived functors $(L_iF)_{i \geq 0}$ form a homological δ-functor.*

Proof. Let

$$0 \to A' \to A \to A'' \to 0$$

be a short exact sequence in $\mathcal{A}$ and let $P' \to A'$, $P'' \to A''$ be projective resolutions. By the Horseshoe Lemma there is a projective resolution $P \to A$ such that the sequence

$$0 \to P' \to P \to P'' \to 0$$

is split exact. Since F is additive, the sequence

$$0 \to F(P') \to F(P) \to F(P'') \to 0,$$

is also split exact. Taking the long exact sequence in homology we get

$$\cdots \to L_{n+1}F(A') \to L_{n+1}F(A) \to L_{n+1}F(A'') \xrightarrow{\partial} L_nF(A') \to L_nF(A) \to \cdots.$$

This proves the existence of a long exact sequence. Now we need to show the naturality of the connecting morphisms. Consider the diagram

$$\begin{tikzcd}
& 0 \arrow[r] & P' \arrow[dl, "\epsilon'"'] \arrow[dd, "F'"] \arrow[rr] & & P \arrow[dl, "\epsilon"'] \arrow[rr] & & P'' \arrow[dl, "\epsilon''"'] \arrow[dd, "F''"] \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[rr] \arrow[dd, "f'"'] & & A \arrow[rr] \arrow[dd, "f"'] & & A'' \arrow[r] \arrow[dd, "f''"'] & 0 & \\
& 0 \arrow[r] & Q' \arrow[rr] \arrow[dl, "\eta'"'] & & Q \arrow[rr] \arrow[dl, "\eta"'] & & Q'' \arrow[r] \arrow[dl, "\eta''"'] & 0 \\
0 \arrow[r] & B' \arrow[rr, "\iota_B"] & & B \arrow[rr, "\pi_B"] & & B'' \arrow[r] & 0, &
\end{tikzcd}$$

where the front of the diagram is given and the two front squares commutes, $\epsilon' : P' \to A'$, $\epsilon'' : P'' \to A''$, $\eta' : Q' \to B'$ and $\eta'' : Q'' \to B''$ are any projective resolutions, $\epsilon : P \to A$, $\eta : Q \to B$ are given by the

Horseshoe Lemma and $F' : P' \to Q'$, $F'' : P'' \to Q''$ are lifts of f' and f'' respectively. We will construct $F : P \to Q$ such that

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d, "F'"] & P \arrow[r] \arrow[d, "F"] & P'' \arrow[r] \arrow[d, "F''"] & 0 \\
0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0
\end{tikzcd}$$

commutes. Remember that $P_n = P'_n \oplus P''_n$ and $Q_n = Q'_n \oplus Q''_n$ for every $n \in \mathbb{N}$ by the Horseshoe Lemma. Hence we can define

$$F_n : P'_n \oplus P''_n \to Q'_n \oplus Q''_n$$

by the matrix

$$\begin{pmatrix} F'_n & \gamma_n \\ 0 & F''_n \end{pmatrix},$$

where $\gamma_n : P''_n \to Q'_n$ will be defined inductively. For F to be a lifting of f we must have $\eta F_0 - f\epsilon = 0$. Denote by λ_P and λ_Q the restrictions of ϵ and η to P''_0 and Q''_0 respectively. Then the equality

$$\iota_B \eta' \gamma_0 = -\lambda_Q f''_0 + f\lambda_P$$

must hold. For more clarity, define $g_0 := -\lambda_Q f''_0 + f\lambda_P$. By diagram chasing, one can see that $\pi_B \circ g_0 = 0$. Hence, since $\mathrm{Hom}(P''_0, -)$ is exact (as P''_0 is projective), there exists a map $\beta : P''_0 \to B'$ such that $\iota_B \circ \beta = g_0$. We can then define γ_0 to be a lift of β

$$\begin{tikzcd}
 & P''_0 \arrow[dl, "\gamma_0"'] \arrow[d, "\beta"] & \\
Q'_0 \arrow[r, "\eta'"] & B' \arrow[r] & 0.
\end{tikzcd}$$

We now want to define γ_n for any $n \in \mathbb{N}$. Assume that there is a $n \in \mathbb{N}$ such that γ_i is defined for every $i \in \{0, \dots, n-1\}$. For F to be a chain map, we must have the equality

$$\begin{aligned} dF - Fd &= \left[\begin{pmatrix} d' & \lambda \\ 0 & d'' \end{pmatrix}, \begin{pmatrix} F' & \gamma \\ 0 & F'' \end{pmatrix} \right] \\ &= \begin{pmatrix} d'F' - F'd' & (d'\gamma - \gamma d'' + \lambda F'' - F'\lambda') \\ 0 & d''F'' - F''d'' \end{pmatrix} = 0, \end{aligned}$$

where $[\cdot, \cdot]$ is the matrix commutator. Since both F' and F'' are chain maps, the only condition that γ_n must satisfy is

$$d'\gamma_n = \gamma_{n-1}d'' - \lambda_n F'_n + F''_{n-1}\lambda_n.$$

Define $g_n := \gamma_{n-1}d'' - \lambda_n F'_n + F'_{n-1}\lambda_n$. Using diagram chasing, one can show that $d'g_n = 0$. Since Q' is exact, g_n factors through a map $\beta : P''_n \to d'(Q'_n)$. Hence we can define γ_n to be a lift of β

$$\begin{tikzcd}
 & P''_n \arrow[dl, "\gamma_0"'] \arrow[d, "\beta"] & \\
Q'_n \arrow[r, "\eta'"] & d'(Q'_n) \arrow[r] & 0.
\end{tikzcd}$$

This concludes the proof. $\square$

Before proving our next result, we will need to briefly discuss the notion of pullback.

Definition 3.23. Let $\mathcal{A}$ be an abelian category and let $f : B \to A$, $g : C \to A$ be morphisms. Consider the morphism

$$\varphi := (f, -g) : B \oplus C \to A,$$

induced by the universal property of the coproduct applied to f and $-g$. The *pullback* of f and g is defined to be the kernel $P = \ker(\varphi)$, with induced morphisms $P \to B$ and $P \to C$ that make the diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & B \arrow[d, "f"] \\
C \arrow[r, "g"] & A
\end{tikzcd}$$

commutes. The key property used in the following proof that will not be discussed here is that if f is an epimorphism, so is the morphism $P \to C$. Similarly, if g is an epimorphism, so is the morphism $P \to B$.

Theorem 3.24. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $(L_i)_{i \geq 0}$ be a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and assume that for all objects A in $\mathcal{A}$ and for all $i > 0$ there is an epimorphism $u : P \to A$ such that $L_i(u) = 0$. Then $(L_i)_{i \geq 0}$ is universal.*

Remark 3.25. In particular, if $\mathcal{A}$ has enough projectives, any left derived functor is universal. One can take P projective with an epimorphism $u : P \to A$. Since $L_i(P) = 0$, the map $L_i(P) \to L_iF(A)$ has to be the zero map.

Proof. Let $(T_i)_{i \geq 0}$ be another homological δ-functor and let $\varphi_0 : T_0 \to L_0$ be a natural transformation. We construct for every $n > 0$ a natural transformation $\varphi_n : T_n \to L_n$ by induction. Suppose that there is some $n \in \mathbb{N}$ such that we constructed $\varphi_i : T_i \to L_i$ for every $i \in \{0, \dots, n-1\}$. Let A be an object in $\mathcal{A}$. By assumption, we can choose a short exact sequence

$$0 \to K \to P \to A \to 0,$$

with $P \to A$ as in the statement. Consider the diagram with exact rows

$$\begin{tikzcd}
 & T_n(A) \arrow[r, "\delta"] & T_{n-1}(K) \arrow[r, "\epsilon"] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(P) \arrow[d, "\varphi_{n-1}"] \\
L_n(P) \arrow[r, "0"] & L_n(A) \arrow[r, hook, "\iota"] & L_{n-1}(K) \arrow[r, "\eta"] & L_{n-1}(P).
\end{tikzcd}$$

Using Freyd-Mitchell, we can define $\varphi_n : T_n(A) \to L_n(A)$ on elements. Let $a \in T_n(A)$. By the commutativity of the right square and exactness of the top row we have

$$\eta \circ \varphi_{n-1} \circ \delta(a) = \varphi_{n-1} \circ \epsilon \circ \delta(a) = 0.$$

By exactness of the bottom row, $\varphi_{n-1} \circ \delta(a)$ is in the image of ι and we can define

$$\varphi_n(a) := \iota^{-1} \circ \varphi_{n-1} \circ \delta(a).$$

Moreover, one can show that this map is the unique map making the above diagram commute. Let us assume for now that φ_n is well-defined (this will make sense later). Let $f : A' \to A$ be a morphism and consider the pullback

$$\begin{tikzcd}
P' \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{P} \arrow[d] \\
 & A' \arrow[d, "f"] \\
P \arrow[r] & A,
\end{tikzcd}$$

where the map $\widetilde{P} \to A'$ is an epimorphism given by the assumption. Notice that the map $P' \to \widetilde{P}$ is an epimorphism as $P \to A$ is an epimorphism. Thus we get a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

As both $(L_i)_{i\geq 0}$ and $(T_i)_{i\geq 0}$ are homological δ-functors, each small quadrilateral commutes in the following diagram:

$$\begin{tikzcd}
T_n(A') \arrow[rrr, "T_n(f)"] \arrow[dr, "\delta"] \arrow[ddd, "\varphi_n(A')"'] & & & T_n(A) \arrow[dl, "\delta"'] \arrow[ddd, "\varphi_n(A)"] \\
& T_{n-1}(K') \arrow[r] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(K) \arrow[d, "\varphi_{n-1}"] & \\
& L_{n-1}(K') \arrow[r] & L_{n-1}(K) & \\
L_n(A') \arrow[rrr, "L_n(f)"] \arrow[ur, "\delta"] & & & L_n(A). \arrow[ul, "\delta"']
\end{tikzcd}$$

Thus, one can see by diagram chasing that the following holds

$$\delta \circ L_n(f) \circ \varphi_n(A') = \delta \circ \varphi_n(A) \circ T_n(f).$$

Since $\delta : L_n(A) \to L_{n-1}(K)$ is monic, we can cancel the above on the left to see the naturality of φ_n. By taking $A = A'$ and $f = \mathrm{id}_A$ this also shows that $\varphi_n(A)$ does not depend on the choice of P, which shows that it is well-defined.

We now want to show that φ_n commutes with δ_n. Let

$$0 \to A' \to A \to A'' \to 0$$

be a short exact sequence. Let $P \to A$ and $\widetilde{P} \to A''$ be epimorphisms given by the assumption. Similarly as before, the pullback

$$\begin{tikzcd}
P'' \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \widetilde{P} \arrow[d] \\
P \arrow[r] & A \arrow[r] & A''
\end{tikzcd}$$

yields a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K'' \arrow[r] \arrow[d, "g"] & P'' \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d, "\mathrm{id}_{A''}"] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0.
\end{tikzcd}$$

Thus, we get another commutative diagram

$$\begin{tikzcd}
T_n(A'') \arrow[r, "\delta"] \arrow[d, "\varphi_n"] & T_{n-1}(K'') \arrow[r, "T(g)"] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(A') \arrow[d, "\varphi_{n-1}"] \\
L_n(A'') \arrow[r, "\delta"] & L_{n-1}(K'') \arrow[r, "L(g)"] & L_{n-1}(A'),
\end{tikzcd}$$

where the right square commutes by naturality of φ_{n-1} and the left square commutes by construction of φ_n. Since $(T_i)_{i\geq 0}$ and $(L_i)_{i\geq 0}$ are both homological δ-functors and by construction of the above morphism of short exact sequences, the horizontal composites are their respective δ maps. This yields the desired commutative relation between δ and φ. ☐

3.5 Injective resolutions

Definition 3.26. An object I in an abelian category $\mathcal{A}$ is *injective* if for any monic $f : A \to B$ and any morphism $\alpha : A \to I$, there exists a morphism $\beta : B \to I$ such that $\alpha = \beta \circ f$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[dl, "\beta"] \\
& I &
\end{tikzcd}$$

The following proposition is immediate and thus, given without proof.

Proposition 3.27. *Let $\mathcal{A}$ be an abelian category and let I be an object in $\mathcal{A}$. Then, I is injective in $\mathcal{A}$ if and only if I is projective in $\mathcal{A}^{\mathrm{op}}$.*

Corollary 3.28. *Let $\mathcal{A}$ be an abelian category and let I be an object in $\mathcal{A}$. Then, I is injective if and only if the functor $\operatorname{Hom}_{\mathcal{A}}(-, I)$ is exact.*

From this proposition, one can dualize several definitions and results from the projective context. For example, we say that an abelian category $\mathcal{A}$ has *enough injectives* if for every object A in $\mathcal{A}$ there is a monic $A \to I$ for some injective object I in $\mathcal{A}$. Similarly, there is a version of the comparison Lemma in the injective context. Moreover, one has the following criterion for the category of right R-modules:

Proposition 3.29 (Baer's Criterion)**.** *A right R-module E is injective if and only if for every right ideal $J \subseteq R$, every map $J \to E$ can be extended to a map $R \to E$.*

Proof. The proof is omitted as it is typically done in a commutative algebra course. ☐

We now wish to show that the category R-**mod** has enough injectives. A first step towards this result is the following important example.

Example 3.30. By Baer's criterion, the abelian group $\mathbb{Q}/\mathbb{Z}$ is injective, see Exercise 6.1.

Lemma 3.31. *Let M be a R-module and let A be an abelian group. The canonical morphism*

$$\tau : \operatorname{Hom}_{\mathbf{Ab}}(M, A) \to \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbf{Ab}}(R, A)),$$

defined for any $m \in M$ by

$$\begin{aligned} \tau f(m) : R &\to A \\ r &\mapsto f(mr), \end{aligned}$$

is an isomorphism.

Proof. The inverse is given by

$$\eta : \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbf{Ab}}(R, A)) \to \operatorname{Hom}_{\mathbf{Ab}}(M, A),$$

where

$$\begin{aligned} \eta(g) : M &\to A \\ m &\mapsto g(m)(1). \end{aligned}$$

It is easy to check that these two maps are mutual inverses. ☐

Proposition 3.32. *Let $R : \mathcal{B} \to \mathcal{A}$ be an additive functor that is right adjoint to some exact functor. Then, for any injective object I in $\mathcal{B}$, the object $R(I)$ is also injective.*

Proof. Denote by $L : \mathcal{A} \to \mathcal{B}$ the left adjoint to R. It suffices to show that $\operatorname{Hom}_{\mathcal{A}}(-, R(I))$ is exact. Given a monic $f : A \to A'$ in $\mathcal{A}$, the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}(A', R(I)) \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_{\mathcal{A}}(A, R(I)) \arrow[d, "\cong"] \\
\operatorname{Hom}_{\mathcal{B}}(L(A'), I) \arrow[r] & \operatorname{Hom}_{\mathcal{B}}(L(A), I)
\end{tikzcd}$$

commutes. Since L is exact and I is injective, the bottom map is surjective. Hence, the top map is also surjective and we are done. ☐

Corollary 3.33. *If I is an injective abelian group, then $\operatorname{Hom}_{\mathbf{Ab}}(R, I)$ is an injective R-module.*

Proof. The functor $\mathrm{Hom}_{\mathbf{Ab}}(R,-)$ is right adjoint to the forgetful functor

$$U : R\text{-}\mathbf{mod} \to \mathbf{Ab}. \qquad \square$$

Proposition 3.34. *The category R-**mod** has enough injectives.*

Proof. Let M be a right R-module. Define

$$I(M) := \prod_{\mathrm{Hom}_R(M,I_0)} I_0,$$

where $I_0 = \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{R}, \mathbb{Q}/\mathbb{Z})$ is injective, since $\mathbb{Q}/\mathbb{Z}$ is injective. One can verify that a product of injective objects is injective. Now define the morphism

$$\begin{aligned} \iota : M &\to I(M) \\ m &\mapsto \prod_{\varphi \in \mathrm{Hom}_R(M,I_0)} \varphi(m). \end{aligned}$$

We show that it is an injective morphism. Let $m \in M \setminus \{0\}$ and consider the subgroup generated by m. It satisfies either $\langle m \rangle \cong \mathbb{Z}/n\mathbb{Z}$ for some $n > 1$, or $\langle m \rangle \cong \mathbb{Z}$. In the first case, define a morphism of abelian groups

$$\begin{aligned} \langle m \rangle &\to \mathbb{Q}/\mathbb{Z} \\ m &\mapsto \frac{1}{n}. \end{aligned}$$

By the injectivity of $\mathbb{Q}/\mathbb{Z}$, it extends to a morphism of groups $\gamma : M \to \mathbb{Q}/\mathbb{Z}$. Thus, through the identification $\mathrm{Hom}_{\mathbf{Ab}}(M, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(M, I_0)$, we get an element $\varphi \in \mathrm{Hom}_R(M, I_0)$ such that $\varphi(m)(1) = \gamma(m) \neq 0$. In particular, $\iota(m) \neq 0$. If $\langle m \rangle \cong \mathbb{Z}$, the exact same proof works by extending the morphism

$$\begin{aligned} \langle m \rangle &\to \mathbb{Q}/\mathbb{Z} \\ m &\mapsto \frac{1}{2}. \end{aligned}$$

This concludes the proof. $\square$

3.6 Exercises Week 6

(i) Use the Baer Criterion to show that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, and then give an injective resolution of $\mathbb{Z}$.

(ii) For A an abelian group, we define its *Pontrjagin dual* as :

$$A^* = \mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}/\mathbb{Z}).$$

Show that when A is finite, we have $(A^*)^* \cong A$, and deduce that there is an isomorphism of categories $\mathrm{FinAb} \cong \mathrm{FinAb}^{\mathrm{op}}$, where FinAb is the category of finite abelian groups. However, find an abelian group such that $(A^*)^*$ is not isomorphic to A.

(iii) Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor and $U : \mathcal{B} \to \mathcal{C}$ be an exact functor. Show that we have a natural isomorphism :

$$L_i(UF) \cong U(L_i F).$$

(iv) Let R be a commutative ring, and N an R-module. Show or recall that $- \otimes_R N : \text{R-}\mathbf{Mod} \to \text{R-}\mathbf{Mod}$ is right exact. We denote by $\mathrm{Tor}_i^R(-, N)$ the associated left derived functor.
Find a projective resolution of $\mathbb{Z}/n\mathbb{Z}$, and use it to compute $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for every $i \geq 0$ and $m \in \mathbb{Z}$.

3.7 Right derived functors

Last time we saw that if $F : \mathcal{A} \to \mathcal{B}$ is an additive right exact functor between abelian categories and $\mathcal{A}$ has enough projectives, then one can define the left derived functors L_*F. Dually, under certain conditions, we may define right derived functors of additive left exact functors.

Definition 3.35. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and assume that $\mathcal{A}$ has enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive left exact functor and let A be an object in $\mathcal{A}$. Fix an injective resolution $A \to I^\bullet$ and define the i-th right derived functor of F evaluated at A by

$$R^iF(A) := H^i(F(I^\bullet)).$$

Note that since

$$0 \to F(A) \to F(I^0) \to F(I^1)$$

is exact, we always have $R^0F(A) \cong F(A)$.
As F also defines a right exact functor $F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$, and $\mathcal{A}^{\mathrm{op}}$ has enough projectives if $\mathcal{A}$ has enough injectives, we can construct the left derived functors L_iF^{op} as well. Since $I^\bullet$ becomes a projective resolution of A in $\mathcal{A}^{\mathrm{op}}$, we observe that

$$R^iF(A) \cong (L_iF^{\mathrm{op}})^{\mathrm{op}}(A).$$

Therefore all results about right exact functors apply to left exact functors. In particular, the objects $R^iF(A)$ are independent of the choice of injective resolutions, R^*F is a universal cohomological δ-functor, and $R^iF(I) = 0$ for $i \neq 0$ whenever I is injective.

3.8 Tor and Ext

As a consequence of the previous section, given an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, we know how to construct left derived functors of F if it is right exact and right derived functors of F if it is left exact under the condition that $\mathcal{A}$ has enough projectives respectively enough injectives. In this section we will encounter two famous examples of derived functors.

3.9 Balancing Tor and Ext

The following proposition is an important source of right and left exact functors.

Proposition 3.36. *Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be an adjoint pair of additive functors between abelian categories. That is, for every object A of the category $\mathcal{A}$ and every object B of $\mathcal{B}$ there is a natural isomorphism*

$$\tau_{A,B} : Hom_{\mathcal{B}}(L(A), B) \xrightarrow{\cong} Hom_{\mathcal{A}}(A, R(B)).$$

Then L is right exact and R is left exact.

Proof. Please see exercise 3 of sheet 2. ☐

This implies that left adjoints have left derived functors, and right adjoints have right derived functors. This of course assumes that $\mathcal{A}$ has enough projectives, and $\mathcal{B}$ has enough injectives for the derived functors to be defined.

Proposition 3.37. *Let R and S be rings. If B is an R-S-bimodule and C is a right S-module, then $Hom_S(B, C)$ is naturally a right R-module via*

$$(fr)(b) = f(rb)$$

for $f \in Hom_S(B, C)$, $r \in R$ and $b \in B$. The functor $Hom_S(B, -)$ from $\mathbf{mod}-S$ to $\mathbf{mod}-R$ is right adjoint to $- \otimes_R B$.

Proof. We need to show that for every R-module A and S-module C there is a natural isomorphism

$$\tau_{A,C} : \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)).$$

Given $f : A \otimes_R B \to C$, we define $(\tau_{A,C} f)(a)$ as the map $b \mapsto f(a \otimes b)$ for each $a \in A$. Given $g : A \to \mathrm{Hom}_S(B, C)$, we define $\tau_{A,C}^{-1} g$ to be the map defined by the bilinear form $a \otimes b \mapsto g(a)(b)$. The verification that $(\tau_{A,C} f)(a)$ is an S-module map, $\tau_{A,C} f$ is an R-module map, $\tau_{A,C}^{-1} g$ is an S-module map, $\tau_{A,C}$ is an isomorphism with inverse $\tau_{A,C}^{-1}$, and that $\tau_{A,C}$ is natural are left as an exercise for the reader. ☐

As a consequence of Proposition 3.36 and Proposition 3.37 we note that the functor $- \otimes_R B$ is right exact for any left R-module B. This justifies the following definition.

Definition 3.38. Let R be a ring and B a left R-module. Then $T(A) = A \otimes_R B$ is a right exact functor from $\mathbf{mod} - R$ to $\mathbf{Ab}$. We define the abelian groups

$$\mathrm{Tor}_n^R(A, B) = (L_n T)(A).$$

In particular, $\mathrm{Tor}_0^R(A, B) \cong A \otimes_R B$.

Recall that these groups are computed by finding a projective resolution $P_\bullet \to A$ and taking the homology of $P_\bullet \otimes_R B$. Hence if A is a projective R-module, then $\mathrm{Tor}_n^R(A, B) = 0$ for $n \neq 0$.
The reader may notice that the functor $A \otimes_R -$ is also right exact for any right R-module A. So we could also form the derived functors $L_*(A \otimes_R -)$. In fact, one can show that this yields nothing new in the sense that

$$L_*(A \otimes_R -)(B) \cong L_*(- \otimes_R B)(A),$$

as it is shown in the next theorem.

Theorem 3.39. *Let R be a ring, A a right R-module and B a left R-module. Then*

$$L_n(A \otimes_R -)(B) \cong L_n(- \otimes_R B)(A) = \mathrm{Tor}_n^R(A, B)$$

for all n.

Before we are able to prove this theorem we have to introduce a few definitions and an important lemma.

Definition 3.40. Let R be a ring. Suppose $P_\bullet$ and $Q_\bullet$ are chain complexes of right and left R-modules, respectively. Form the double complex

$$P_\bullet \otimes_R Q_\bullet$$

with the family $\{P_p \otimes_R Q_q\}_{p,q}$ of objects using the sign trick, that is, with horizontal differentials $d^P \otimes \mathrm{id}$ and vertical differentials $(-1)^p \mathrm{id} \otimes d^Q$. The double complex $P_\bullet \otimes_R Q_\bullet$ is called the tensor product double complex, and $\mathrm{Tot}^\oplus(P_\bullet \otimes_R Q_\bullet)$ is called the (total) tensor product chain complex of $P_\bullet$ and $Q_\bullet$.

Lemma 3.41 (Acyclic Assembly Lemma)**.** *Let R be a ring and $C_{\bullet,\bullet}$ be a double complex in* $\mathbf{mod} - R$. *Then*

- $\mathrm{Tot}^\Pi(C_{\bullet,\bullet})$ *is an acyclic chain complex, assuming either of the following:*
  - *(i) $C_{\bullet,\bullet}$ is an upper half-plane complex with exact columns.*
  - *(ii) $C_{\bullet,\bullet}$ is a right half-plane complex with exact rows.*
- $\mathrm{Tot}^\oplus(C_{\bullet,\bullet})$ *is an acyclic chain complex, assuming either of the following:*
  - *(iii) $C_{\bullet,\bullet}$ is an upper half-plane complex with exact rows.*
  - *(iv) $C_{\bullet,\bullet}$ is a right half-plane complex with exact columns.*

Proof. We first show that it suffices to establish case (i). Interchanging rows and columns also interchanges (i) and (ii), as well as (iii) and (iv), so (i) implies (ii) and (iv) implies (iii). Suppose we are in case (iv), and let $\tau_n C_{\bullet,\bullet}$ be the double subcomplex of $C_{\bullet,\bullet}$ obtained by truncating each column at level n:

$$(\tau_n C)_{p,q} = \begin{cases} C_{p,q} & \text{if } q > n \\ \ker(d^v : C_{p,n} \to C_{p,n-1}) & \text{if } q = n \\ 0 & \text{if } q < n. \end{cases}$$

Each $\tau_n C_{\bullet,\bullet}$ is, up to vertical translation, a first quadrant double complex with exact columns, so (i) implies that

$$\mathrm{Tot}^{\oplus}(\tau_n C_{\bullet,\bullet}) = \mathrm{Tot}^{\Pi}(\tau_n C_{\bullet,\bullet})$$

is acyclic. This implies that $\mathrm{Tot}^{\oplus}(C_{\bullet,\bullet})$ is acyclic, as every cycle of $\mathrm{Tot}^{\oplus}(C_{\bullet,\bullet})$ is a cycle (and hence also a boundary) in some subcomplex $\mathrm{Tot}^{\oplus}(\tau_n C_{\bullet,\bullet})$. Therefore (i) implies (iv) as well.

We are left to prove case (i). Translating $C_{\bullet,\bullet}$ left and right, it suffices to prove that $H_0(\mathrm{Tot}^{\Pi}(C_{\bullet,\bullet}))$ is zero. Let

$$c = (\cdots, c_{-p,p}, \cdots, c_{-2,2}, c_{-1,1}, c_{0,0}) \in \prod_{p \geq 0} C_{-p,p} = \mathrm{Tot}^{\Pi}(C_{\bullet,\bullet})_0$$

be a 0-cycle. Our goal is to find elements $b_{-p,p+1}$ by induction on p so that

$$d^v(b_{-p,p+1}) + d^h(b_{-p+1,p}) = c_{-p,p}.$$

Assembling those elements will yield an element b of $\prod_{p \geq -1} C_{-p,p+1}$ such that $d(b) = c$, proving that $H_0(\mathrm{Tot}^{\Pi}(C_{\bullet,\bullet})) = 0$. The following schematic should help give the idea.

$$\begin{tikzcd}
b_{-p,p+1} \arrow[d, mapsto] & & & & & \\
c_{-p,p} & b_{-p+1,p} \arrow[l, mapsto] \arrow[d, mapsto] & & & & \\
 & c_{-p+1,p-1} & b_{-p+2,p-1} \arrow[l, mapsto] \arrow[d, mapsto] & & & \\
 & & \vdots \arrow[d, mapsto] & & & \\
 & & c_{-2,2} & b_{-1,2} \arrow[l, mapsto] \arrow[d, mapsto] & & \\
 & & & c_{-1,1} & b_{0,1} \arrow[l, mapsto] \arrow[d, mapsto] & \\
 & & & & c_{0,0} \arrow[d, mapsto] & b_{1,0} = 0 \arrow[l, mapsto] \\
 & & & & 0 &
\end{tikzcd}$$

We begin the induction by choosing $b_{1,0} = 0$ for $p = -1$. Since $C_{0,-1} = 0$, we have $d^v(c_{0,0}) = 0$ and because the 0-th column is exact, there is $b_{0,1} \in C_{0,1}$ such that $d^v(b_{0,1}) = c_{0,0}$. Inductively, we compute that

$$\begin{aligned} d^v(c_{-p,p} - d^h(b_{-p+1,p})) &= d^v(c_{-p,p}) + d^h d^v(b_{-p+1,p}) \\ &= d^v(c_{-p,p}) + d^h(c_{-p+1,p-1}) - d^h d^h(b_{-p+2,p-1}) \\ &= 0 \end{aligned}$$

where we used the properties of the differentials of a double complex in the first equality, the induction hypothesis in the second equality and the fact that c is a cycle in the last equality. Since the $-p$-th column is exact by assumption, there is $b_{-p,p+1} \in C_{-p,p+1}$ such that

$$d^v(b_{-p,p+1}) = c_{-p,p} - d^h(b_{-p+1,p})$$

as desired. This concludes the induction and hence the proof. $\square$

Now we are ready to tackle the proof of Theorem 3.39.

Proof of Theorem 3.39. Choose a projective resolution $P_\bullet \xrightarrow{\epsilon} A$ in $\mathbf{mod} - R$ and a projective resolution $Q_\bullet \xrightarrow{\eta} B$ in $R - \mathbf{mod}$. Thinking of A and B as complexes concentrated in degree zero, we can form the three tensor product double complexes $P_\bullet \otimes_R Q_\bullet$, $A \otimes_R Q_\bullet$ and $P_\bullet \otimes_R B$. The augmentation maps ϵ and η induce maps from $P_\bullet \otimes_R Q_\bullet$ to $A \otimes_R Q_\bullet$ and $P_\bullet \otimes_R B$. The situation is represented in the following diagram.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \\
A \otimes_R Q_2 \arrow[d, "\mathrm{id}\otimes d"'] & P_0 \otimes_R Q_2 \arrow[l, bend right, "\epsilon\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes d"'] & P_1 \otimes_R Q_2 \arrow[l, "d\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes -d"'] & \cdots \arrow[l] \arrow[d] & \\
A \otimes_R Q_1 \arrow[d, "\mathrm{id}\otimes d"'] & P_0 \otimes_R Q_1 \arrow[l, bend right, "\epsilon\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes d"'] & P_1 \otimes_R Q_1 \arrow[l, "d\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes -d"'] & P_2 \otimes_R Q_1 \arrow[l, "d\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes d"'] & \cdots \arrow[l] \\
A \otimes_R Q_0 & P_0 \otimes_R Q_0 \arrow[l, bend right, "\epsilon\otimes\mathrm{id}"'] \arrow[d, bend right, "\mathrm{id}\otimes\eta"] & P_1 \otimes_R Q_0 \arrow[l, "d\otimes\mathrm{id}"'] \arrow[d, bend right, "\mathrm{id}\otimes\eta"] & P_2 \otimes_R Q_0 \arrow[l, "d\otimes\mathrm{id}"'] \arrow[d, bend right, "\mathrm{id}\otimes\eta"] & \cdots \arrow[l] \\
 & P_0 \otimes_R B & P_1 \otimes_R B \arrow[l, "d\otimes\mathrm{id}"'] & P_2 \otimes_R B \arrow[l, "d\otimes\mathrm{id}"'] & \cdots \arrow[l]
\end{tikzcd}$$

Note that the diagram doesn't represent one big double complex, however there are the three distinct double complexes $P_\bullet \otimes_R Q_\bullet$, $A \otimes_R Q_\bullet$ and $P_\bullet \otimes_R B$ and the curved arrows represent double complex morphisms. Using the Acyclic Assembly Lemma 3.41, we will show that the maps

$$\mathrm{Tot}^\oplus(P_\bullet \otimes_R Q_\bullet) \xrightarrow{\epsilon\otimes\mathrm{id}_{Q_\bullet}} \mathrm{Tot}^\oplus(A \otimes_R Q_\bullet) = A \otimes_R Q_\bullet$$

and

$$\mathrm{Tot}^\oplus(P_\bullet \otimes_R Q_\bullet) \xrightarrow{\mathrm{id}_{P_\bullet}\otimes\eta} \mathrm{Tot}^\oplus(P_\bullet \otimes_R B) = P_\bullet \otimes_R B$$

are quasi-isomorphisms, inducing the promised isomorphisms on homology as wanted

$$L_*(A \otimes_R -)(B) \xleftarrow{\cong} H_*(\mathrm{Tot}^\oplus(P_\bullet \otimes_R Q_\bullet)) \xrightarrow{\cong} L_*(- \otimes_R B)(A).$$

We consider the double complex $C_{\bullet,\bullet}$ one obtains from $P_\bullet \otimes_R Q_\bullet$ by adding $(A \otimes Q_\bullet)[-1]$ in the column $p = -1$.

The translate $\mathrm{Tot}^\oplus(C_{\bullet,\bullet})[-1]$ is the mapping cone of the map $\epsilon \otimes \mathrm{id}_{Q_\bullet}$ from $\mathrm{Tot}^\oplus(P_\bullet \otimes_R Q_\bullet)$ to $A \otimes_R Q_\bullet$ because

$$\begin{aligned}
\mathrm{Tot}^\oplus(C_{\bullet,\bullet})[-1]_n &= \mathrm{Tot}^\oplus(C_{\bullet,\bullet})_{n-1} \\
&= \mathrm{Tot}^\oplus(P_\bullet \otimes_R Q_\bullet)_{n-1} \oplus (A \otimes_R Q_n) \\
&= \mathrm{cone}(\epsilon \otimes \mathrm{id}_{Q_\bullet})_n
\end{aligned}$$

for all n. Thus in order to show that $\epsilon \otimes \mathrm{id}_{Q_\bullet}$ is a quasi-isomorphism, it suffices to show that $\mathrm{Tot}^\oplus(C_{\bullet,\bullet})$ is acyclic by Proposition 3.12 of the lecture notes of week 4. Since each $- \otimes_R Q_q$ is an exact functor (Q_q is projective), every row of $C_{\bullet,\bullet}$ is exact, so $\mathrm{Tot}^\oplus(C_{\bullet,\bullet})$ is exact by the Acyclic Assembly Lemma 3.41.

Similarly, the mapping cone of $\mathrm{id}_{P_\bullet} \otimes \eta : \mathrm{Tot}^\oplus(P_\bullet \otimes_R Q_\bullet) \to P_\bullet \otimes_R B$ is a translate of the complex $\mathrm{Tot}^\oplus(B_{\bullet,\bullet})$, where $D_{\bullet,\bullet}$ is the double complex obtained from $P_\bullet \otimes_R Q_\bullet$ by adding $(P_\bullet \otimes_R B)[-1]$ in the row $q = -1$. Since each $P_p \otimes_R -$ is an exact functor (P_p is projective), every column of $D_{\bullet,\bullet}$ is exact, so $\mathrm{Tot}^\oplus(D_{\bullet,\bullet})$ is exact by the Acyclic Assembly Lemma 3.41. Hence $\mathrm{cone}(\mathrm{id}_{P_\bullet} \otimes \eta)$ is acyclic, and thus $\mathrm{id}_{P_\bullet} \otimes \eta$ is a quasi-ismomorphism as well. This concludes the proof. □

Another consequence of Proposition 3.36 and Proposition 3.37 is the fact that for each R-module A the functor $\mathrm{Hom}_R(A, -)$ is left exact. This motivates the next definition.

Definition 3.42. Let R be a ring and A an R-module. Then $F(B) = \mathrm{Hom}_R(A,B)$ is a left exact functor from $\mathbf{mod} - R$ to $\mathbf{Ab}$. We define the abelian groups

$$\mathrm{Ext}^n_R(A,B) = R^n\mathrm{Hom}_R(A,-)(B).$$

In particular, $\mathrm{Ext}^0_R(A,B) = \mathrm{Hom}_R(A,B)$.

Recall that these groups are computed by finding an injective resolution $B \to I^\bullet$ and taking homology of $\mathrm{Hom}_R(A, I^\bullet)$.
The notion of derived functor has obvious variations for contravariant functors. For example, let F be a contravariant left exact functor from $\mathcal{A}$ to $\mathcal{B}$. This is the same as a covariant left exact functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$, so if $\mathcal{A}$ has enough projectives, we can define the right derived functors $R^*F(A)$ where A is an object of $\mathcal{A}$ to be the cohomology of $F(P^\bullet)$, $P^\bullet \to A$ being a projective resolution in $\mathcal{A}$. This is a universal δ-functor as well with $R^0F(A) \cong F(A)$, and $R^iF(P) = 0$ for $i \neq 0$ whenever P is projective.
One may notice that the functor $\mathrm{Hom}_R(-,B)$ is contravariant and left exact for each R-module B. It is therefore entitled to right derived functors $R^*\mathrm{Hom}_R(-,B)(A)$ where A is also an R-module. The next theorem shows that this yields nothing new in the sense that

$$R^*\mathrm{Hom}_R(-,B)(A) \cong R^*\mathrm{Hom}_R(A,-)(B) = \mathrm{Ext}^*_R(A,B).$$

Theorem 3.43. *Let R be a ring and A, B a pair of R-modules. Then*

$$\mathrm{Ext}^n_R(A,B) = R^n\mathrm{Hom}(A,-)(B) \cong R^n\mathrm{Hom}(-,B)(A)$$

for all n.

We omit the proof of this theorem, as it is similar to the one of Theorem 3.39.
In view of Theorem 3.39 and Theorem 3.43, the following definition seems natural.

Definition 3.44. Let R be a ring and T a left exact functor of p "variable" R-modules, some covariant and some contravariant. The functor T is called right balanced if it meets the following conditions:

(i) When any one of the covariant variables of T is replaced by an injective module, T becomes an exact functor in each of the remaining variables.

(ii) When any one of the contravariant variables of T is replaced by a projective module, T becomes an exact functor in each of the remaining variables.

The functor $\mathrm{Hom}_R(-,-)$ is an example of a right balanced functor which is contravariant in its first variable and covariant in its second variable. Another example is provided by the functor $\mathrm{Hom}_R(- \otimes_R -, -)$ since it is naturally isomorphic to $\mathrm{Hom}_R(-, \mathrm{Hom}_R(-,-))$ by Proposition 3.37.
The following result which we won't prove explains the interest behind the definition of right balanced functors.

Proposition 3.45. *Let R be a ring and T a right balanced functor of p variable R-modules. Then all p of the right derived functors*

$$R^*T(A_1, \cdots, \hat{A}_i, \cdots, A_p)(A_i)$$

of T are naturally isomorphic where $A_1, \ldots, A_p$ are R-modules.

A similar discussion applies to right exact functors T which are left balanced. The prototype left balanced functor is $- \otimes_R -$. In particular all of the left derived functors associated to a left balanced functor are naturally isomorphic.

3.10 Tor and flatness

We saw previously that if A is a right R-module and B is a left R-module, then $\mathrm{Tor}^R_*(A,B)$ may be computed either as the left derived functors of $A \otimes_R -$ evaluated at B or as the left derived functors of $- \otimes_R B$ evaluated at A. It follows that if either A or B is projective, then $\mathrm{Tor}^R_n(A,B) = 0$ for $n \neq 0$.

Definition 3.46. Let R be a ring. A left R-module B is flat if the functor $- \otimes_R B$ is exact. Similarly, a right R-module A is flat if the functor $A \otimes_R -$ is exact.

The above remarks show that projective modules are flat. The example $R = \mathbb{Z}$, $B = \mathbb{Q}$ shows that flat modules need not be projective. We will now state three important lemmas concerning flat R-modules.

Lemma 3.47. *Let R be a ring. The following are equivalent for every left R-module B:*

(i) *B is flat.*

(ii) $\operatorname{Tor}_n^R(A, B) = 0$ *for all $n \neq 0$ and all right R-modules A.*

(iii) $\operatorname{Tor}_1^R(A, B) = 0$ *for all right R-modules A.*

Proof. Please see exercise 1 of sheet 7. ☐

Lemma 3.48. *Let R be a ring. If*

$$0 \to A \to B \to C \to 0$$

is an exact sequence of R-modules and both B and C are flat, then A is also flat.

Proof. Please see exercise 1 of sheet 7. ☐

Lemma 3.49 (Flat Resolution Lemma)**.** *Let R be a ring, A a right R-module and B a left R-module. The abelian groups $\operatorname{Tor}_*^R(A, B)$ may be computed using resolutions by flat modules. That is, if $F_\bullet \to A$ is a resolution of A with the F_n being flat modules, then*

$$\operatorname{Tor}_*^R(A, B) = H_*(F_\bullet \otimes_R B).$$

Similarly, if $F'_\bullet \to B$ is a resolution of B by flat modules, then

$$\operatorname{Tor}_*^R(A, B) = H_*(A \otimes_R F'_\bullet).$$

Proof. Please see exercise 3 of sheet 7. ☐

3.11 Universal coefficient theorem

Using the lemmas of the previous subsection we are able to derive a very useful formula for using the homology of a chain complex $P_\bullet$ of right R-modules to compute the homology of a complex $P_\bullet \otimes_R M$ where M is a left R-module. Here is the most general formulation one can give.

Theorem 3.50 (Künneth formula)**.** *Let R be a ring and $P_\bullet$ a chain complex of flat right R-modules such that each submodule $d(P_n)$ of P_{n-1} is also flat. Then for every n and every left R-module M, there is an exact sequence*

$$0 \to H_n(P_\bullet) \otimes_R M \to H_n(P_\bullet \otimes_R M) \to \operatorname{Tor}_1^R(H_{n-1}(P_\bullet), M) \to 0.$$

Proof. By Lemma 3.48 and the short exact sequence

$$0 \to Z_n \to P_n \to d(P_n) \to 0$$

we observe that Z_n is also flat. Since $\operatorname{Tor}_1^R(d(P_n), M) = 0$, the short exact sequence

$$0 \to Z_n \otimes_R M \to P_n \otimes_R M \to d(P_n) \otimes_R M \to 0$$

is exact for every n. These assemble to give a short exact sequence of chain complexes

$$0 \to Z_\bullet \otimes_R M \to P_\bullet \otimes_R M \to d(P_\bullet) \otimes_R M \to 0.$$

Since the differentials in the $Z_\bullet$ and $d(P_\bullet)$ complexes are zero, the long exact sequence of homology associated to the above short exact sequence of chain complexes is

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
H_{n+1}(d(P_\bullet) \otimes_R M) \arrow[d, "\partial"'] & \cong & d(P_{n+1}) \otimes_R M \\
H_n(Z_\bullet \otimes_R M) \arrow[d, "\alpha"'] & \cong & Z_n \otimes_R M \\
H_n(P_\bullet \otimes_R M) \arrow[d, "\beta"'] & & \\
H_n(d(P_\bullet) \otimes_R M) \arrow[d, "\partial"'] & \cong & d(P_n) \otimes_R M \\
H_{n-1}(Z_\bullet \otimes_R M) \arrow[d] & \cong & Z_{n-1} \otimes_R M. \\
\vdots & &
\end{tikzcd}$$

Using the definition of the connecting homomorphism ∂, it is immediate that $\partial = \iota \otimes \mathrm{id}_M$, where ι is the inclusion of $d(P_{n+1})$ in Z_n. On the other hand,

$$0 \to d(P_{n+1}) \xrightarrow{\iota} Z_n \to H_n(P_\bullet) \to 0$$

is a flat resolution of $H_n(P_\bullet)$, so by the Flat Resolution Lemma 3.49

$$\mathrm{Tor}_*^R(H_n(P_\bullet), M)$$

is the homology of

$$0 \to d(P_{n+1}) \otimes_R M \xrightarrow{\partial} Z_n \otimes_R M \to 0.$$

We conclude the proof by looking at the short exact sequence

$$0 \to \ker(\beta) \to H_n(P_\bullet \otimes_R M) \to \mathrm{im}(\beta) \to 0$$

and observe that

$$\ker(\beta) = \mathrm{im}(\alpha) \cong Z_n \otimes_R M / \ker(\alpha) \cong Z_n \otimes_R M / \mathrm{im}(\partial) \cong H_n(P_\bullet) \otimes_R M$$

and

$$\mathrm{im}(\beta) = \ker(\partial) = \mathrm{Tor}_1^R(H_{n-1}(P_\bullet), M).$$

□

If we work with $R = \mathbb{Z}$ and the chain complex $P_\bullet$ consists of free abelian groups the short exact sequence given by the Künneth formula splits, as shown in the next theorem.

Theorem 3.51 (Universal Coefficient Theorem for Homology). *Let $P_\bullet$ be a chain complex of free abelian groups. Then for every n and every abelian group M the Künneth formula 3.50 splits non-canonically, yielding a direct sum decomposition*

$$H_n(P_\bullet \otimes_{\mathbb{Z}} M) \cong H_n(P_\bullet) \otimes_{\mathbb{Z}} M \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(P_\bullet), M).$$

Proof. Recall that every subgroup of a free abelian group is free abelian. Since $d(P_n)$ is a subgroup of P_{n-1}, it is free abelian. Hence the surjection $P_n \to d(P_n)$ splits, giving a non-canonical decomposition

$$P_n \cong Z_n \oplus d(P_n).$$

Applying $- \otimes_{\mathbb{Z}} M$, we see that $Z_n \otimes_{\mathbb{Z}} M$ is a direct summand of the intermediate group

$$\ker(d_n \otimes \mathrm{id}_M : P_n \otimes_{\mathbb{Z}} M \to P_{n-1} \otimes_{\mathbb{Z}} M).$$

Modding out $Z_n \otimes_{\mathbb{Z}} M$ and $\ker(d_n \otimes \mathrm{id}_M)$ by the common image of $d_{n+1} \otimes \mathrm{id}_M$, we see that $H_n(P_\bullet) \otimes_{\mathbb{Z}} M$ is a direct summand of $H_n(P_\bullet \otimes_{\mathbb{Z}} M)$. Since $P_\bullet$ and $d(P_\bullet)$ are flat, the Künneth formula 3.50 tells us that the other summand is given by $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(P_\bullet, M))$ which concludes the proof. ☐

3.12 Exercises Week 7

(i) Let $E_{\bullet,\bullet}$ be a first quadrant double complex with horizontal differentials d^h and vertical differentials d^v. Define the page E^0 as follows: for integers $p, q \geq 0$, let $E^0_{p,q} = E_{p,q}$ and $d^0_{p,q} = d^v_{p,g}$. Each column $E^0_{p,\bullet}$ forms a chain complex, so we may define $E^1_{p,q} = H_q(E^0_{p,\bullet})$ and let $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ be the map that $d^h_{p,q}$ induces on homology. Lastly, note that each row $E^1_{\bullet,q}$ forms a chain complex, so we may define $E^2_{p,q} = H_p(E^1_{\bullet,q})$. The purpose of this exercise is to calculate the differentials $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$.

  (a) Show that $E^2_{p,q}$ can be presented as the group of all pairs (a, b) in $E_{p-1,q+1} \times E_{p,q}$ such that $0 = d^v b = d^v a + d^h b$, modulo the relation that these pairs are trivial: $(a, 0)$; $(d^h x, d^v x)$ for $x \in E_{p,q+1}$; and $(0, d^h c)$ for all $c \in E_{p+1,q}$ with $d^v c = 0$.

  (b) If $d^h(a) = 0$, show that such a pair (a, b) determines an element of $H_{p+q}(T)$, where T is the total complex of the double complex $E_{\bullet,\bullet}$.

  (c) Show that the formula $d(a, b) = (0, d^h(a))$ determines a well-defined map

$$d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}.$$

(ii) In this exercise, $E_{\bullet,\bullet}$ is still a first quadrant double complex, and E^0, E^1 and E^2 are the pages of the same spectral sequence described in the previous exercise. Let T be the total complex of the double complex $E_{\bullet,\bullet}$. By diagram chasing, show that $E^2_{0,0} = H_0(T)$, and that there is an exact sequence

$$H_2(T) \to E^2_{2,0} \xrightarrow{d} E^2_{0,1} \to H_1(T) \to E^2_{1,0} \to 0.$$

(iii) Suppose that a spectral sequence converging to H_* has $E^2_{pq} = 0$ unless $q = 0, 1$. Show that there is a long exact sequence

$$\cdots \to H_{p+1} \to E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \to H_p \to E^2_{p,0} \xrightarrow{d} E^2_{p-2,1} \to H_{p-1} \to \dots$$

(iv) (Mapping Lemma for E^∞) Let $f : \{E^r_{pq}\} \to \{E'^r_{pq}\}$ be a morphism of spectral sequences such that for some r, $f^r : E^r_{pq} \cong E'^r_{pq}$ is an isomorphism for all p and q. Show that $f^\infty_{pq} : E^\infty_{pq} \cong E'^\infty_{pq}$ as well.

3.13 Applications of the universal coefficient theorem

Example 3.52. Singular Homology of a Topological Space X

Let $S(X) \in Ch(\mathcal{A}b)$ be the singular complex of X, then each $S_n(X)$ is a free abelian group. For $M \in \mathcal{A}b$, define homology of X with coefficients in M:

$$H_\bullet(X;M) := H_\bullet(S(X) \otimes M)$$

By the Universal Coefficient Theorem for Homology, for each n, we have the following:

$$H_n(X;m) \cong H_n(X;\mathbb{Z}) \otimes M \oplus Tor_1^{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), M).$$

Theorem 3.53 (Künneth formula for complexes)**.** *Let R be a PID, $P_\bullet \in Ch(mod_R)$, then each P_n is free. Let $Q_\bullet \in Ch({}_Rmod)$. Then for each n there exists a split short exact sequence:*

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes_R H_q(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{p+q=n-1} Tor_1^R(H_p(P), H_q(Q)) \to 0.$$

Example 3.54. Let X, Y be topological spaces. Consider $X \times Y$. **Eilenberg Zilber Theorem** gives the following isomorphism:

$$H_\bullet(S(X \times Y)) \cong H_\bullet(S(X) \otimes S(Y)).$$

Then by 3.53, we have

$$\begin{aligned} &H_n(X \times Y; R) \cong \\ &\left[\bigoplus_i (H_i(X;R) \otimes_R H_{n-i}(Y;R))\right] \oplus \left[\bigoplus_i Tor_1^R(H_i(X;R), H_{n-i-1}(Y;R))\right] \end{aligned}$$

When R is a field, Tor groups vanish and we obtain the following isomorphism:

$$H_n(X \times Y; R) \cong \bigoplus_i (H_i(X;R) \otimes_R H_{n-i}(Y;R)).$$

Chapter 4. Spectral Sequences

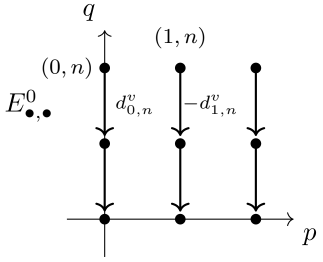
Our goal is to compute homology groups in a systematic way. Generally, a spectral sequence is a sequence of "pages", and each "page" has a grid of objects that approximate homology. Move from "page n" to "page $n+1$" by calculating homology.

4.1 Introduction

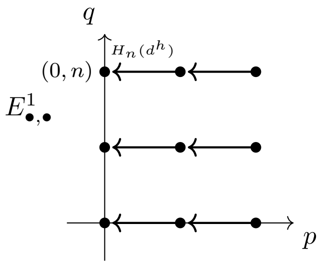
Begin with the following example.

Example 4.1. Let $E_{\bullet,\bullet}$ be a 1st quadrant double complex. To compute the homology of $Tot_\bullet(E)$, use the following spectral sequence.

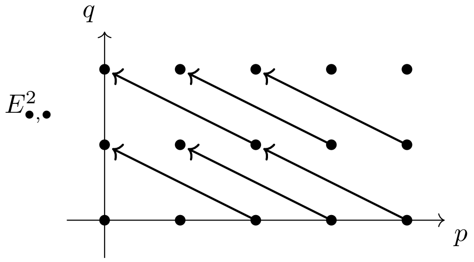
Define "**page 0**" as follows: $E^0_{p,q} := E_{p,q}$, $d^0_{p,q} := (-1)^p d^v_{p,q}$.



Each column is a chain complex, thus we can compute vertical homology at each point.
Then define "**page 1**" to be $E^1_{p,q} := H_q(E^0_{p,\bullet})$. For each p, $d^h_{p,\bullet} : E^0_{p,\bullet} \to E^0_{p-1,\bullet}$ is a morphism of chain complexes, then the morphisms in "page 1" are naturally defined to be $d^1_{p,q} := H_q(d^h_{p,\bullet}) : H_q(E^0_{p,\bullet}) \to H_q(E^0_{p-1,\bullet})$.



Each row is a chain complex. At each point, compute horizontal homology and define "**page 2**" as follows: $E^2_{p,q} := H_p(E^1_{\bullet,q})$.



Define morphism $d^2_{\bullet,\bullet}$ as the first exercise in the sheet 8 or the exercise 5.1.2 in the textbook, so that each line of slope $-\frac{1}{2}$ is chain complex. Repeat this process and we can define "**page n**".
To see why we are doing this, and how this spectral sequence is an approximation of homology of $Tot(E_{\bullet,\bullet})$, we will see another example.

Example 4.2. Suppose the only non-zero columns of $E_{\bullet,\bullet}$ are $A_\bullet = E_{0,\bullet}$ and $B_\bullet = E_{1,\bullet}$. Then the above spectral sequence computes $H_\bullet(Tot(E))$ up to extension in the sense that ∀n there is a short exact sequence

$$0 \to E^2_{0,2} \to H_n(T) \to E^2_{1,n-1} \to 0.$$

Proof. First, calculate $E^2_{0,n}$ and $E^2_{1,n-1}$:

$$E^2_{0,n} = H_0(E^1_{\bullet,n}) = coker(d^1_{1,n})$$

$$E^2_{1,n-1} = H_1(E^1_{\bullet,n-1}) \cong ker(d^1_{1,n-1}).$$

For each n, there is a short exact sequence:

$$0 \to \underbrace{E_{0,n}}_{A_n} \xrightarrow{i_n} \underbrace{E_{0,n} \oplus E_{1,n-1}}_{Tot(E)_n} \xrightarrow{\pi_n} \underbrace{E_{1,n-1}}_{B[-1]_n} \to 0$$

and thus the short exact sequence of chain complexes:

$$0 \to A \xrightarrow{i} Tot(E) \xrightarrow{\pi} B[-1] \to 0,$$

yielding the following long exact sequence:

$$\cdots \to \underbrace{H_{n+1}(B[-1])}_{H_n(B)} \xrightarrow{\partial_n} H_n(A) \xrightarrow{\widetilde{i_n}} H_n(Tot(E)) \xrightarrow{\widetilde{\pi_n}} \underbrace{H_n(B[-1])}_{H_{n-1}(B)} \to \dots$$

Thus there exists an exact short sequence for each n:

$$0 \to coker(\partial_n) \xrightarrow{\widetilde{i_n}} H_n(Tot(E)) \xrightarrow{\widetilde{\pi_n}} \underbrace{im(\widetilde{\pi_n})}_{ker(\partial_{n-1})} \to 0$$

Now we claim $\partial_n = H_n(d^h_{1,\bullet})$. In fact, if $[b] \in H_n(B)$, then $\widetilde{\pi}_{n+1}(0,[b]) = [b]$, and $d^{Tot}_{n+1}(0,b) = (d^h_{1,n}(b), d^v_{1,n}(b)) = (d^h_{1,n}, 0)$. Since $i_n(d^h_{1,n}) = (d^h_{1,n}, 0)$, therefore $\partial_n([b]) = [d^h_{1,n}] = H_n(d^h_{1,\bullet})([b])$. Hence the claim holds and we finish the proof. ☐

4.2 Homological and cohomological spectral sequences

Definition 4.3. A homology spectral sequence (starting with E^a) in an abelian category $\mathcal{A}$ consists of the following data:

(i) For each $r \geq a$, a family $\{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ of objects in $\mathcal{A}$;

(ii) For each $r \geq a$, a family $\{\ d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\ \}$ of morphisms in $\mathcal{A}$, such that $d^r d^r = 0$, i.e. each line of slope $-\frac{r-1}{r}$ is a chain complex;

(iii) For each $r \geq a$, $\forall p, q \in \mathbb{Z}$, we have $E^{r+1}_{p,q} \cong ker(d^r_{p,q})/im(d^r_{p+r.q-r+1})$.

The **total degree** of $E^r_{p,q}$ is $p+q$. Each $d^n_{p,q}$ decreases the total degree by 1.

Definition 4.4. Let E, E' be spectral sequences over $\mathcal{A}$, a morphism $f : E \to E'$ is a family of morphisms in $\mathcal{A}$: $f^r_{p,q} : E^r_{p.q} \to E'^r_{p,q}$ where r is suitably large such that $d^r f^r = f^r d^r$ and $f^{r+1}_{p,q}$ is induced by $f^r_{p,q}$ on homology.

Remark 4.5. There is a category of homology spectral sequences.

Dually we can define **cohomology spectral sequence**.

Lemma 4.6 (Mapping lemma)**.** *Let $f : E \to E'$ be a morphism between two spectral sequences such that for some fixed r and each pair p, q, $f^r_{p,q}$ is an isomorphism. Then for each $s \geq r$, $f^s_{p,q}$ is an isomorphism.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
B^r_{p,q} \arrow[r] \arrow[d, "f^r_{p,q}"'] & Z^r_{p,q} \arrow[r] \arrow[d, "f^r_{p,q}"'] & E^{r+1}_{p,q} \arrow[r] \arrow[d, "f^{r+1}_{p,q}"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
B'^r_{p,q} \arrow[r] & Z'^r_{p,q} \arrow[r] & E'^{r+1}_{p,q} \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

By five lemma, $f^{r+1}_{p,q}$ is an isomorphism. By induction, for each $s \geq r$, $f^s_{p,q}$ is an isomorphism. ☐

Definition 4.7. A homology spectral sequence E is bounded if for each n, there are finitely many non zero terms of total degree n in $E^a_{\bullet,\bullet}$.

Lemma 4.8. *If E is bounded, for each p and q, there exists r_0 such that $E^r_{p,q} \cong E^{r_0}_{p,q}$ for every $r \geq r_0$.*

Proof. First notice that if $E^a_{p,q} = 0$ for some p, q, then by definition and induction we have for each $r \geq a$, $E^r_{p,q} = 0$.
For each fixed p, q, choose r_0 large enough such that $p + r_0$ is sufficiently large and $p - r_0$ is sufficiently small, such that for every $r \geq r_0$, $E^a_{p+r,q-r+1} = E^a_{p-r,q+r-1} = 0$, thus for each $r \geq r_0$, $E^r_{p+r,q-r+1} = E^r_{p-r,q+r-1} = 0$.
For each $r \geq r_0$ consider the chain complex

$$\cdots \to E^r_{p+r,q-r+1} \to E^r_{p,q} \to E^r_{p-r,q+r-1} \to \ldots$$

we have $E^{r+1}_{p,q} \cong E^r_{p,q}$. Thus $E^r_{p,q} \cong E^{r_0}_{p,q}$ for each $r \geq r_0$. ☐

We write $E^\infty_{p,q}$ for this stable value of $E^r_{p,q}$.

Definition 4.9. Let E be bounded. We say E converges to $H_\bullet$ if we are given a finite filtration of subobjects

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1}H_n \subseteq F_p H_n \subseteq \cdots \subseteq F_t H_n = H_n$$

such that $E^\infty_{p,q} = F_p H_{p+q}/F_{p-1}H_{p+q}$.

We denote this by $E^a_{p,q} \to H_{p+q}$.

Definition 4.10. A spectral sequence E collapses at E^r if there is only one non-zero row or column in E^r. If E collapses and converges to $H_\bullet$, we have $H_n = E^r_{p,q}$ where $p+q=n$.

Definition 4.11. (E^∞ terms) Let E be a spectral sequence. There is a sequence:

$$0 = B^a_{p,q} \subseteq \cdots \subseteq B^r_{p,q} \subseteq B^{r+1}_{p,q} \subseteq \cdots \subseteq Z^{r+1} \subseteq Z^r_{p,q} \subseteq \cdots \subseteq Z^a_{p,q} = E^a_{p,q}$$

such that $B^r_{p,q}/B^{r-1}_{p,q} \cong im(d^r)$, $Z^r_{p,q}/Z^{r+1}_{p,q} \cong ker(d^r)$, and $E^r_{p,q} \cong Z^r_{p,q}/B^r_{p,q}$. Introduce the intermediate objects:

$$B^\infty_{p,q} = \bigcup_{r=a}^{\infty} B^r_{p,q} \quad \text{and} \quad Z^\infty_{p,q} = \bigcap_{r=a}^{\infty} Z^r_{p,q}$$

and define $E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$. This definition is compatible with the bounded case.

Remark 4.12. In an unbounded spectral sequence we tacitly assume that $B^\infty_{p,q}$, $Z^\infty_{p,q}$ and $E^\infty_{p,q}$ exist. It is true for the category of modules.

Definition 4.13. We say a spectral sequence E is bounded below if for every n there exists an integer $s(n)$, such that $p < s(n)$ implies $E^a_{p,n-p} = 0$. Dually change "$<$" into "$>$" for the cohomology case.

Definition 4.14. We say a spectral sequence E is regular if for each p, q there exists an integer r_0, such that $d^r_{p,q} = 0$ for every $r \geq r_0$.

Lemma 4.15. *Bounded below spectral sequences are regular.*

Proof. Let E be a bounded below spectral sequence. For each fixed p, q, choose r_0 such that $r_0 > p - s(p+q-1)$. Thus for each $r \geq r_0$, $E^a_{p-r,q+r-1} = 0$. Then for every $r \geq r_0$ we have $E^r_{p-r,q+r-1} = 0$, which implies that the map $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1} = 0$ is just 0. ☐

4.3 Exercises Week 8

(i) In this exercise we will show the snake lemma using the notion of convergence of first quadrant spectral sequences. Recall:

Lemma 4.16 (Snake Lemma). *Consider a commutative diagram in an abelian category $\mathcal{A}$*

$$\begin{tikzcd} & A' \arrow[r, "\alpha"] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r, "\beta"'] & C & \end{tikzcd}$$

such that the rows are exact. Then there is an exact sequence

$$\ker f \to \ker g \to \ker h \to coker\ f \to coker\ g \to coker\ h$$

Moreover if α is monic, then so is $\ker f \to \ker g$, and if β is epi, then so is $coker\ g \to coker\ h$.

For this, consider the first quadrant double complex with bottom left corner being

$$\begin{tikzcd}
0 & C' \arrow[l] \arrow[d, "h"] & B' \arrow[l] \arrow[d, "g"] & A' \arrow[l, "\alpha"'] \arrow[d, "f"] & K' \arrow[l] \\
K & C \arrow[l] & B \arrow[l, "\beta"] & A \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where $K' = \ker \alpha, K = \text{coker } \beta$, and define the associated spectral sequence as in Exercise 1 of Sheet 8. Throughout this exercise, we will assume the true fact that this spectral sequence converges to the homology of the total complex $H_*(T)$ (see for example Section 5.6 of Weibel's book).

  (a) Show that every term of the E^2 page is 0 except for $E^2_{3,0}$ and $E^2_{1,1}$ which are isomorphic one to each other.
  (b) Show the snake lemma.

(ii) Let $\{E^r_{pq}\}$ be a regular upper half-plane spectral sequence, that is $E^r_{pq} = 0$ whenever $q < 0$, such that for any p, q with $q \geq 0$, $E^\infty_{pq} \cong \mathbb{Z}/2\mathbb{Z}$. Show that E approaches H_* and converges to H'_*, where $H_n := \mathbb{Z}, H'_n := \mathbb{Z}_2$ (the 2-adic numbers) for any n.

(iii) In this exercise we will construct the Wang sequence. Suppose that we have a fibration

$$F \xrightarrow{i} E \xrightarrow{\pi} S^n$$

where the base space is an n-sphere for $n \geq 2$.

  (a) Show that we have an exact sequence

$$0 \to E^\infty_{n,q} \to H_q(F) \overset{d^n}{\to} H_{q+n-1}(F) \to E^\infty_{0,q+n-1} \to 0$$

  for each $q \geq 0$, where d^n is a differential from the Leray-Serre spectral sequence.

  (b) Show that for each $q \geq 0$ we have a short exact sequence

$$0 \longrightarrow E^\infty_{0,q} \longrightarrow H_q(E) \longrightarrow E^\infty_{n,q-n} \longrightarrow 0.$$

  (c) Use the last two exercises to show that there is a long exact sequence

$$... \to H_q(F) \overset{i_*}{\to} H_q(E) \to H_{q-n}(F) \overset{d^n}{\to} H_{q-1}(F) \overset{i_*}{\to} H_{q-1}(E) \to ...$$

(iv) In this exercise we will calculate the homology groups of the loop space ΩS^n for $n \geq 2$.

  (a) Use the Serre long exact sequence to calculate $H_p(\Omega S^n)$ for $p \leq n - 1$.
  (b) Use induction and the Wang sequence to calculate $H_p(\Omega S^n)$ for $p \geq n$.

4.4 Notions of convergence of spectral sequences

The main goal of this lecture is to extend the notion of convergence of a spectral sequence to the unbounded case. We first state the following example which will be relevant in the second half of these notes.

Example 4.17. Let E be a 1st quadrant spectral sequence, i.e. $E^a_{pq} = 0$ whenever p or q is negative. Suppose that $E^a_{pq} \Rightarrow H_{p+q}$. Then the filtration on H_n is

$$0 = F_{-1}H_n \subset \cdots \subset F_nH_n = H_n.$$

The morphisms (on the E^r page) whose codomain is on the x-axis, or whose domain is on the y-axis must be the 0 map, as a direct consequence of being a first quadrant spectral sequence. In particular this implies $E^r_{p0} \subset E^a_{p0}$ and $E^a_{0q} \hookrightarrow E^r_{0q}$. And so in particular we have for the E^∞-page $E^\infty_{p0} \subset E^a_{p0}$ and $E^a_{0q} \hookrightarrow E^\infty_{0q}$. By definition of convergence, this implies we have maps $H_n \to E^\infty_{n0} \subset E^a_{n0}$ and $E^q_{0n} \hookrightarrow E^\infty_{0q} \subset H_n$. These morphisms are called *edge homomorphisms*. Because historically spectral sequences first appeared in the case of fiber sequences of topological spaces we call the E^r_{0q} *fiber terms* and we call the E^r_{p0} *base terms.*

The terminology of fiber and base terms will be retroactively motivated in the next section. We now introduce our first notion of convergence.

Definition 4.18. We say that a spectral sequence E weakly converges to H_* if we are given $H_n \in \mathrm{Ob}(\mathcal{A})$ each having a filtration

$$\cdots \subset F_{p-1}H_n \subset F_pH_n \subset \cdots \subset H_n$$

such that for all integer p, q there are isomorphisms $\beta_{pq} : E^\infty_{pq} \to F_pH_{p+q}/F_{p-1}H_{p+q}$.

In order to introduce our second, better, notion of convergence, we introduce the following terminology on filtrations.

Definition 4.19. Let $H \in \mathrm{Ob}(\mathcal{A})$ with a filtration

$$\cdots \subset F_{p-1}H \subset F_pH \subset \cdots \subset H.$$

We call this filtration *exhaustive* if $\bigcup_p F_pH = H$, *Hausdorff* if $\bigcap_p F_pH = 0$ and *complete* if $H \cong \varprojlim H/F_pH$.

With this in hand, we are able to state the following notion of convergence.

Definition 4.20. We say E *approaches* (or abuts to) H_* if it weakly converges to H_* with filtrations

$$\cdots \subset F_{p-1}H_n \subset F_pH_n \subset \cdots \subset H_n$$

that are exhaustive and Hausdorff.

Remark 4.21. If E weakly converges to H_*, one can observe that it must abut to $\bigcup_p F_pH_*/\bigcap_p F_pH_*$.

The final notion of convergence is the following.

Definition 4.22. We say that E *converges* to H_* if E abuts to H_*, E is regular and the filtrations of H_* are complete.

We introduced the three notions of convergence from weakest to strongest, luckily for nice enough spectral sequences we have results such as the following, which allows us to promote weaker notions of convergence.

Lemma 4.23. *Let E be a spectral sequence that is bounded below. Then E approaches H_* if and only if E converges to H_*.*

Proof. ($\Leftarrow$) This is immediate as the definition of convergence includes "approaches to H_*".
($\Rightarrow$) Bounded below implies regularity by a lemma from the previous week. For all n there exists s such that $E^a_{pq} = 0$ for all $p < s$ and such that $p + q = n$ by definition of bounded below. This implies that the same holds at the E^∞-page which by definition of approaching implies that $F_pH_n/F_{p-1}H_n = 0$. This in turn implies that for all $p < s$ we have that $F_pH_n = \bigcap_p F_pH_n$, which is equal to 0 because the filtration is Hausdorff by assumption. This implies that $H_n \cong \varprojlim H_n/F_pH_n$, i.e. that the filtration is regular. This concludes the proof. □

Even when a spectral sequence converges, it need not give complete information. Thus, it can be useful to have other methods to relate the information on the pages of the spectral sequence with the abutment. In line with categorical thinking, a first attempt at this is the following definition.

Definition 4.24. Let E, E' be two spectral sequences that converge to H_*, H'_* respectively with isomorphisms $\beta_{pq} : E^{\infty}_{pq} \to F_pH_{p+q}/F_{p-1}H_{p+q}$ and isomorphisms $\beta'_{pq} : E'^{\infty}_{pq} \to F_pH'_{p+q}/F_{p-1}H'_{p+q}$.
We call a morphism $h : H_* \to H'_*$ compatible with a morphism $f : E \to E'$ if h preserves the filtration (i.e. $h(F_pH_n) \subset F_pH'_n$) such that the following diagram commutes

$$\begin{tikzcd}
E^{\infty}_{pq} \arrow[r, "\beta_{pq}"] \arrow[d, "f^{\infty}_{pq}"'] & F_pH_{p+q}/F_{p-1}H_{p+q} \arrow[d, "\bar{h}"] \\
E'^{\infty}_{pq} \arrow[r, "\beta'_{pq}"] & F_pH'_{p+q}/F_{p-1}H'_{p+q}
\end{tikzcd} .$$

The above definition is used in the following theorem to assist in spectral sequence computations.

Theorem 4.25. *(Comparison theorem) Let E, E' converge to H_*, H'_* respectively, and $h : H_* \to H'_*$ be a morphism which is compatible with $f : E \to E'$. If for all p, q there is an r such that $f^r_{pq} : E^r_{pq} \to E'^r_{pq}$ is an isomorphism, then the map $h : H_* \to H'_*$ is an isomorphism.*

Proof. By the mapping lemma f^{∞}_{pq} is an isomorphism. Weak convergence and compatibility of h and f yields (via a simple application of the third isomorphism theorem) that for all s, p we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{p-1}H_n/F_sH_n \arrow[r] \arrow[d] & F_pH_n/F_sH_n \arrow[r] \arrow[d] & E^{\infty}_{p,n-p} \arrow[r] \arrow[d, "f^{\infty}_{p,n-p}"] & 0 \\
0 \arrow[r] & F_{p-1}H'_n/F_sH'_n \arrow[r] & F_pH'_n/F_sH'_n \arrow[r] & E'^{\infty}_{p,n-p} \arrow[r] & 0
\end{tikzcd} .$$

Fixing s, we get by induction on p and by the 5 lemma that $F_pH_n/F_sH_n \cong F_pH'_n/F_sH'_n$ for all p. By exhaustivity, this implies that we have an isomoprhism $H_n/F_sH_n \cong H'_n/F_sH'_n$, we conclude taking inverse limits (with respect to s) by completeness of the filtration. ☐

Remark 4.26. A spectral sequence E may converge to two different limits, so H_* can be difficult to reconstruct from this data.

Example 4.27. Let E be a first quadrant spectral sequence such that $E^{\infty}_{pq} \cong \mathbb{Z}/2\mathbb{Z}, \forall p, q \geq 0$. Then without further information, we do not know whether H_2 is $\mathbb{Z}/8\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^{\otimes 3}$. The comparison theorem can help us in situations such as these.

4.5 The Leray-Serre spectral sequence

The goal of this section is to define the Lerray-Serre spectral sequence and show how it can be useful for computing homology groups of spaces.

Definition 4.28. Let Z be a pointed topological space and $I = [0, 1]$ the unit interval. We say that a morphism $f : X \to Y$ of pointed topological spaces satisfies the *homotopy lifting property* for Z if given a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "- \times 0"] & X \arrow[d, "f"] \\
Z \times I \arrow[r, "H"'] & Y
\end{tikzcd}$$

there always exists $G : Z \times I \to X$ such that

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "- \times 0"] & X \arrow[d, "f"] \\
Z \times I \arrow[r, "H"'] \arrow[ur, "G"'] & Y
\end{tikzcd}$$

commutes.

Definition 4.29. We call a sequence of composable maps $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ in the category of pointed topological spaces a Serre fibration if ι is the inclusion of $\pi^{-1}(*_B)$, the fiber of π over the base point of B, into E and π satisfies the homotopy lifting property for all CW-complexes.

Example 4.30. For B a pointed topological spaces we have a Serre fibration $\Omega B \xrightarrow{\iota} B^I \xrightarrow{ev_1} B$, where B^I is the space of maps $I \to B$ starting at the base point of B, with the compact open topology and $ev_1 : B^I \to B$ is the map given by evaluating at 1. We omit the proof that $B^I \to B$ satisfies the appropriate homotopy lifting property.

With the appropriate topological and homological language, we can now state the following theorem of Serre.

Theorem 4.31. *(Serre Spectral Sequence) Let $F \xrightarrow{\iota} E \xrightarrow{\pi} B$ be a Serre fibration. Assume further that B is simply connected. Then there is a first quadrant spectral sequence*

$$E^2_{p,q} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E).$$

Remark 4.32. The edge map $H_q(F) \hookrightarrow E^\infty_{0,q} \subset H_q(E)$ is the map $H_q(\iota)$ and if we also assume that F is connected, the other edge map $H_p(E) \hookrightarrow E^\infty_{p,0}(B)$ is $H_*(\pi)$.

There are many application in homology computations of the above result, we present here a specific one

Proposition 4.33. *With the assumptions of the theorem and further assuming that F is connected, if there exist $n_1 \geq 1$ and $n_2 \geq 2$ such that, for any abelian group A*

$$H_i(F, A) = 0, \forall 0 < i < n_1,$$

$$H_j(B, A) = 0, \forall 0 < j < n_2,$$

then there is a long exact sequence

$$H_{n_1+n_2-1}(F) \xrightarrow{\iota_*} H_{n_1+n_2-1}(E) \xrightarrow{\pi_*} H_{n_1+n_2-1}(B) \longrightarrow H_{n_1+n_2-2}(F) \longrightarrow \cdots .$$

Proof. We look at the Leray-Serre Spectral sequence in this case. On the y-axis we have $E^2_{0q} = H_0(B, H_q(F)) \cong H_q(F)$ and similarly on the x-axis we have $E^2_{p0} = H_p(B, H_0(F)) \cong H_p(B)$ where the isomorphisms follow from the fact that $H_0(F) \cong \mathbb{Z}$ and $H_0(B) \cong \mathbb{Z}$ which are the homological incarnation of the connectivity assumptions on B and F. The vanishing assumptions on the homology of F and B (with arbitrary coefficients) imply, by the universal coefficient theorem, that $E^2_{pq} = 0$ for $p < n_2$ or $q < n_1$ and one of them non-zero.
We get vanishing of the corresponding terms of the E^∞ page. In particular, this implies that the filtration quotients of F_pH_k are trivial for $1 \leq k \leq n_1 + n_2 - 1$, as under these conditions, when $k = p + q$, $p < n_2$ or $q < n_1$ and at least one of p and q is non zero, thus $F_pH_{p+q}/F_{p-1}H_{p+q} \cong E^\infty_{pq} \cong 0$. This in particular implies that in this range we have isomorphism $F_0H_k(E) \cong E^\infty_{0,k}$ and $E^\infty_{k,0} \cong H_k(E)/E^\infty_{0,k}$, which follow from definition of convergence. This gives the following short exact sequence

$$0 \to E^\infty_{0,k} \to H_k(E) \to E^\infty_{k,0} \to 0.$$

Looking at the kth page, we have a differential $E^k_{k,0} \to E^k_{0,k-1}$, by turning the page and recalling that we are working with a first quadrant spectral sequence we see that this map fits in the following exact sequence

$$0 \to E^{k+1}_{k,0} \to E^k_{k,0} \to E^k_{0,k-1} \to E^{k+1}_{0,k-1} \to 0.$$

For first quadrant spectral sequences, one can notice that $E^{k+1}_{k,0} \cong E^\infty_{k,0}$ and $E^{k+1}_{0,k-1} \cong E^\infty_{0,k-1}$ because all the differentials that can modify these groups live on earlier pages. Similarly, one can notice that $H_k(B) \cong E^2_{k,0} \cong E^k_{k,0}$ and $H_{k-1}(F) \cong E^2_{0,k-1} \cong E^k_{0,k-1}$ by using that $E^2_{pq} = 0$ for $p < n_2$ or $q < n_1$ and one of them non-zero. This allows to rewrite the above exact sequence as

$$0 \to E^\infty_{k,0} \to H_k(B) \to H_{k-1}(F) \to E^\infty_{0,k-1} \to 0.$$

We can splice this exact sequence with the following short exact sequence

$$0 \to E^\infty_{0,k} \to H_k(E) \to E^\infty_{k,0} \to 0$$

in order to obtain the desired long exact sequence

$$H_{n_1+n_2-1}(F) \xrightarrow{\iota_*} H_{n_1+n_2-1}(E) \xrightarrow{\pi_*} H_{n_1+n_2-1}(B) \longrightarrow H_{n_1+n_2-2}(F) \longrightarrow \cdots .$$

$\square$

Remark 4.34. We can of course extend the above long exact sequence "naively" by $0 \to \ker(H_{n_1+n_2-1}(F) \xrightarrow{\iota_*} H_{n_1+n_2-1}(E))$. It would be nice to have a more direct understanding of how to extend the above long exact sequence. Inspecting the spectral sequence once more, we see that a potential obstruction comes from the image of $E^{n_1}_{n_2,n_1}$ in $H_{n_1+n_2-1}(F)$. By the universal coefficient theorem, we have in general

$$E^2_{p,q} = H_p(B, H_q(F)) \cong H_p(B) \otimes H_q(F) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(B), H_q(F)).$$

In particular, due to vanishing assumptions, in the context of interest to us the first obstruction to a long exact sequence in homology should be related to $H_{n_2}(B, H_{n_1}(F)) \cong H_{n_2}(B) \otimes H_{n_1}(F)$.

Remark 4.35. We resume the example of the path space fibration $\Omega B \to B^I \to B$. Recall that B is simply connected, so that, ignoring differentials and only making explicit the terms we are interested in, the E_2 page will look like

$$\begin{array}{|lllll}
H_2(\Omega B) & E^2_{12} & E^2_{22} & E^2_{32} & E^2_{42} \\
 & & & & \\
H_1(\Omega B) & H_1(B, H_1(\Omega B)) & H_2(B, H_1(\Omega B)) & E^2_{31} & E^2_{41} \\
 & & & & \\
\mathbb{Z} & E^2_{10} & H_2(B) & H_3(B) & H_4(B) \quad . \\
\hline
\end{array}$$

We know that this spectral sequence converges to the homology of a contractible space. This in particular means that $H_2(B, H_1(\Omega B))$ has to vanish after turning to the E_3 page, in particular we have an exact sequence

$$H_4(B) \to H_2(B, H_1(\Omega B)) \to H_2(\Omega B).$$

The cokernel of the last map is E^3_{02} and we see, due to lack of space for further differentials, that it must be killed upon turning to the 4th page, so that it must be isomorphic to E^3_{30}. However, by the universal coefficient theorem and because B is simply connected we have $H_1(B, H_1(\Omega B)) \cong 0$, so that $E^3_{30} \cong E^2_{30} \cong H_3(B)$. This gives us an exact sequence

$$H_4(B) \to H_2(B, H_1(\Omega B)) \to H_2(\Omega B) \to H_3(B) \to 0.$$

Because the term $H_2(B, H_1(\Omega B))$ is a bit awkward we would like to make it a bit clearer. We first do this via an application of the universal coefficient theorem. Before writing out what this yields, we notice that a combination of the Hurewicz isomorphism and the isomorphism $\pi_i(\Omega X) \cong \pi_{i+1}(X)$ gives us

$$H_1(\Omega X) \cong \pi_1(\Omega X)^{ab} \cong \pi_2(B) \cong H_2(B).$$

Now an application of the universal coefficient theorem gives

$$H_2(B, H_1(\Omega B)) \cong H_2(B, H_2(B))$$
$$\cong H_2(B) \otimes H_2(B) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_1(B), H_2(B)) \cong H_2(B) \otimes H_2(B).$$

Putting all of this together gives us an exact sequence

$$H_4(B) \to H_2(B) \otimes H_2(B) \to H_2(\Omega B) \to H_3(B) \to 0.$$

4.6 Exercises Week 9

Throughout the sheet we fix abelian categories $\mathcal{A}$ and $\mathcal{B}$, assumed to have enough projectives. We let $A_\bullet \in \mathrm{Ch}(\mathcal{A})$, and fix a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \to A_\bullet$ with augmentation $\varepsilon : P_{\bullet,0} \to A_\bullet$.

(i) Recall that by definition we have that $B_p(\varepsilon) : B_p(P, d^h) \to B_p(A)$ and $H_p(\varepsilon) : B_p(P, d^h) \to H_p(A)$ are projective resolutions.

  (a) Show that $Z_p(\varepsilon) : Z_p(P, d^h) \to Z_p(A)$ and $\varepsilon_p : P_{p,\bullet} \to A_p$ are projective resolutions.

  (b) Suppose that $A_\bullet$ is bounded below. Show that $\varepsilon : \mathrm{Tot}^{\oplus}(P_{\bullet,\bullet}) \to A.$ is a quasi-isomorphism.

  (c) Show (b) but now supposing that $\mathcal{A}$ satisfies axiom (AB4) (ie $\mathcal{A}$ is cocomplete and arbitrary direct sums of monics are monics). If you cannot show that ε gives a quasi-isomorphism, show via a spectral sequence argument that $\mathrm{Tot}^{\oplus}(P_{\bullet,\bullet})$ and $A_\bullet$ have the same homologies.

(ii) Let $B_\bullet \in \mathrm{Ch}(\mathcal{A})$, and $f : A_\bullet \to B_\bullet$ be a chain map. If $Q_{\bullet,\bullet} \to B_\bullet$ is a Cartan-Eilenberg resolution, show that f induces a map of double complexes $\tilde{f} : P_{\bullet,\bullet} \to Q_{\bullet,\bullet}$ (as mentioned in class, this map is unique up to homotopy).

(iii) Fix a right exact functor $F : \mathcal{A} \to \mathcal{B}$

  (a) Suppose $A_\bullet$ is concentrated in degree 0. Show that $\mathbb{L}_i(F(A)) = L_iF(A_0)$.

  (b) Let $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ be the subcategory of chain complexes in $\mathcal{A}$ trivial below degree 0, ie $A_\bullet \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ if $A_p = 0\ \forall p < 0$. Show that the hyper-left derived functors $\mathbb{L}_iF$, when restricted to $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, are given by the ordinary derived functors of degree 0 homology, L_iH_0F (observe that H_0F is a right exact functor).

  (c) Show the dimension shifting formula $\mathbb{L}_iF(A[n]) = \mathbb{L}_{i+n}F(A)$

(iv) Let $A_\bullet$ be the mapping cone complex of f, $0 \longrightarrow A_1 \xrightarrow{f} A_0 \longrightarrow 0$. Show there is a long exact sequence

$$...\mathbb{L}_iF(A) \longrightarrow L_iF(A_1) \overset{L_iF(f)}{\longrightarrow} L_iF(A_0) \longrightarrow \mathbb{L}_iF(A) \longrightarrow \mathbb{L}_{i-1}F(A) \longrightarrow ...$$

(v) Let $f : X \to Y$ be a morphism of ringed spaces and $\mathcal{F}$ be an $\mathcal{O}_X$-module.

  (a) Suppose that $R^qf_*\mathcal{F} = 0$ for $q > 0$. Then show that $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$ for all p.

  (b) Suppose that $H^p(Y, R^qf_*\mathcal{F}) = 0$ for all q and $p > 0$. Then show that $H^q(X, \mathcal{F}) = H^0(Y, R^qf_*\mathcal{F})$ for all q.

4.7 Hyperhomology

Let $\mathcal{A}$ be an abelian category with enough projectives.

Definition 4.36. Let $A_\bullet$ be a chain complex in $\mathcal{A}$. A (left) *Cartan-Eilenberg* (C-E) resolution is an upper half-plane double complex $P_{\bullet,\bullet}$ (i.e. $P_{pq} = 0$ if $q < 0$) consisting of projective objects of $\mathcal{A}$, together with a chain map ("augmentation") $\varepsilon : P_{\bullet 0} \to A_\bullet$ such that

(i) If $A_p = 0$, then the column $P_{p\bullet} = 0$,

(ii) The maps on boundaries and homology

$$B_p(\varepsilon) : B_p(P, d^h) \to B_p(A)$$
$$H_p(\varepsilon) : H_p(P, d^h) \to H_p(A)$$

are projective resolutions. We recall that $B_p(P, d^h)_q = \operatorname{Im}(d^h_{p+1,q})$ where $d^h_{p+1,q} : P_{p+1,q} \to P_{pq}$, and similarly for $H_p(P, d^h)_q$.

Remark 4.37. Here are some remarks that are proven in Exercise 1 of the exercise sheet 10.

(i) The induced maps $Z_p(\varepsilon) : Z_p(P, d^h) \to Z_p(A)$ and $\varepsilon^p : P_{p,\bullet} \to A_p$ are also projective resolutions.

(ii) The augmentation $\varepsilon : \operatorname{Tot}^{\oplus}(P_{\bullet,\bullet}) \to A$ is a quasi-isomorphism in $\mathcal{A}$.

We have seen that any object in $\mathcal{A}$ admits a projective resolution in $\mathbf{Ch}(\mathcal{A})$. Similarly, we will prove in the following lemma that any chain complex in $\mathbf{Ch}(\mathcal{A})$ admits a Cartan-Eilenberg resolution.

Lemma 4.38. *Every chain complex $A_\bullet$ in $\mathbf{Ch}(\mathcal{A})$ admits a Cartan-Eilenberg resolution $\varepsilon : P_{\bullet,\bullet} \to A_\bullet$.*

Proof. For each p, we consider $Z_p := Z_p(A_\bullet), B_p := B_p(A_\bullet)$ and $H_p := H_p(A_\bullet)$. Select projective resolutions $P^B_{p\bullet}$ of B_p and $P^H_{p\bullet}$ of H_p. Since we have a short exact sequence

$$0 \longrightarrow B_p \longrightarrow Z_p \longrightarrow H_p \longrightarrow 0,$$

by the Horseshoe lemma, there is a projective resolution $P^Z_{p\bullet}$ of Z_p such that

$$0 \longrightarrow P^B_{p\bullet} \longrightarrow P^Z_{p\bullet} \longrightarrow P^H_{p\bullet} \longrightarrow 0$$

is a short exact sequence of chain complexes. Now since

$$0 \longrightarrow Z_p \longrightarrow A_p \longrightarrow B_{p-1} \longrightarrow 0$$

is a short exact sequence, by the Horseshoe lemma, there is a projective resolution of $P^A_{p\bullet}$ of A_p such that the sequence

$$0 \longrightarrow P^Z_{p\bullet} \longrightarrow P^A_{p\bullet} \longrightarrow P^B_{p-1,\bullet} \longrightarrow 0$$

is a short exact sequence of chain complexes. We define $P_{\bullet,\bullet}$ to be the double complex whose p-th column is $P^A_{p\bullet}$. Using the sign trick, we define the vertical differentials to be $d^v_p = (-1)^p d^A_{p\bullet}$. We define the horizontal differentials d^h_{p+1} as the following composition

$$P^A_{p+1,\bullet} \longrightarrow P^B_{p\bullet} \hookrightarrow P^Z_{p\bullet} \hookrightarrow P^A_{p\bullet}.$$

By construction of $P_{\bullet,\bullet}$, the maps $\varepsilon_p : P_{p,0} \to A_p$ assemble to give a chain map, and each $B_p(\varepsilon)$ and $H_p(\varepsilon)$ give projective resolutions. ☐

Remark 4.39. If $f : A_\bullet \to B_\bullet$ is a chain map and $P_{\bullet,\bullet}, Q_{\bullet,\bullet}$ are Cartan-Eilenberg resolutions of $A_\bullet, B_\bullet$ respectively, then there is a double complex map $\tilde{f} : P_{\bullet,\bullet} \to Q_{\bullet,\bullet}$ over f.

Definition 4.40. Let $f, g : D_{\bullet,\bullet} \to E_{\bullet,\bullet}$ be maps of double complexes. A *chain homotopoy* from f to g consists of maps

$$s^h_{pq} : D_{pq} \to E_{p+1,q}, \quad s^v_{pq} : D_{pq} \to E_{p,q+1}$$

such that

$$g - f = (s^h d^h + d^h s^h) + (d^v s^v + s^v d^v)$$

and

$$s^v d^h + d^h s^v = s^h d^v + d^v s^h = 0.$$

Note that this definition is set up so that the collection $\{s := s^h + s^v : \mathrm{Tot}(D)_n \to \mathrm{Tot}(E)_{n+1}\}$ forms an ordinary chain homotopy between the induced maps $\mathrm{Tot}(f)$ and $\mathrm{Tot}(g)$ from $\mathrm{Tot}^{\oplus}(D_{\bullet,\bullet})$ to $\mathrm{Tot}^{\oplus}(E_{\bullet,\bullet})$, i.e

$$\mathrm{Tot}(g) - \mathrm{Tot}(f) = sd + ds$$

where d denotes the differential of the total complex.

Remark 4.41. Let $f, g : A_\bullet \to B_\bullet$ be maps of chain complexes and $\varepsilon : P_{\bullet,\bullet} \to A_\bullet, \varepsilon' : Q_{\bullet,\bullet} \to B_\bullet$ be Cartan-Eilenberg resolutions.

1) One can show that if f, g are chain homotopic maps of complexes, then $\tilde{f}, \tilde{g}$ are chain homotopic maps of double complexes.

2) Any two Cartan-Eilenberg resolutions of $A_\bullet$ are chain homotopy equivalent, i.e if we have

$$\begin{tikzcd}
P_{\bullet,\bullet} \arrow[r, bend right, "p"'] \arrow[d] & Q_{\bullet,\bullet} \arrow[l, bend right, "q"'] \arrow[d] \\
A_\bullet \arrow[r, "\mathrm{Id}_{A_\bullet}"'] & A_\bullet
\end{tikzcd}$$

then $p \circ q$ and $\mathrm{Id}_{Q_{\bullet,\bullet}}$ are homotopic, as well as $q \circ p$ and $\mathrm{Id}_{P_{\bullet,\bullet}}$. In particular, for any additive functor F, the chain complexes $\mathrm{Tot}^{\oplus}(F(P_{\bullet,\bullet}))$ and $\mathrm{Tot}^{\oplus}(F(Q_{\bullet,\bullet}))$ are chain homotopy equivalent.

Definition 4.42. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor, $A_\bullet$ be a chain complex in $\mathcal{A}$ and $P_{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $A_\bullet$. We define

$$\mathbb{L}_i F(A_\bullet) := H_i \mathrm{Tot}^{\oplus}(F(P_{\bullet,\bullet})).$$

One can show that this definition does not depend on the choice of $P_{\bullet,\bullet}$.
If $f : A_\bullet \to B_\bullet$ is a chain map and $\tilde{f} : P_{\bullet,\bullet} \to Q_{\bullet,\bullet}$ a map of Cartan-Eilenberg resolution over f, we define

$$\mathbb{L}_i(f) := H_i \mathrm{Tot}^{\oplus}(\tilde{f}) : \mathbb{L}_i F(A_\bullet) \to \mathbb{L}_i F(B_\bullet).$$

Then, $\mathbb{L}_i F$ defines a functor from $\mathbf{Ch}(\mathcal{A})$ to $\mathcal{B}$ if $\mathcal{B}$ is cocomplete (otherwise $\mathrm{Tot}^{\oplus}(F(P_{\bullet,\bullet}))$ and $\mathbb{L}_i F(A_\bullet)$ may not exist for all $A_\bullet$ in $\mathbf{Ch}(\mathcal{A})$). In this case, we call $\mathbb{L}_i F$ the *left hyper-derived functors* of F.

Lemma 4.43. *For $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ a short exact sequence of bounded below chain complexes, there is a long exact sequence*

$$\cdots \longrightarrow \mathbb{L}_{i+1} F(C_\bullet) \xrightarrow{\delta} \mathbb{L}_i F(A_\bullet) \longrightarrow \mathbb{L}_i F(B_\bullet) \longrightarrow \mathbb{L}_i F(C_\bullet) \xrightarrow{\delta} \cdots.$$

For the proof, we need the two following results that will be proven in the exercise sheet 10.

Lemma 4.44. *Let $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ be the subcategory of complexes $A_\bullet$ with $A_p = 0$ for $p < 0$. Then for any $A_\bullet$ in $\mathbf{Ch}_{\geq 0}(\mathcal{A})$, we have*

$$\mathbb{L}_i F(A_\bullet) = L_i(H_0 F)(A_\bullet).$$

Lemma 4.45 (Dimension shifting)**.** *For any n, we have*

$$\mathbb{L}_i F(A_\bullet[n]) = \mathbb{L}_{n+i} F(A_\bullet)$$

where we recall that $A[n]_i = A_{n+i}$.

Proof of Lemma 4.43. Let $\tilde{p} := \min\{p \mid A_q = B_q = C_q = 0 \ \forall q \leq p\}$. Consider $A_\bullet[\tilde{p}], B_\bullet[\tilde{p}], C_\bullet[\tilde{p}]$. Then, using the two previous lemmas, we have

$$\mathbb{L}_{i+1} F(A_\bullet) = \mathbb{L}_{i+1-\tilde{p}} F(A_\bullet[\tilde{p}]) = L_{i+1-\tilde{p}} H_0 F(A_\bullet[\tilde{p}])$$

and similarly

$$\mathbb{L}_{i+1} F(B_\bullet) = L_{i+1-\tilde{p}} H_0 F(B_\bullet[\tilde{p}])$$

and

$$\mathbb{L}_{i+1} F(C_\bullet) = L_{i+1-\tilde{p}} H_0 F(C_\bullet[\tilde{p}]).$$

Thus, we may assume, without loss of generality, that $A_\bullet, B_\bullet, C_\bullet$ belong to $\mathbf{Ch_{\geq 0}}(\mathcal{A})$. Therefore, the sequence from the lemma becomes the long exact sequence for the derived functors of the right exact functor $H_0 F$. ☐

In view of a further theorem (Theorem 4.47), we need the following result that we are not going to prove.

Theorem 4.46. *Let $C_{\bullet,\bullet}$ be a double complex. If $C_{\bullet,\bullet}$ is zero in the second quadrant, then there is a convergent spectral sequence*

$$'E^2_{pq} = H^h_p H^v_q(C_{\bullet,\bullet}) \Rightarrow H_{p+q}(Tot^{\oplus}(C_{\bullet,\bullet})).$$

If instead, $C_{\bullet,\bullet}$ is zero in the fourth quadrant, then there is a convergent spectral sequence

$$''E^2_{pq} = H^v_p H^h_q(C_{\bullet,\bullet}) \Rightarrow H_{p+q}(Tot^{\oplus}(C_{\bullet,\bullet})).$$

Now we can state the following result.

Theorem 4.47. *Let $A_\bullet$ be a chain complex and let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Assume that $\mathcal{A}$ has enough projectives. Then, there is always a convergent spectral sequence*

$$''E^2_{pq} = (L_p F)(H_q(A_\bullet)) \Rightarrow \mathbb{L}_{p+q} F(A_\bullet).$$

If $A_\bullet$ is bounded below, then there is a convergent spectral sequence

$$'E^2_{pq} = H_p(L_q F(A_\bullet)) \Rightarrow \mathbb{L}_{p+q} F(A_\bullet).$$

Proof. For the second spectral sequence, if $A_\bullet$ is bounded below, consider $'E^2_{pq}$ arising from the double complex $F(P_{\bullet,\bullet})$ where $P_{\bullet,\bullet}$ is a Cartan-Eilenberg resolution of $A_\bullet$. Then by definition, $H_p(L_q F(A_\bullet)) = H^h_p H^v_q(F(P_{\bullet,\bullet}))$. Thus, by Theorem 4.46, we have the desired convergent spectral sequence.
For the first spectral sequence, choose the Cartan-Eilenberg resolution $P_{\bullet,\bullet}$ of $A_\bullet$ that we have constructed in the proof of Lemma 4.38. Then it follows from the construction of $P_{\bullet,\bullet}$ that for any p, the chain complex $P_{p\bullet}$ is split. In particular, since F is an additive functor, we have that $H_p F(P_{n\bullet}) = F H_p(P_{n\bullet})$ for any non-negative integers p, n. From Theorem 4.46, we have a convergent spectral sequence

$$''E^2_{pq} = L_p(H^h_q F)(A_\bullet) \Rightarrow \mathbb{L}_{p+q} F(A_\bullet).$$

Since, $H_p F(P_{n\bullet}) = F H_p(P_{n\bullet})$, it follows that $''E^2_{pq} = (L_p F)(H_q(A_\bullet))$, allowing us to conclude. ☐

Corollary 4.48. *Let $A_\bullet$ be a chain complex.*

(i) If $A_\bullet$ is exact, then $\mathbb{L}_i F(A_\bullet) = 0$.

(ii) Any quasi-isomorphism $f : A_\bullet \to B_\bullet$ induces an isomorphism $\mathbb{L}_* F(f) : \mathbb{L}_* F(A_\bullet) \to \mathbb{L}_* F(B_\bullet)$.

(iii) If each A_p is F-acyclic (i.e. $L_q F(A_p) = 0$ for $q \neq 0$) and $A_\bullet$ is bounded below, then $\mathbb{L}_p F(A_\bullet) = H_p(F(A_\bullet))$.

Proof.

(i) If $A_\bullet$ is exact, then $H_q(A_\bullet) = 0$ for $q \neq 0$. Hence ${}''E^2_{pq}$ degenerates and is equal to 0. Then $\mathbb{L}_{p+q}F(A_\bullet) = 0$.

(ii) The isomorphisms $H_q(f) : H_q(A_\bullet) \to H_q(B_\bullet)$ induce isomorphisms

$$L_pF(H_q(A_\bullet)) \xrightarrow{\cong} L_pF(H_q(B_\bullet))$$

which induce isomorphisms $\mathbb{L}_{p+q}F(A_\bullet) \cong \mathbb{L}_{p+q}F(B_\bullet)$.

(iii) If each A_n is F-acyclic, then ${}'E^2_{pq} = H_p(L_qF(A_\bullet)) = 0$ for any $q \neq 0$. At $q = 0$, we recover $H_*F(A_\bullet)$. ☐

Example 4.49 (Hypertor)**.** Let R be a ring and B a right R-module. The *hypertor groups* $\mathbf{Tor^R_i(A_\bullet, B)}$ of a chain complex $A_\bullet$ are the hyperderived functors $\mathbb{L}_iF(A_\bullet)$ where the functor $F = - \otimes_R B$.
Applying Theorem 4.47, we have the following convergent spectral sequence

$$''E^2_{pq} = \mathrm{Tor}_p(H_q(A_\bullet), B) \Rightarrow \mathbf{Tor}^R_{p+q}(A_\bullet, B)$$

and for $A_\bullet$ bounded below, we obtain

$$'E^2_{pq} = H_p\mathrm{Tor}_q(A_\bullet, B) \Rightarrow \mathbf{Tor}^R_{p+q}(A_\bullet, B).$$

4.8 Cohomology variant

Let $\mathcal{A}$ be an abelian category that has enough injectives. A (right) *Cartan-Eilenberg* resolution of a cochain complex $A^\bullet$ in $\mathcal{A}$ is an upper half-plane double complex $I^{\bullet\bullet}$ consisting of injective objects of $\mathcal{A}$, together with an augmentation map $\varepsilon : A^\bullet \to I^{\bullet 0}$ such that

(i) If $A^p = 0$, then the column $I^{p\bullet} = 0$,

(ii) The maps on coboundaries and cohomology

$$B^p(\varepsilon) : B^p(A) \to B^p(I, d^h)$$
$$H^p(\varepsilon) : H^p(A) \to H^p(I, d^h)$$

are injective resolutions.

For $F : \mathcal{A} \to \mathcal{B}$ a left exact functor, we can define the *right hyper-derived functors* of F as

$$\mathbb{R}^iF(A^\bullet) := H^i\mathrm{Tot}^\Pi(F(I^{\bullet\bullet}))$$

for $A^\bullet$ a cochain complex in $\mathcal{A}$ and $I^{\bullet\bullet}$ its Cartan-Eilenberg resolution.
Similarly as in Theorem 4.47, if $A^\bullet$ is a cochain complex, we have a convergent spectral sequence

$$''E^{pq}_2 = (R^pF)(H^q(A^\bullet)) \Rightarrow \mathbb{R}^{p+q}F(A^\bullet),$$

and if $A^\bullet$ is bounded below, we have

$$'E^{pq}_2 = H^p(R^qF(A^\bullet)) \Rightarrow \mathbb{R}^{p+q}F(A^\bullet).$$

4.9 Grothendieck spectral sequence

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. We are givent left exact functors $G : \mathcal{A} \to \mathcal{B}, F : \mathcal{B} \to \mathcal{C}$. We assume that G sends injective to F-acyclic objects. Fix A an object in $\mathcal{A}$.

Theorem 4.50 (Grothendieck Spectral Sequence Theorem)**.** *With this setup, there exists a first quadrant cohomological spectral sequence*

$$'E^{pq}_2 = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

Proof. Consider an injective resolution $A \to I^\bullet$. We build a Cartan-Eilenberg resolution of $G(I^\bullet)$. By the cohomological version of Theorem 4.47, we get two convergent spectral sequences. The first one is given by

$$'E_2^{pq} = H^p\big(R^qF(GI^\bullet)\big) \Rightarrow (\mathbb{R}^{p+q}F)(GI^\bullet).$$

By hypothesis, each $G(I^p)$ is F-acyclic, so $R^qF(G(I^p)) = 0$ for $q \neq 0$. By Corollary 4.48, we have that

$$(\mathbb{R}^pF)(GI^\bullet) \cong H^p(FG(I^\bullet)) = R^p(FG)(A).$$

Now the second convergent spectral sequence is given by

$$''E_2^{pq} = (R^pF)(H^q(GI^\bullet)) \Rightarrow (\mathbb{R}^{p+q}F)(GI^\bullet).$$

Since $R^{p+q}(FG)(A) \cong (\mathbb{R}^{p+q}F)(GI^\bullet)$ and $R^pG(A) = H^q(GI^\bullet)$, we have found the desired convergent spectral sequence. ☐

Let us apply this theorem to two different examples. But in view of the following examples, we still need the following proposition.

Proposition 4.51. *Let $L : \mathcal{A} \to \mathcal{B}, R : \mathcal{B} \to \mathcal{A}$ be additive functors between abelian categories such that L is the left adjoint of R.*

- *If L is exact, then R preserves injectives, i.e. for any injective object I of $\mathcal{B}$, $R(I)$ is injective.*
- *Dually, if R is exact, then L preserves projectives.*

Proof. We only prove the first case, as the second one is dual. Let I be an injective object of $\mathcal{B}$. We want to show that $\operatorname{Hom}_{\mathcal{A}}(-, R(I))$ is exact. Let $f : A \to A'$ be an injection in $\mathcal{A}$. We want to show that the map $f^* : \operatorname{Hom}_{\mathcal{A}}(A', R(I)) \to \operatorname{Hom}_{\mathcal{A}}(A, R(I))$ is a surjection. Since L is the left adjoint of R, we have the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{B}}(L(A'), I) \arrow[r, "Lf^*"] \arrow[d, "\cong"] & \operatorname{Hom}_{\mathcal{B}}(L(A), I) \arrow[d, "\cong"] \\
\operatorname{Hom}_{\mathcal{A}}(A', R(I)) \arrow[r, "f^*"] & \operatorname{Hom}_{\mathcal{A}}(A, R(I))
\end{tikzcd}.$$

As L is left exact, the map $Lf : L(A) \to L(A')$ is an injection. Moreover, since I is injective, the map Lf^* is a surjection. Therefore, the map f^* is also a surjection, proving that $R(I)$ is an injective object in $\mathcal{A}$. ☐

Example 4.52 (Base Change Spectral Sequence). Let R, S be rings and $\varphi : R \to S$ be a ring homomorphism. Let B be an S-module. Consider the composite

$$R - \mathbf{Mod} \xrightarrow{\operatorname{Hom}_R(S,-)} S - \mathbf{Mod} \xrightarrow{\operatorname{Hom}_S(B,-)} \mathbf{Ab}.$$

Note that $\operatorname{Hom}_R(S, -)$ and $\operatorname{Hom}_S(B, -)$ are both left exact. The left adjoint of $\operatorname{Hom}_R(S, -)$ is given by the functor $\tilde{\varphi} : S - \mathbf{Mod} \to R - \mathbf{Mod}$ which sends an S-module M to the same group M but seen as an R-module via the map φ. Note that $\tilde{\varphi}$ is exact. By Proposition 4.51, we have that $\operatorname{Hom}_R(S, -)$ preserves injectives. Since injectives are $\operatorname{Hom}_S(B, -)$-acyclic objects, we can apply Theorem 4.50, and we get for any R-module A the following convergent spectral sequence

$$E_2^{pq} = \operatorname{Ext}^p_S(B, \operatorname{Ext}^q_R(S, A)) \Rightarrow \operatorname{Ext}^{p+q}_R(B, A)$$

where, in the RHS, B inherits to the R-module structure induced by the map φ.

Example 4.53 (Leray Spectral Sequence). Let $f : X \to Y$ be a continuous map of topological spaces. The direct image sheaf functor $f_* : \operatorname{Sh}(X) \to \operatorname{Sh}(Y)$ has the inverse image functor f^{-1} as a right adjoint. Moreover, f_* is left exact. By Proposition 4.51, since f^{-1} is exact, we find that f_* preserves injectives.

Now let $\mathcal{F}$ be a sheaf of abelian groups on X. We observe that

$$\Gamma(Y, f_*\mathcal{F}) = f_*\mathcal{F}(Y) = \mathcal{F}(f^{-1}(Y)) = \mathcal{F}(X) = \Gamma(X, \mathcal{F}),$$

that is we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}(X) \arrow[rr, "f_*"] \arrow[dr, "\Gamma"'] & & \mathrm{Sh}(Y) \arrow[dl, "\Gamma"] \\
& \mathbf{Ab} &
\end{tikzcd}.$$

Define $H^p := R^p\Gamma$ the right derived functor of Γ. Then, Theorem 4.50 provides the following convergent spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

This spectral sequence is called the *Leray spectral sequence*.

4.10 Exercises Week 10

(i) (**Mapping cylinder**) Let $\mathcal{A}$ be an abelian category and consider a morphism $f : B \to C$ between cochain complexes in B. We define the mapping cylinder of f to be the cochain complex $\mathrm{cyl}(f)$ given by

- $\mathrm{cyl}(f)^n := B^n \oplus B^{n+1} \oplus C^n$,
- differential map $d = \mathrm{cyl}(f)^n \to \mathrm{cyl}(f)^{n+1}$ given by the matrix

$$\begin{bmatrix} d_B & \mathrm{id}_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{bmatrix}.$$

(a) Show that $\mathrm{cyl}(f)$ is a well-defined cochain complex.

In the following, we denote by $\mathrm{cyl}(B)$ the mapping cylinder of the identity map id_B on the object B.

(b) Given $f, g : B \to C$, show that f and g are chain homotopic if and only if there exists a family of maps $\{s^n : B^{n+1} \to C^n\}$ such that $(f, s, g) : \mathrm{cyl}(B) \to C$ is a morphism of cochain complexes.

(c) Consider the maps $\alpha : B \to \mathrm{cyl}(B)$, given by $\alpha(b) = (b, 0, 0)$, and $\beta : \mathrm{cyl}(B) \to B$, given by $\beta(b', b'', b) = b' + b$. Show that the map $s(b', b'', b) = (0, b, 0)$ defines a chain homotopy between $\mathrm{id}_{\mathrm{cyl}(B)}$ and $\alpha\beta$. Deduce that α is a chain homotopy equivalence.

(d) Now consider $\alpha' : B \to \mathrm{cyl}(B)$, given by $\alpha'(b) = (0, 0, b)$. Using the universal property of the category $\mathbf{K}(\mathcal{A})$, deduce that α' is also a chain homotopy equivalence. Find an explicit map s defining a chain homotopy between $\mathrm{id}_{\mathrm{cyl}(B)}$ and $\alpha'\beta$. (Hint: consider $F : \mathbf{Ch}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ in the universal property diagram.)

(ii) (**Examples of exact triangles**) Consider an abelian category $\mathcal{A}$ and objects A, B and C in $\mathcal{A}$.

(a) Find exact triangles for the maps 0_A and 1_A. (Hint: use the mapping cones in the category $\mathbf{K}(\mathcal{A})$.)

(b) Given an exact triangle (u, v, w) on A, B, C, show that the rotates

$$\begin{tikzcd}
& A[-1] \arrow[dl, "-u[-1]"'] & \\
B \arrow[rr, "v"] & & C \arrow[ul, "w"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[dl, "v"'] & \\
C[+1] \arrow[rr, "-w[-1]"] & & A \arrow[ul, "u"']
\end{tikzcd}$$

are also exact triangles.

(iii) Show that there is no morphism $w : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ making the short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \overset{\cdot 2}{\to} \mathbb{Z}/4\mathbb{Z} \overset{\cdot 1}{\to} \mathbb{Z}/2\mathbb{Z} \to 0$$

into an exact triangle $(2, 1, w)$.

(iv) Let $\mathcal{D}$ be a triangulated category and let the two rows below be two exact triangles, and $g : B \to B'$ a morphism. Show that if $v'gu = 0$ then there exists $f : A \to A', h : C \to C'$ such that (f, g, h) is a morphism of triangles (i.e. the diagram below commutes).

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, dotted, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, dotted, "h"'] & TA \arrow[d, dotted, "f"'] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'
\end{tikzcd}$$

Hint: consider cohomological functor $\mathrm{Hom}(X, \cdot)$.

(v) Let $\mathcal{A}^{\mathbb{Z}}$ be the category of graded objects in an abelian category $\mathcal{A}$ (i.e. $\{A_n\}_{n\in\mathbb{Z}}$ with A_n an object in $\mathcal{A}$), a morphism form $A = \{A_n\}$ to $B = \{B_n\}$ being a family of morphisms $f_n : A_n \to B_n$. Define $T(A)$ to be the translated graded object such that $T(A)_n = A_{n-1}$ and call (u, v, w) an exact triangle on (A, B, C) if for all n the sequence

$$A_n \overset{u}{\to} B_n \overset{v}{\to} C_n \overset{w}{\to} A_{n-1} \overset{u}{\to} B_{n-1}$$

is exact in $\mathcal{A}$.

- (a) If $\mathcal{A} = \mathbf{Ab}$ the category of abelian group, show that axiom (TR1) and (TR2) hold but (TR3) fails for $\mathcal{A}^{\mathbb{Z}}$. Hint: consider group extension.
- (b) If $\mathcal{A}$ is the category of vector space over a field, show that $\mathcal{A}^{\mathbb{Z}}$ is a triangulated category.

(vi) Show that in a triangulated category, every commutative square in the upper diagram below can be completed to the lower diagram, in which all the rows and columns are exact triangles and all the squares commute, except the bottom right corner anticommutes.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "u"'] & B \arrow[d] \\
A' \arrow[r] & B'
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "u"'] & B \arrow[r, "j"] \arrow[d] & C \arrow[r, "k"] \arrow[d] & T(A) \arrow[d, "Tu"] \\
A' \arrow[r] \arrow[d, "v"'] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & T(A') \arrow[d, "Tv"] \\
A'' \arrow[r] \arrow[d, "w"'] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & T(A'') \arrow[d, "Tw"] \\
T(A) \arrow[r, "Ti"] & T(B) \arrow[r, "Tj"] & T(C) \arrow[r, "Tk"] & T^2(A)
\end{tikzcd}$$

Hint: use (TR1) to construct everything except the third column, and construct an exact triangle on (A, B', D), then use (TR4) to construct exact triangles on (C, D, B''), (A'', D, C'), and finally (C', C'', C).

Chapter 5. Derived Categories

5.1 The category $\mathbf{K}(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category. We consider $\mathbf{Ch}(\mathcal{A})$ the category of cochain complexes in $\mathcal{A}$. We start by defining the category $\mathbf{K}(\mathcal{A})$.

Definition 5.1. The category $\mathbf{K} = \mathbf{K}(\mathcal{A})$ is the quotient category of $\mathbf{Ch}(\mathcal{A})$ defined as follows: the objects are the cochain complexes in $\mathcal{A}$, and the morphisms are the chain homotopy equivalence classes of morphisms of chain complexes. In the sense that

$$\mathrm{Hom}_{\mathbf{K}}(A, B) = \mathrm{Hom}_{\mathrm{Ch}}(A, B)/\sim$$

where $f \sim g \iff f - g = ds + sd$ for some $s : A^{n+1} \to B^n$.

By exercise 1.4.5 in Weibel's book, this category is well defined. It also is additive but is not abelian in general. The additivity follows from the fact that if we have chain homotopy equivalences $f_1 \sim g_1, f_2 \sim g_2$ then the sum is also an equivalence i.e. $f_1 + f_2 \sim g_1 + g_2$. Furthermore, this construction is closed under taking full subcategory in the sense that if $\mathcal{C}$ is a full subcategory of $\mathrm{Ch}(\mathcal{A})$ and $\mathcal{K}$ is the full subcategory of $\mathbf{K}(\mathcal{A})$ with $\mathrm{Ob}\mathcal{K} = \mathrm{Ob}\mathcal{C}$, then

$$\mathrm{Hom}_{\mathcal{K}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)/\sim .$$

Lemma 5.2. *Cohomology induces well defined functors*

$$H^i : \mathbf{K}(\mathcal{A}) \to \mathcal{A}.$$

Proof. If $f \sim g$, we have $f - g \sim 0$. In particular $H^i(f - g) = 0$ so we conclude that $H^i(f) = H^i(g)$. □

Proposition 5.3 (Universal property). *Let $F : \mathrm{Ch}(\mathcal{A}) \to \mathcal{D}$ be a functor that sends chain homotopy equivalences to isomorphisms, then F factors uniquely through $\mathbf{K}(\mathcal{A})$:*

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{A}) \arrow[r, "F"] \arrow[d] & \mathcal{D} \\
\mathbf{K}(\mathcal{A}) \arrow[ur, "\exists!"] &
\end{tikzcd}$$

To prove this proposition, we need to define the mapping cylinder of a morphism of cochain complexes.

Definition 5.4. Let $f : B \to C$ be a morphism of cochain complexes. The mapping cylinder $\mathrm{Cyl}(f)$ of f is the cochain complex defined as $\mathrm{Cyl}(f)^n = B^n \oplus B^{n+1} \oplus C^n$, where the differentials are given by

$$\partial = \begin{pmatrix} d_B & d_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{pmatrix}.$$

We will also denote by $\mathrm{Cyl}(B) := \mathrm{Cyl}(\mathrm{id}_B)$ the mapping cylinder of the identity of B

We can now proof the universal property.

Proof. Let $f \sim g : B \to C$ and let s be the chain homotopy morphism, we want to show that $F(f) = F(g)$. Consider the morphisms

$$\alpha : B \to \mathrm{Cyl}(B) : b \mapsto (0, 0, b),$$
$$\beta : \mathrm{Cyl}(B) \to B : (b', b'', b) \to b' + b.$$

We have $\beta\alpha = \mathrm{id}_B$, $\alpha\beta \sim \mathrm{id}_{\mathrm{Cyl}(B)}$ by exercise 1c of sheet 11. So by our hypothesis on the functor F, $F(\alpha)$ is an isomorphism of inverse $F(\beta)$. Now define α' as

$$\alpha' : B \to \mathrm{Cyl}(B) : b \mapsto (b, 0, 0),$$

this morphism is a left inverse of β i.e. $\beta\alpha' = \mathrm{id}_B$, so

$$F(\alpha') = F(\alpha)F(\beta)F(\alpha') = F(\alpha).$$

By exercise 1b of sheet 11, $f \sim g$ implies that the map $\gamma = (f, s, g) : \mathrm{Cyl}(B) \to C$ is a chain complex morphism. Moreover, $\gamma\alpha' = f$, $\gamma\alpha = g$, so we conclude that

$$F(f) = F(\gamma)F(\alpha') = F(\gamma)F(\alpha) = F(g).$$

□

Definition 5.5. Let $u : A \to B$ be a morphism of cochain complexes. We have seen that this gives us a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "v"] & \mathrm{cone}(u) \arrow[r, "\delta"] & A[-1] \arrow[r] & 0 . \end{tikzcd}$$

The strict triangle on u is the triplet (u, v, δ) of maps in $\mathbf{K}(\mathcal{A})$ usually denoted as

$$\begin{tikzcd} & \mathrm{cone}(u) \arrow[dl, "\delta"] & & . \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] & \end{tikzcd}$$

Definition 5.6. Let A, B and C be cochain complexes and $u : A \to B$, $v : B \to C$, $w : C \to A[-1]$ be morphisms in $\mathbf{K}(\mathcal{A})$. We say that (u, v, w) is an exact triangle on (A, B, C) if it is isomorphic to a strict triangle, i.e. there exists some strict triangle

$$\begin{tikzcd} & \mathrm{cone}(u') \arrow[dl, "\delta'"] & & . \\ A' \arrow[rr, "u'"] & & B' \arrow[ul, "v'"'] & \end{tikzcd}$$

and chain homotopy equivalences (i.e. isomorphisms in $\mathbf{K}(\mathcal{A})$) f, g, h such that the following diagram commutes in $\mathbf{K}(\mathcal{A})$(i.e $u'f \sim gu$, ...)

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & A[-1] \arrow[d, "{f[-1]}"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & \mathrm{cone}(u') \arrow[r, "\delta"] & A'[-1] \end{tikzcd} .$$

Definition 5.7. Let (u, v, w) be an exact triangle on (A, B, C), then we have a long exact sequence of cohomology

$$\begin{tikzcd} \ldots \arrow[r] & H^i(A) \arrow[r, "u^*"] & H^i(B) \arrow[r, "v^*"] & H^i(C) \arrow[r, "w^*"] & H^{i+1}(A) \arrow[r] & \ldots \end{tikzcd}$$

Proof. We have already seen that the cone short exact sequence induces a long exact sequence, using the "triangle isomorphism" diagram and the functoriality of H^i, we get the long exact sequence for exact triangles. □

Example 5.8. The morphisms 0_A and 1_A induce exact triangles

$$\begin{tikzcd} & A \oplus A[-1] \arrow[dl] & \\ A \arrow[rr, "0_A"] & & A \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl] & \\ A \arrow[rr, "1_A"] & & A \arrow[ul] \end{tikzcd}$$

Example 5.9. If (u, v, w) is an exact triangle on (A, B, C), then so are

$$\begin{tikzcd} & A[-1] \arrow[dl, "{-u[-1]}"'] & \\ B \arrow[rr, "v"] & & C \arrow[ul, "w"'] \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "v"'] & \\ C[-1] \arrow[rr, "{-w[-1]}"] & & A \arrow[ul, "u"'] \end{tikzcd}$$

Remark 5.10. Given a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there may not be a cochain map $w : C \to A[-1]$ such that (u, v, w) is an exact triangle, even though the long exact sequence in homology exists. The long exact sequence is induced by the exact triangle

$$\begin{tikzcd}
& \mathrm{cone}(u) \arrow[dl, "w"'] & \\
A \arrow[rr] & & \mathrm{Cyl}(u) \arrow[ul]
\end{tikzcd}$$

5.2 Triangulated categories

The triangulated categories are additive categories equipped with an additional structure that allows us to work on exact triangles in $\mathbf{K}(\mathcal{A})$ like we usually do with short exact sequence.

Definition 5.11. Let $\mathcal{A}$ be an additive category equipped with an automorphism $T : \mathcal{A} \to \mathcal{A}$ (e.g. the $[-1]$ functor for $\mathbf{K}(\mathcal{A})$). A triangle on (A, B, C) is a triple (u, v, w), that we may denote $(A, B, C)(u, v, w)$,

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA \ .$$

A morphism of triangles is a triple (f, g, h) that makes the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'.
\end{tikzcd}$$

A triangulated category $\mathbf{K}$ is an additive category equipped with an automorphism $T : \mathbf{K} \to \mathbf{K}$ (called the translation functor) and with a distinguished family of triangles (u, v, w), called the exact triangles in $\mathbf{K}$, which satisfies the following axioms

TR1 For every morphism $u : A \to B$ in $\mathbf{K}$, there exists an exact triangle (u, v, w).

The triangle $(A, A, 0)(\mathrm{id}_A, 0, 0)$ is exact.

If (u, v, w) and (u', v', w') are isomorphic triangles such that (u, v, w) is exact, then (u', v', w') is also exact.

TR2 (Rotation) If $(A, B, C)(u, v, w)$ is exact then so are $(T^{-1}C, A, B)(-T^{-1}w, u, v)$ and $(B, C, TA)(v, w, -Tu)$.

TR3 (Morphisms) Given two exact triangles

$$\begin{tikzcd}
& C \arrow[dl, "w"'] & & & C' \arrow[dl, "w'"'] & \\
A \arrow[rr, "u"] & & B \arrow[ul, "v"'] & A' \arrow[rr, "u'"] & & B' \arrow[ul, "v'"']
\end{tikzcd}$$

with morphism $f : A \to A'$, $g : B \to B'$ such that $gu = u'f$, there exists a morphism $h : C \to C'$ such that (f, g, h) is a morphism of triangles

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dotted, "\exists h"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'.
\end{tikzcd}$$

TR4 (Octahedral axiom) Given A, B, C, A', B', C' objects in $\mathbf{K}$ such that there exists three exact triangles $(A, B, C')(u, j, \partial)$, $(B, C, A')(v, x, i)$ and $(A, C, B')(vu, y, \delta)$. Then there exists a fourth exact triangle $(C', B', A')(f, g, (Tj)i)$ such that in the following octahedron diagram, the four exact triangle form four faces and the four other faces commute

$$\begin{tikzcd}
 & & & C \arrow[dl, "x"] \arrow[ddr, "y"] & \\
 & & A' \arrow[dll, "i"] \arrow[ddd, "(Tj)i"] & & \\
B' \arrow[uurrr, "v"] \arrow[ddrr, "j"] & & & & B' \arrow[ull, "\exists g"] \arrow[dl, dashed, "\delta"] \\
 & & & A \arrow[ulll, dashed, "u"] \arrow[uuu, dashed, "vu"] & \\
 & & C' \arrow[ur, dashed, "\partial"] \arrow[uurr, "\exists f"] & &
\end{tikzcd}$$

Our goal will now be to prove that $\mathbf{K}(\mathcal{A})$ is a triangulated category.

Proposition 5.12. *If $(A, B, C)(u, v, w)$ is exact, then $vu = wv = (Tu)v = 0$.*

Proof. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "u"] & 0 \arrow[r] & TA \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & TA
\end{tikzcd}$$

The first square commutes, so by TR3, there exists a morphism $f : 0 \to C$ such that $vu = f0 = 0$. The other cases are similar. ☐

Proposition 5.13. *Let X be an object in a triangulated category $\mathbf{K}$ and let $(A, B, C)(u, v, w)$ be an exact triangle. Then there is an exact sequence in **Ab***

$$\cdots \mathrm{Hom}(X, A) \xrightarrow{u^*} \mathrm{Hom}(X, B) \xrightarrow{v^*} \mathrm{Hom}(X, C) \xrightarrow{w^*} \mathrm{Hom}(X, TA) \cdots$$

Proof. By TR2, it suffices to show the exactness at $\mathrm{Hom}(X, B)$. We have seen that $vu = 0$, so $v^*u^* = 0$. Let $g \in \ker(v^*)$ i.e. $vg = 0$, and consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, dotted, "\exists f"] & X \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d, "0"] & TX \arrow[d, dotted, "\exists Tf"] \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & TA
\end{tikzcd} .$$

The middle square commutes, so by using TR2 and TR3, there exists a map f such that $uf = u^*(f) = g$. ☐

Lemma 5.14 (5-lemma)**.** *If (f, g, h) is a morphism of exact triangles such that two of the morphisms are isomorphisms, then the third is also an isomorphism.*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\
A \arrow[r, "u'"] & B \arrow[r, "v'"] & C \arrow[r, "w'"] & TA'
\end{tikzcd}$$

Proof. We do the case where f and g are isomorphisms, the other cases follow from TR2. By the last proposition there exist exact sequences

$$\mathrm{Hom}(C', A) \to \mathrm{Hom}(C', B) \to \mathrm{Hom}(C', C) \to \mathrm{Hom}(C', TA) \to \mathrm{Hom}(C', TB),$$

$$\mathrm{Hom}(C', A') \to \mathrm{Hom}(C', B') \to \mathrm{Hom}(C', C') \to \mathrm{Hom}(C', TA') \to \mathrm{Hom}(C', TB')$$

There are vertical maps f^*, g^*, h^*, Tf^* and Tg^* between these exact sequences. By our hypothesis, the two left maps and the two right maps are isomorphisms, so h^* is an isomorphism by the 5-lemma on abelian groups, and we can conclude that h is also an isomorphism by Yoneda's lemma. ☐

Corollary 5.15. *Given a morphism $u : A \to B$ there exists an unique exact triangle (u, v, w) up to isomorphism.*

Proof. Assume there exists two exact triangles containing u, then by TR3 there exists a morphism $C \to C'$ and by last proposition, this morphism is an isomorphism.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\mathrm{id}"] & B \arrow[r, "v"] \arrow[d, "\mathrm{id}"] & C \arrow[r, "w"] \arrow[d, dotted, "\exists"] & TA \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "u"] & B \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA
\end{tikzcd}$$

☐

Remark 5.16. We give another visualization of the octahedral axiom, we write the triangle as straight lines omitting the morphism $C \to TA$:

$$\begin{tikzcd}
 & & C' \arrow[dr, "\exists f"] & & \\
 & B \arrow[ur, "j"] \arrow[dr, "v"] & & B' \arrow[ddr, "\exists g"] & \\
 & & C \arrow[ur, "y"] \arrow[drr, "x"] & & \\
A \arrow[uur, "u"] \arrow[urr, "vu"] & & & & A
\end{tikzcd}$$

The octahedral axiom says that the line (C', B', A') is induced by the three other lines.

Proposition 5.17. *$\mathbf{K}(\mathcal{A})$ is a triangulated category, with the translation being the $[-1]$ functor and the exact triangles being the class of triangles isomorphic to*

$$A \xrightarrow{u} B \to \mathit{cone}(u) \to A[-1]$$

Proof. We only prove the first axiom. By definition, every morphism $u : A \to B$ can be embedded in a triangle $A \to B \to \mathrm{cone}(u)$, and the third property holds by definition of the exact triangles, so we just have to show the second property. Consider the following diagram,

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[r] \arrow[d] & A[-1] \arrow[d, "\mathrm{id}"] & {} \arrow[d, "f"] \\
A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{cone}(A) \arrow[r] \arrow[d] & A[-1] \arrow[d, "\mathrm{id}"] & {} \arrow[d, "g"] \\
A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[r] \arrow[d] & A[-1] \arrow[d, "\mathrm{id}"] & {} \arrow[d, "f"] \\
A \arrow[r, "\mathrm{id}"] & A \arrow[r] & \mathrm{cone}(A) \arrow[r] & A[-1] & {}
\end{tikzcd}$$

we want to show that $f = g^{-1}$ in $\mathbf{K}(\mathcal{A})$. This is enough to show that $gf \sim \mathrm{id}$, since $fg = \mathrm{id}$. We have that $gf = 0 \sim \mathrm{id} : \mathrm{cone}(A) \to \mathrm{cone}(A)$ with $s = \begin{pmatrix} -d & 0 \\ -1 & d \end{pmatrix}$, so g is indeed an isomorphism, and $(\mathrm{id}_A, 0, 0)$ is an exact triangle. ☐

5.3 Exercises Week 11

(i) The goal of this exercise is to complete the proof showing that $K(\mathcal{A})$ is triangulated. Recall that to prove (TR4) one needs to show the following diagram commutes and that $(\mathrm{cone}(u), \mathrm{cone}(vu), \mathrm{cone}(v))$ is an exact triangle:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal] & B \arrow[r] \arrow[d, "v"] & \mathrm{cone}(u) \arrow[r] \arrow[d] & A[-1] \arrow[d, equal] \\
A \arrow[r, "vu"] & C \arrow[r] \arrow[d] & \mathrm{cone}(vu) \arrow[r] \arrow[d] & A[-1] \\
 & \mathrm{cone}(v) \arrow[r, equal] \arrow[d] & \mathrm{cone}(v) \arrow[d] & \\
 & B[-1] \arrow[r] & \mathrm{cone}(u[-1]). &
\end{tikzcd}$$

The only part remaining to show that $(\mathrm{cone}(u), \mathrm{cone}(vu), \mathrm{cone}(v))$ is indeed a triangle. By definition of triangles in $\mathcal{K}(\mathcal{A})$, it suffices to show the commutativity of the following diagram in $\mathcal{K}(\mathcal{A})$, plus that γ is a homotopy equivalent in $\mathbf{Ch}((\mathcal{A}))$. We have defined the maps $v \oplus \mathrm{id}, \mathrm{id} \oplus u, \pi_B, \gamma$ in the lecture and one may easily verify every required condition except for the following:

(a) $\gamma \circ (\mathrm{id} \oplus u)$ is homotopic to ι.

(b) γ defines a homotopy equivalence between the two chain complexes.

$$\begin{tikzcd}
B \oplus A[-1] \arrow[r, "v\oplus\mathrm{id}"] \arrow[d, equal] & C \oplus A[-1] \arrow[r, "\mathrm{id}\oplus u"] \arrow[d, equal] & C \oplus B[-1] \arrow[r, "\pi_B"] \arrow[d, "\gamma"] & B[-1] \oplus A[-2] \arrow[d, equal] \\
B \oplus A[-1] \arrow[r, "v\oplus\mathrm{id}"] & C \oplus A[-1] \arrow[r, "\iota"] & (C \oplus A[-1]) \oplus (B[-1] \oplus A[-2]) \arrow[r] & B[-1] \oplus A[-2]
\end{tikzcd}$$

As mentioned one may construct γ as the natural inclusion and the homotopy inverse of γ as

$$g : (c, a', b', a'') \to (c, u(a') + b').$$

By representing the morphisms as matrices, find the suitable chain homotopies and show (a) and (b).

Hint: You may need the matrix corresponding to the boundary map of $(C \oplus A[-1]) \oplus (B[-1] \oplus A[-2])$, *which is given by* $\begin{pmatrix} \partial & -vu & -v & 0 \\ 0 & -\partial & 0 & \mathrm{id} \\ 0 & 0 & -\partial & u \\ 0 & 0 & 0 & \partial \end{pmatrix}$. *The other matrices you need should be rather trivial.*

(ii) Let S be a locally small multiplicative system of morphisms in a category $\mathcal{C}$. Let $q : \mathcal{C} \to S^{-1}\mathcal{C}$ be the localization constructed in class.

(a) Let Z be a zero object in $\mathcal{C}$. Show that $q(Z)$ is a zero object. Deduce that for every X in $\mathcal{C}$ we have

$$q(X) \cong 0 \iff S \text{ contains the zero map } X \xrightarrow{0} X.$$

(b) Assume that the product $X \times Y$ exists in $\mathcal{C}$. Show that

$$q(X \times Y) \cong q(X) \times q(Y)$$

in $S^{-1}\mathcal{C}$.

(c) Assume now that C is an additive category. Show that $S^{-1}\mathcal{C}$ is also additive and that q is an additive functor.

(iii) Let R be a commutative ring and $S \subseteq R$ be multiplicatively closed subset. Let **mod-**R be the category of R-modules. Let Σ be the collection of all morphisms $A \to B$ in **mod-**R such that the induced morphism on localizations $S^{-1}A \to S^{-1}B$ is an isomorphism. Show that **mod-**$S^{-1}R$ is a localizing subcategory of **mod-**R and

$$\mathbf{mod}\text{-}S^{-1}R \cong \Sigma^{-1}\mathbf{mod}\text{-}R.$$

(iv) Let $\mathcal{A}$ be an abelian category. An abelian subcategory $\mathcal{B}$ of $\mathcal{A}$ is called a *Serre subcategory* if it is closed under sub-objects, quotients, and extensions. In other words, this means that for every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, the object A is in $\mathcal{B}$ if and only if both A' and A'' are in $\mathcal{B}$.

Suppose that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ and let Σ be the family of all morphisms f in $\mathcal{A}$ with $\ker(f)$ and $\operatorname{coker}(f)$ in $\mathcal{B}$.

(a) Show that Σ is a multiplicative system in $\mathcal{A}$. We write $\mathcal{A}/\mathcal{B}$ for the localization $\Sigma^{-1}\mathcal{A}$ (provided that it exists).

(b) Show that $q(X) \cong 0$ in $\mathcal{A}/\mathcal{B}$ if and only if X is in $\mathcal{B}$.

(c) Assume that $\mathcal{B}$ is a small category, and show that Σ is locally small. This is one case in which $\mathcal{A}/\mathcal{B} = \Sigma^{-1}\mathcal{A}$ exists.

(d) Show that $\mathcal{A}/\mathcal{B}$ is an abelian category and that $q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is an exact functor.

(e) Let S be a multiplicative subset of a commutative ring R (in the usual sense), and let $\mathbf{mod}_S R$ be the full subcategory of R-modules A such that $S^{-1}A \cong 0$. Show that $\mathbf{mod}_S R$ is a Serre subcategory of **mod-**R. Conclude that $\mathbf{mod}\text{-}S^{-1}R \cong \mathbf{mod}\text{-}R/\mathbf{mod}_S R$.

(v) The goal of this exercise is to show that $\mathbf{K}(\mathrm{Ab})$ is not abelian. Let $\mathcal{K}$ be a triangulated category.

(a) Let

$$A \to B \to C \to TA$$

and

$$A' \to B' \to C' \to TA'$$

be exact triangles in $\mathcal{K}$. Show that there is an exact triangle

$$A \oplus A' \to B \oplus B' \to C \oplus C' \to T(A \oplus A').$$

(b) Let

$$A \overset{u}{\to} B \overset{v}{\to} C \overset{w}{\to} TA$$

be a triangle in $\mathcal{K}$ and assume that w is zero. Deduce from (a) that

$$B \cong A \oplus C.$$

Hint: Show that the triangles

$$A \overset{\mathrm{id}}{\to} A \to 0 \to TA$$

and

$$0 \to C \overset{\mathrm{id}}{\to} C \to 0$$

are exact using (TR4)*, or any other way. Conclude using* (TR3).

(c) Let $f : A \to B$ be a monic in $\mathcal{K}$. Deduce from (b) that

$$B \cong A \oplus C$$

for some C in $\mathcal{K}$.

(d) We now restrict to $\mathcal{K} = \mathbf{K}(\mathrm{Ab})$. Show that $\mathbf{K}(\mathrm{Ab})$ is not abelian by considering the map

$$\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}.$$

From last time, we still need to prove the following :

Proposition 5.18. *Let $\mathcal{A}$ be an abelian category. Then $\boldsymbol{K}(\mathcal{A})$ is triangulated.*

Proof. Axioms (TR1), (TR2) and (TR3) were already discussed last time, so it only remains to prove (TR4). Let $A \xrightarrow{u} B \longrightarrow C'$, $B \xrightarrow{v} C \longrightarrow A'$ and $A \xrightarrow{vu} C \longrightarrow B'$ be triangles, we need to show that there exists a fourth triangle $C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{h} C'[-1]$ making the suitable diagram commute. By construction of triangles in $\mathbf{K}(A)$, we may replace C', B', A' by cone(u), cone(v), cone(vu). Note that h is determined by the morphisms $A' \overset{i}{\to} B[-1]$ and $B \overset{j}{\to} \text{cone}(u)$ via $h := j[-1] \circ i$. A natural guess for the maps f and g is $f = v \oplus \text{Id}_A$ and $g = \text{Id}_C \oplus u[-1]$, as they fit into the commutative diagram

$$\begin{tikzcd}
 & C^n \arrow[d] \arrow[dr] & \\
B^n \oplus A^{n+1} \arrow[r, "f^n"] \arrow[dr] & C^n \oplus A^{n+1} \arrow[r, "g^n"] \arrow[d] & C^n \oplus B^{n+1} \;. \\
 & A^{n+1} &
\end{tikzcd}$$

To prove (TR4), we just need to show that (f, g, h) is an exact triangle on (cone(u), cone(vu), cone(v)). In turn, it is sufficient to construct a map $\gamma : \text{cone}(v) \to \text{cone}(f)$ such that the suitable diagram commutes in $\mathbf{Ch}(\mathcal{A})$ and γ is an isomorphism in $\mathbf{K}(\mathcal{A})$. Set

$$\gamma^n : \begin{array}{lcl} C^n \oplus B^{n+1} & \to & (C^n \oplus A^{n+1}) \oplus (B^{n+1} \oplus A^{n+2}) \\ (c, b) & \mapsto & (c, 0, b, 0), \end{array} .$$

It can be shown that this choice makes the following diagram commutative up to chain homotopy equivalence:

$$\begin{tikzcd}
\text{cone}(u) \arrow[r, "f"] \arrow[d, equal] & \text{cone}(vu) \arrow[r] \arrow[d, equal] & \text{cone}(v) \arrow[r] \arrow[d, dotted, "\gamma"] & \text{cone}(u)[-1] \arrow[d, equal] \\
\text{cone}(u) \arrow[r, "f"] & \text{cone}(vu) \arrow[r] & \text{cone}(f) \arrow[r] & \text{cone}(u)[-1].
\end{tikzcd}$$

It remains to show that γ is an isomorphism in $\mathbf{K}(\mathcal{A})$. We define $\varphi : \text{cone}(f) \to \text{cone}(v)$ by

$$\varphi^n : \begin{array}{lcl} (C^n \oplus A^{n+1}) \oplus (B^{n+1} \oplus A^{n+2}) & \to & C^n \oplus B^{n+1} \\ (c, a, b, a') & \mapsto & (c, u(a) + b), \end{array}$$

It is immediate that $\varphi\gamma = \text{Id}_{\text{cone}(v)}$. Exercise 1 week 12 demonstrates that there is a suitable homotopy showing that $\gamma\varphi \simeq \text{Id}_{\text{cone}(f)}$, which means that $\varphi = \gamma^{-1}$ in $\mathbf{K}(\mathcal{A})$.

□

Using this proposition, we can deduce the following

Corollary 5.19. *Let $\mathcal{C}$ be a subcategory of $\boldsymbol{Ch}(\mathcal{A})$ and $\mathcal{K}$ the corresponding subcategory of $\boldsymbol{K}(\mathcal{A})$. Suppose that $\mathcal{C}$ is full, additive, closed under translation and the mapping* cone *construction. Then $\mathcal{K}$ is triangulated. In particular, the quotient categories $\boldsymbol{K}^b(\mathcal{A}), \boldsymbol{K}^{\pm}(\mathcal{A})$ are triangulated.*

Proof. The proof is essentially the same as the above, so we do not write the details. □

Definition 5.20. A morphism of triangulated categories $\mathcal{F} : \mathbf{K}' \to \mathbf{K}$ is an additive functor between triangulated categories such that $\mathcal{F}T_{\mathbf{K}} = T_{\mathbf{K}'}\mathcal{F}$, and $\mathcal{F}$ preserves exact triangles.
We say that $\mathbf{K}'$ is a triangulated subcategory of $\mathbf{K}$ if

(i) $\mathbf{K}'$ is a full subcategory of $\mathbf{K}$.

(ii) $\iota : \mathbf{K}' \to \mathbf{K}$ is a morphism of triangulated categories.

(iii) Any exact triangle in $\mathbf{K}$ consisting of objects in $\mathbf{K}'$ is also exact in $\mathbf{K}'$.

For example $\mathbf{K}^b(\mathcal{A}), \mathbf{K}^{\pm}(\mathcal{A})$ are triangulated subcategories of $\mathbf{K}(\mathcal{A})$.

Definition 5.21. Let $\mathbf{K}$ be a triangulated category, and $\mathcal{A}$ be an abelian category. A covariant cohomological functor $\mathcal{H} : \mathbf{K} \to \mathcal{A}$ is a functor sending exact triangles in $\mathbf{K}$ to long exact sequences in $\mathcal{A}$. More explicitly, for

$$X \xrightarrow{u} Y \xrightarrow{v} W \xrightarrow{w} TW$$

an exact triangle in $\mathbf{K}$, the following sequence is exact in $\mathcal{A}$:

$$... \longrightarrow \mathcal{H}(T^iX) \longrightarrow \mathcal{H}(T^iY) \longrightarrow \mathcal{H}(T^iW) \longrightarrow \mathcal{H}(T^{i+1}W) \longrightarrow ...$$

Example 5.22. For X in $\mathbf{Ch}(\mathcal{A})$, it is not hard to see that $\mathrm{Hom}_{K(\mathcal{A})}(-, X)$ is a covariant cohomological functor.

Similarly, one can define the dual notion of contravariant cohomological functor.

5.4 Localization of a category

In this section, we introduce the notion of localization of a category with respect to a collection S of morphisms, which will all become isomorphisms in the resulting construction. This process is analogous to the localization of a commutative ring with respect to a multiplicative set. In fact, as any ring can be seen as a (locally small) preadditive category with a unique object, this construction will serve a as broad generalization of the usual localization.

Definition 5.23. Let $\mathcal{C}$ be a category, and S a collection of morphisms of C (not necessarily a set). A localization of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$ together with a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that the following holds :

(i) For any s in S, $Q(s)$ is an isomorphism.

(ii) Any functor $F : \mathcal{C} \to \mathcal{D}$ satisfying (1) factors uniquely through $S^{-1}\mathcal{C}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q"] & \mathcal{D} \\ S^{-1}\mathcal{C} \arrow[ur, dashed, "\exists!"] & \end{tikzcd} .$$

Remark 5.24. It follows from the above universal property that $S^{-1}\mathcal{C}$ is unique up to unique isomorphism, so that we may abuse the notations as usual and speak of "the" localization.

Example 5.25. (i) Let S be the collection of chain homotopy equivalences in $\mathbf{Ch}(\mathcal{A})$. As seen earlier, $K(\mathcal{A})$ satisfies the above universal property and is thus isomorphic to $S^{-1}\mathbf{Ch}(\mathcal{A})$.

(ii) Let $\tilde{Q}$ be the collection of quasi-isomorphisms in $\mathbf{Ch}(\mathcal{A})$. The localization $\tilde{Q}^{-1}\mathbf{Ch}(\mathcal{A})$, when it exists, is called the derived category of $\mathcal{A}$. We denote it by $\mathbf{D}(\mathcal{A})$.

Definition 5.26. Let $S, \mathcal{C}$ be as above. The collection S is said to be a multiplicative system if the following conditions are satisfied :

(i) S contains all identity morphisms of $\mathcal{C}$, and for any s, t in S such that their composition makes sense, we also have that $t \circ s$ is in S.

(ii) (Ore condition) For $t : Z \to Y$ in S, and $g : X \to Y$ any morphism, there exists W in $\mathcal{C}$, $s : W \to X$ in S and a morphism $f : W \to Z$ such that

$$\begin{tikzcd}
W \arrow[r, dotted, "\exists f"] \arrow[d, dotted, "\exists s"] & Z \arrow[d, "t"] \\
X \arrow[r, "g"] & Y
\end{tikzcd}$$

commutes. Moreover, the symmetric situation

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "s"] & Z \arrow[d, dotted, "\exists t"] \\
X \arrow[r, dotted, "\exists g"] & Y
\end{tikzcd}$$

is also true.

(iii) (Cancellation) Given $f, g : X \to Y$, the following are equivalent :
(i) $s \circ f = s \circ g$ for some s in S.
(ii) $f \circ t = g \circ t$ for some t in S.

Remark 5.27. Note that the above definition recover the notion of a multiplicative set from commutative algebra when $\mathcal{C}$ is the category associated to a commutative ring.

We now make the following construction : given S a multiplicative system, a fraction in S is a diagram of the shape

$$X \xleftarrow{s} X_1 \xrightarrow{f} Y,$$

where s is in S. We denote this by fs^{-1}. We say that two fractions fs^{-1} and gt^{-1} are equivalent if there exists a third fraction hr^{-1} such that the three fit into a commutative diagram of the shape

$$\begin{tikzcd}
& X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & X_3 \arrow[l, "r"'] \arrow[r, "h"] \arrow[u] \arrow[d] & Y. \\
& X_2 \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

To show this is indeed an equivalence relation, note that reflexivity and symmetry are immediate. Transivity follows by applying the Ore condition twice.
Let $\mathrm{Hom}_S(X, Y)$ be the family of equivalences classes of fractions $X \overset{s}{\leftarrow} X_1 \overset{f}{\to} Y$ under the above equivalence relation.

Definition 5.28. The collection S is said to be locally small if for any X in $\mathcal{C}$, there exists a subcollection S_X of S consisting of morphisms with target X satisfying :

(i) S_X is a set.

(ii) For any $X_1 \to X$ in S, there exists a morphism $X_2 \to X_1$ in $\mathcal{C}$ such that $X_2 \to X_1 \to X$ is in S_X.

Remark 5.29. This precisely means that every fraction $X \xleftarrow{s} X_1 \xrightarrow{f} Y$ is equivalent to a fraction $X \xleftarrow{t} X_2 \xrightarrow{g} Y$ with t in S_X :

$$\begin{tikzcd}
 & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & X_2 \arrow[l] \arrow[r] \arrow[u, dotted] & Y. \\
 & X_2 \arrow[ul, "t"] \arrow[ur, "g"'] \arrow[u, equal] &
\end{tikzcd}$$

Proposition 5.30. *If S is locally small,* $\mathrm{Hom}_S(X,Y)$ *is a set for any X, Y in $\mathcal{C}$.*

Proof. We give a sketch of proof. First we make S_X into a category by defining a morphism from $X_1 \xrightarrow{s} X$ to $X_2 \xrightarrow{t} X$ in S_X to be a morphism $X_2 \to X_1$ such that

$$\begin{tikzcd}
X_1 \arrow[r, "s"] & X \\
X_2 \arrow[u, dotted] \arrow[ur, "t"] &
\end{tikzcd}$$

commutes.
Then using the Ore condition, one can show that up to enlarging S_X a bit, it is a filtered category. Fix Y in $\mathcal{C}$. We can then define the functor

$$F : \begin{array}{ccl} S_X & \to & \mathbf{Sets} \\ s & \mapsto & \text{Set of fractions } X \xleftarrow{s} X_1 \xrightarrow{f} Y. \end{array}$$

Now we claim that $\operatorname*{colim}_{s \in S_X} F(s) = \mathrm{Hom}_S(X,Y)$. Indeed, the colimit can be described as the quotient of $\coprod_{s \in S_X} F(s)$ by a certain equivalence relation, which by chasing definitions is exactly what we constructed earlier. ☐

Given two fractions $X \xleftarrow{r} X' \xrightarrow{g} Y$ and $Y \xleftarrow{t} Y' \xrightarrow{h} Z$, we would like to find a suitable notion of morphism between them. By the Ore condition, we have the existence of morphisms f, s such that following diagram commutes :

$$\begin{tikzcd}
 & W \arrow[r, dotted, "f"] \arrow[d, dotted, "s"] & Y' \arrow[r, "h"] \arrow[d, "t"] & Z \\
X & X' \arrow[l, "r"'] \arrow[r, "g"] & Y. &
\end{tikzcd}$$

We then define $ht^{-1} \circ gr^{-1} = (hf)(rs)^{-1}$.

Proposition 5.31. *(i) The composition constructed above is well defined.*

(ii) $S^{-1}\mathcal{C}$ is a category with objects Ob $S^{-1}\mathcal{C}$ =Ob $\mathcal{C}$, morphisms given by $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y) := \mathrm{Hom}_S(X,Y)$, *and identities $X = X = X$.*

Proof. We only show independence of the choice of W and the associated morphisms, the other cases being similar. Assume we have another diagram

$$\begin{tikzcd}
 & \tilde{W} \arrow[r, dotted, "\tilde{f}"] \arrow[d, dotted, "\tilde{s}"] & Y' \arrow[r, "h"] \arrow[d, "t"] & Z \\
X & X' \arrow[l, "r"'] \arrow[r, "g"] & Y &
\end{tikzcd} .$$

Applying the Ore condition, we get a commutative square :

$$\begin{tikzcd}
W' \arrow[r, dotted, "\tilde{u}"] \arrow[d, dotted, "u"'] & \tilde{W} \arrow[d, "\tilde{s}"] \\
W \arrow[r, "s"] & X'
\end{tikzcd}$$

We compute that $t \circ f \circ u = t \circ \tilde{f} \circ \tilde{u}$, so that there exists a morphism $v : A \to W'$ in S with $f \circ u \circ v = \tilde{f} \circ \tilde{u} \circ v$ by cancellation. The following diagram then shows that $(hf)(rs)^{-1}$ and $(h\tilde{f})(r\tilde{s})^{-1}$ are equivalent :

$$\begin{tikzcd}
 & X' \arrow[dl, "r"'] & W \arrow[l, "s"'] \arrow[r, "f"] & Y' \arrow[dr, "h"] & \\
X & & A \arrow[ll] \arrow[rr] \arrow[u, "u\circ v"'] \arrow[d, "\tilde{u}\circ v"] & & Z \\
 & X' \arrow[ul, "r"] & \tilde{W} \arrow[l, "\tilde{s}"'] \arrow[r, "f"] & Y'. \arrow[ur, "h"'] &
\end{tikzcd}$$

The second point is clear. ☐

We are finally ready to show that the localizing category exists under the assumption that S is a locally small multiplicative system.

Theorem 5.32. *(Gabriel-Zisman)*
Let $\mathcal{C}$ be a category, and S a locally small multiplicative system. Then the functor

$$q : \begin{array}{ccc} \mathcal{C} & \to & S^{-1}\mathcal{C} \\ X \xrightarrow{f} Y & \mapsto & X = X \xrightarrow{f} Y \end{array}$$

(defined as the identity on the objects) is a localization.

Proof. We first check that q is functorial. It is clear that identities are preserved. Moreover, the diagram

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "h"] \arrow[d, equal] & Z \\
X \arrow[r, equal] & X \arrow[r, "f"] & Y &
\end{tikzcd}$$

shows that the composition of $X = X \xrightarrow{f} Y$ and $Y = Y \xrightarrow{h} Z$ is given by $X = X \xrightarrow{hf} Z$.
We now need to check that q sends elements of S to isomorphisms. But this is rather clear, the inverse of $X = X \xrightarrow{s} Y$ being given by $X \xrightarrow{s} X = Y$.
Lastly, we verify the universal property. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor sending elements of S to isomorphisms. Define $S^{-1}F : S^{-1}\mathcal{C} \to \mathcal{D}$ to be the functor which coincides with F on the objects, and sends a fraction fs^{-1} to $F(f)F(s)^{-1}$. It is clear that $S^{-1}F \circ q = F$, and that any functor satisfying this property must coincide with $S^{-1}F$, so we just have to check that $S^{-1}F$ is well-defined and indeed functorial. But both these properties follow from the functoriality of F and the assumption that $F(s)$ is invertible for any s in S. ☐

Remark 5.33. We could have constructed $S^{-1}\mathcal{C}$ using right fractions instead, using the dual notion of "locally small on the right" (i.e S^{op} is locally small in $\mathcal{C}^{op}$).

Corollary 5.34. *Two morphisms $f, g : X \to Y$ in $\mathcal{C}$ satisfy $q(f) = q(g)$ if and only if there is some s in S with $s \circ f = s \circ g$*

Proof. Indeed, both conditions are equivalent to the commutativity of

$$\begin{tikzcd}
 & X \arrow[dl, equal] \arrow[dr, "f"] & \\
X & X' \arrow[u, "s"] \arrow[l, "s"'] \arrow[r] \arrow[d, "s"] & Y. \\
 & X \arrow[ul, equal] \arrow[ur, "g"'] &
\end{tikzcd}$$

□

5.5 Exercises Week 12

(i) Give examples of maps f, g in $\mathrm{Ch}(\mathcal{A})$ such that

  (a) $f = 0$ in $\mathbb{D}(\mathcal{A})$, but f is not null homotopic, and

  (b) g induces the zero map on cohomology, but $g \neq 0$ in $\mathbb{D}(\mathcal{A})$.

  Hint: For (ii), try $X : 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0, Y : 0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z}/3 \to 0, g = (1, 2)$.

(ii) *(Proof of proposition 10.4.1.(2)).* Let $\mathcal{K}$ be a triangulated category and S a multiplicative system arising from a cohomological functor. We prove that $S^{-1}\mathcal{K}$ is triangulated. In the lectures,

- We defined the translation T' by taking the translation $T : \mathcal{K} \to \mathcal{K}$ of $\mathcal{K}$ and set T' as

$$T'(C) = T(C) \text{ for all } C \in \mathrm{Ob}\mathcal{K}$$

  and on morphisms

$$T'(X \xleftarrow{s} X' \xrightarrow{f} Y) = T(X) \xleftarrow{T(s)} T(X') \xrightarrow{T(f)} T(Y).$$

  (Which is well defined because S arises from a cohomological functor and by functoriality of T.)

- We defined the exact triangles as follows. Consider fractions

$$A \xleftarrow{s_1} A \xrightarrow{u} B$$
$$B \xleftarrow{s_2} B \xrightarrow{v} C$$
$$C \xleftarrow{s_3} C' \xrightarrow{w} T(A).$$

  Composition is given by the Øre condition

$$\begin{tikzcd}
A' \arrow[r, "u'"] \arrow[dd] \arrow[ddd, bend right, "t_1"'] & B' \arrow[r, "v'"] \arrow[d] \arrow[dd, bend right, "t_2"'] & C' \arrow[r, "w"] \arrow[d, "s_3"] & T(A) \\
 & B \arrow[r, "v"'] \arrow[d, "s_2"] & C & \\
A \arrow[r, "u"'] \arrow[d, "s_1"] & B & & \\
A & & &
\end{tikzcd}$$

  so that

$$vs_2^{-1} \cong s_3 v' t_2^{-1} \text{ and } us_1^{-1} \cong t_2 u' t_1^{-1}.$$

  We say that $(us_1^{-1}, vs_2^{-1}, ws_3^{-1})$ is an exact triangle in $S^{-1}\mathcal{K}$ just in case (u', v', w) is an exact triangle in $\mathcal{K}$.

Verify that $S^{-1}\mathcal{K}$ satisfies the axioms TR1, TR2, TR3 of a triangulated category.

(iii) This is a follow up of Exercise 12.4. Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$ and let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the quotient map constructed in Exercise 12.4.

- (a) Show that $H = \pi H^0 : \mathcal{K}(\mathcal{A}) \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is a cohomological functor, so that $\mathcal{K}_H(\mathcal{A})$ is a triangulated category. (See exercise 10.2.5 in Weibel's book).
- (b) Show that X is in $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ iff the cohomology $H^i(X)$ is in $\mathcal{B}$ for all i.
- (c) Show that $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ is a localizing subcategory of $\mathcal{K}(\mathcal{A})$, and conclude that its localization $\mathbb{D}_{\mathcal{B}}(\mathcal{A})$ is a triangulated subcategory of $\mathbb{D}(\mathcal{A})$.
- (d) Suppose that $\mathcal{B}$ has enough injectives and that every injective object of $\mathcal{B}$ is also injective in $\mathcal{A}$. Show that there is an equivalence
$$\mathbb{D}^+(\mathcal{B}) \cong \mathbb{D}^+_{\mathcal{B}}(\mathcal{A}).$$

(iv) Let R be a Noetherian ring, and let $\mathbb{M}(R)$ denote the category of all finitely generated R-modules. Let $\mathbb{D}_{\mathrm{fg}}(R)$ denote the full subcategory of $\mathbb{D}(\mathrm{Mod} - R)$ consisting of complexes A whose cohomology modules $H^i(A)$ are all finitely generated, that is, the category $\mathbb{D}_{\mathbb{M}(R)}(\mathrm{Mod} - R)$ of exercise 13.3.

Show that $\mathbb{D}_{\mathrm{fg}}(R)$ is a triangulated category and that there is an equivalence
$$\mathbb{D}^-(\mathbb{M}(R)) \cong \mathbb{D}^-_{\mathrm{fg}}(\mathbb{M}(R)).$$

Hint. By exercise 12.4, $\mathbb{D}(R)$ is a Serre subcategory of Mod-R.

(v) *(Albelian derived category).* The goal is to prove the following lemma.

Definition 5.35 (Semisimple abelian category)**.** An abelian category is called semisimple if all short exact sequence split.

Lemma 5.36. *Let $\mathcal{A}$ be an abelian category. The derived category $\mathbb{D}(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is semisimple.*

- (a) Show that if $\mathcal{A}$ is semisimple, for every morphism $f : A \to B$, there is a morphism $g : B \to A$ such that
$$fgf = f \text{ and } gfg = g.$$
- (b) Prove the following lemma.

  Lemma 5.37 (Verdier)**.** *A triangulated category $\mathcal{K}$ is abelian if and only if every morphism $f : A \to B$ is isomorphic to*
$$A' \oplus I \xrightarrow{\begin{pmatrix} 0 & 1_I \\ 0 & 0 \end{pmatrix}} I \oplus B'.$$
- (c) Using Lemma 5.37, prove that $\mathbb{D}(\mathcal{A})$ abelian implies that $\mathcal{A}$ is semisimple.
- (d) Prove that if $\mathcal{A}$ is semisimple, then $\mathbb{D}(\mathcal{A})$ is abelian.

5.6 Localization

Let $\mathcal{B}$ be a subcategory of $\mathcal{C}$ and S a multiplicative system in $\mathcal{C}$. We may want to compare the localization of $\mathcal{B}$ with $S^{-1}\mathcal{C}$ and, with this goal in mind, we introduce the following notion :

Definition 5.38. Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$, and S a locally small multiplicative system such that $S \cap \mathcal{B}$ is also a multiplicative system. Then, we write

$$S^{-1}\mathcal{B} := (S \cap \mathcal{B})^{-1}\mathcal{B}$$

and we define $S^{-1}\mathcal{B}$ to be a *localizing subcategory* of $\mathcal{C}$ (for S) if the natural functor

$$S^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$$

is fully faithful.

The following lemma gives useful characterizations of localizing subcategory :

Lemma 5.39. *Consider the following statements : 1. $\forall B, B'$ in $\mathcal{B}$, then*

$$\mathrm{Hom}_{S\cap\mathcal{B}}(B, B') \cong \mathrm{Hom}_S(B, B')$$

where the left hand side is thought in $\mathcal{B}$ and the right hand side is thought in $\mathcal{C}$.
2. Whenever $C \to B$ is a morphism in S with B in $\mathcal{B}$, there is a morphism $B' \to C$ in $\mathcal{C}$ with B' in $\mathcal{B}$ such that the composition

$$B' \to C \to B$$

is in S.
3. Whenever $B \to C$ is a morphism in S with B in $\mathcal{B}$, there is a morphism $C \to B'$ in $\mathcal{C}$ with B' in $\mathcal{B}$ such that the composition

$$B \to C \to B'$$

is in S.
Then we have that a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is localizing if and only if 1. holds. If S is locally small of the left 2. implies that $\mathcal{B}$ is localizing. And finally if S is locally small on the right 3. implies that $\mathcal{B}$ is localizing.

Proof. First, we prove that a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is localizing if and only if 1. holds :
By definition $\mathcal{B}$ is localizing if and only if

$$S^{-1}\mathcal{B} \to S^{1}\mathcal{C}$$

is fully faithful, i.e. $\forall B, B'$ in $S^{-1}\mathcal{B}$

$$\mathrm{Hom}_{S^{-1}\mathcal{B}} = \mathrm{Hom}_{S^{-1}\mathcal{B}}(B, B') \cong \mathrm{Hom}_{S^{-1}\mathcal{C}}(B, B') = \mathrm{Hom}_S(B, B')$$

and we conclude by the Gabriel-Zisman Theorem.
Now since the second and third statements are dual, we only prove that if S is locally small on the left, 2. implies 1.
Let

$$B \leftarrow C \to B''$$

be a left fraction in $S^{-1}\mathcal{C}$. Then by 2., there exists B' in $\mathcal{B}$ and a morphism $B' \to C$ in $\mathcal{C}$ such that

$$B' \to C \to B$$

is in S, i.e. we have an equivalent fraction

$$B \leftarrow B' \to B''$$

that must lie in $\mathcal{B}$ since it is a full subcategory. Thus, we conclude as desired that

$$\mathrm{Hom}_{S^{-1}\mathcal{B}}(B, B') \cong \mathrm{Hom}_{S^{-1}\mathcal{C}}(B, B') \ .$$

□

Remark 5.40. In particular, we showed that if two fractions in $\mathcal{B}$ are equivalent via a fraction

$$B \leftarrow C \rightarrow B''$$

with C in $\mathcal{C}$, then they are equivalent via a fraction with C in $\mathcal{B}$.

Moreover, under some additional assumptions, we have the following strong properties of localizing subcategories :

Lemma 5.41. *1) If $\mathcal{B}$ is a localizing subcategory of $\mathcal{C}$ such that for all objects C in $\mathcal{C}$, there is a morphism $C \to B$ in S with B in $\mathcal{B}$, then*

$$S^{-1}\mathcal{B} \cong S^{-1}\mathcal{C}$$

2) Suppose in addition that $S \cap \mathcal{B}$ consists of isomorphisms, then

$$\mathcal{B} \cong S^{-1}\mathcal{B} \cong S^{-1}\mathcal{C}$$

Proof. 1) Since $\mathcal{B}$ is localizing

$$S^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$$

is fully faithful. Now by hypothesis for all objects C in $\mathcal{C}$, there is a morphism $C \to B$ in S with B in $\mathcal{B}$, i.e.

$$S^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$$

is also essentialy surjective and we conclude as desired that

$$S^{-1}\mathcal{B} \cong S^{-1}\mathcal{C} \ .$$

2) We prove that

$$S^{-1}\mathcal{B} \cong \mathcal{B} \ .$$

By universal property of localization, we have the following diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\mathrm{Id}_{\mathcal{B}}"] \arrow[d, "q"] & \mathcal{B} \\
S^{-1}\mathcal{B} \arrow[ur, dashed, "\exists ! F"] &
\end{tikzcd}$$

i.e. there exists F such that $F \circ q = \mathrm{Id}_{\mathcal{B}}$. Conversely, again by the universal property, we have the following diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\mathrm{Id}_{\mathcal{B}}"] \arrow[d, "q"] & \mathcal{B} \arrow[r, "q"] & S^{-1}\mathcal{B} \\
S^{-1}\mathcal{B} \arrow[ur] \arrow[urr, dashed, "\exists ! G"] & &
\end{tikzcd}$$

And by unicity, we conclude that we also have

$$q \circ F = \mathrm{Id}_{S^{-1}\mathcal{B}}$$

which concludes. ☐

5.7 The derived category

Let $\mathcal{A}$ be an abelian category and Q the class of quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$. We recall that the derived category $\mathcal{D}(\mathcal{A})$ is defined to be the localization $Q^{-1}\mathcal{K}(\mathcal{A})$ of the triangulated category $\mathcal{K}(\mathcal{A})$.

The goal of this section is to prove the following results :

- $\mathcal{D}(\mathcal{A})$ is a triangulated category.
- $\mathcal{D}^+(A)$ is determined by maps between bounded complexes of injective objects.

First, we generalize our framework of localization :

Definition 5.42. Let $\mathcal{K}$ be a triangulated category. Then the system S *arising from the cohomological functor* $H : \mathcal{K} \to \mathcal{A}$ is given by

$$S := \{s \text{ morphism in } \mathcal{K} : H^i(s) \text{ is an isomorphism } \forall i\}$$

Example 5.43. The quasi-isomorphisms Q arise from the cohomological functor H^0.

The following proposition illustrates in what sense the definition is a generalization :

Proposition 5.44. *If S arises from a cohomological functor, then*
1) S is a multiplicative system.
2) If it exists, $S^{-1}\mathcal{K}$ is triangulated and

$$q : \mathcal{K} \to S^{-1}\mathcal{K}$$

is a morphism of triangulated categories.

Proof. 1) The first axiom of multiplicative system is trivial to check by functoriality.
Now, to prove the second axiom, let $f : X \to Y$ in $\mathcal{K}$ and $s : Z \to Y$ in S. Since $\mathcal{K}$ is triangulated, we can embed s in an exact triangle by axiom (TR1) :

$$Z \xrightarrow{s} Y \xrightarrow{u} C \xrightarrow{\delta} TZ$$

Hence we have $uf : X \to C$ in $\mathcal{K}$, and again by (TR1), we can embed it in an exact triangle

$$W \xrightarrow{t} X \xrightarrow{uf} C \xrightarrow{\nu} TW$$

Putting everything together yields the following diagram using (TR3)

$$\begin{tikzcd}
W \arrow[r, "t"] \arrow[d, dashed, "\exists g"] & X \arrow[r, "uf"] \arrow[d, "f"] & C \arrow[r, "\nu"] \arrow[d, equal] & TW \arrow[d, "Tg"] \\
Z \arrow[r, "s"] & Y \arrow[r, "u"] & C \arrow[r, "\delta"] & TZ ,
\end{tikzcd}$$

which is a morphism of triangles. Thus we only need to prove that t is in S. Let H^* be the cohomological functor S is arising from. Then, by definition we need to show that $H^*(t)$ is an isomorphism. But since by hypothesis s is in S, then $H^*(s)$ is an isomorphism, and thus the long exact sequence induced by H^* on our first triangle

$$\cdots \to H^*(Z) \xrightarrow{H^*(s)} H^*(Y) \xrightarrow{H^*(u)} H^*(C) \to \dots$$

yields $H^*(C) = 0$. Therefore, similarly, the long exact sequence induced by H^* on the second triangle

$$\cdots \to 0 \to H^*(W) \xrightarrow{H^*(t)} H^*(Y) \xrightarrow{H^*(uf)} 0 \to \dots$$

shows that $H^*(t)$ is an isomorphism as desired.
We now prove that we have cancellation, and by duality we only prove cancellation on the left.

Let $f, g : X \to Y$, such that there is $s : Y \to Y'$ in S with $sf = sg$. Again, by (TR1), we can embed s in an exact triangle

$$Z \xrightarrow{u} Y \xrightarrow{s} Y' \xrightarrow{\delta} TZ \ .$$

Similarly to the previous point, by examining the long exact sequence induced by H^* and using that s is in S, we deduce that $H^*(Z) = 0$. Now using that, by (10.2.8), $\mathrm{Hom}_{\mathcal{K}}(X, -)$ is a cohomological functor, we have the following exact sequence

$$\mathrm{Hom}_{\mathcal{K}}(X, Z) \xrightarrow{u} \mathrm{Hom}_{\mathcal{K}}(X, Y) \xrightarrow{s} \mathrm{Hom}_{\mathcal{K}}(X, Y') \ .$$

Thus, by exactness, since

$$s(f - g) = 0$$

there exists $g' : X \to Z$ such that

$$f - g = ug'$$

Using once again (TR1), we embed g' in an exact triangle

$$X' \xrightarrow{t} X \xrightarrow{g'} Z \xrightarrow{w} TX'$$

Finally, by commutativity of the next diagram

$$\begin{tikzcd}
X' \arrow[r, "t"] \arrow[d, equal] & X \arrow[r, "g'"] & Z \arrow[r, "w"] & TX' \arrow[d, equal] \\
X' \arrow[r, equal] & X' \arrow[u, "t"] \arrow[r] & 0 \arrow[u] \arrow[r] & TX' \ ,
\end{tikzcd}$$

we conclude that $g't = 0$, and putting everything together

$$(f - g)t = ug't = 0$$

and hence $f = g$ as desired. Checking that t is indeed in S is done simiarly as the previous point by examinating the long exact sequence induced by H^*, and using that we remarked previously that $H^*(Z) = 0$.
2) Suppose that $S^{-1}\mathcal{K}$ exists. We give an explicit description of the translation functor and the class of exact triangles, the proof that $S^{-1}\mathcal{K}$ is indeed a triangulated category is left as exercise.
We define the translation functor as sending the fraction

$$X \xleftarrow{s} C \xrightarrow{f} Y$$

to

$$T(X) \xleftarrow{s} T(C) \xrightarrow{f} T(Y)$$

where T is the translation functor of $\mathcal{K}$, or equivalently

$$T(fs^{-1}) = T(f)T(s)^{-1} \ .$$

We define exact triangles by making use of the Ore condition for S. Given

$$\begin{tikzcd}
 & & C' \arrow[r, "w"] \arrow[d, "s_3"] & T(A) \\
 & B' \arrow[r, "v"] \arrow[d, "s_2"] & C & \\
A' \arrow[r, "u"] \arrow[d, "s_1"] & B & & \\
A & & &
\end{tikzcd}$$

by applying the Ore condition for S two times, we get

$$t_1 : A'' \to A$$

$$t_2 : B'' \to B$$

in S, and

$$u' : A'' \to B''$$

$$v' : B'' \to C'$$

in $\mathcal{C}$ such that

$$us_1^{-1} \cong t_2 u' t_1^{-1}$$

and

$$vs_2^{-1} \cong s_3 v' t_2^{-1}$$

Finally, we say that

$$(us_1^{-1}, vs_2^{-1}, ws_3^{-1})$$

is an *exact triangle* in $S^{-1}\mathcal{K}$ just in case (u', v', w) is an exact triangle in $\mathcal{K}$. □

This yields to fulfill our first goal :

Corollary 5.45. *If they exist,* $\mathcal{D}(\mathcal{A})$*,* $\mathcal{D}^b(\mathcal{A})$*,* $\mathcal{D}^+(\mathcal{A})$ *and* $\mathcal{D}^-(\mathcal{A})$ *are triangulated categories.*

Moreover, in some "concrete" contexts, we always have existence :

Proposition 5.46. *Let* R *be a ring and* X *a topological space. Then if* $\mathcal{A} = R$*-mod,* $\mathcal{A} = \mathrm{Sh}(X)$ *or* $\mathcal{A} = \mathrm{PSh}(X)$*, then* $\mathcal{D}(\mathcal{A})$ *exists.*

Remark 5.47. It suffices to prove that the multiplicative system of quasi-isomorphism Q is locally small in theses cases.

Aiming towards our second goal, we prove the following useful lemma :

Lemma 5.48. *Let* Y *be a bounded below co-chain complex of injectives. Then every quasi-isomorphism* $t : Y \to Z$ *is a split injection in* $\mathcal{K}(\mathcal{A})$*, i.e.* $\exists s : Z \to Y$ *such that*

$$st \cong \mathrm{Id}_Y$$

Proof. First, since t is a quasi-isomorphism, we have that the mapping cone

$$\mathrm{Cone}(t) = T(Y) \oplus Z$$

is exact. Moreover, there is then a natural map

$$\varphi : \mathrm{Cone}(t) \to T(Y)$$

By the comparison lemma of 2.3.7, since Y is bounded below, we have that φ is null-homotopic, i.e. there exists $v = (k, s)$ such that

$$\varphi = vd + dv$$

Hence, we compute

$$-y = \varphi(y, z) = (vd + dv)(y; z)$$

$$= v(-dy, dz - ty) + d(ky + sz)$$

$$= -kdy + sdz - sty + dky + dsz$$

where we recalled that

$$d_{\mathrm{Cone}(t)} = \begin{pmatrix} -d_Y & 0 \\ -t & d_Z \end{pmatrix}$$

and we conclude that

$$y = (kdy + sty - dky) + (dsz - sdz)$$

Thus

$$ds = sd$$

i.e. s is a morphism of chain complexes and

$$st = \mathrm{Id}_Y + dk - kd$$

i.e. st is chain homotopic to Id_Y. Hence $st = \mathrm{Id}_Y$ in $\mathcal{K}^+(\mathcal{A})$. ☐

Remark 5.49. 1) The localization $S^{-1}\mathcal{C}$ can be constructed using equivalence classes of right fractions

$$t^{-1}g : X \xrightarrow{g} Z \xleftarrow{t} Y$$

where t is in S, provided that S is locally small.
2) By a corollary from last week, two parallel maps $f, g : X \to Y$ in $\mathcal{C}$ become identified if and only if there exists $s : W \to X$ in S such that

$$sf = sg$$

Corollary 5.50. *If I is a bounded below cochain complex of injectives, then $\forall X$*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, I) \cong \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, I) \ .$$

Similarly, if P is a bounded above cochain complex of projectives, then $\forall X$

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(P, X) \cong \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P, X) \ .$$

Proof. By duality, we prove only the first statement.
By the previous remark first point, we realize $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, I)$ as equivalence classes of right fractions. Thus, we define the map

$$\Phi : \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, I) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, I)$$

$$f \longrightarrow X \xrightarrow{f} I \overset{\mathrm{Id}_I}{\cong} I$$

and we prove it is the desired isomorphism.
Surjectivity:
Let $t^{-1}g : X \xrightarrow{g} Z \xleftarrow{t} I$ in $\mathcal{D}(\mathcal{A})$. By hypothesis, we can apply the previous lemma and hence there exists a section $s : Z \to I$ such that

$$st = \mathrm{Id}_I$$

Thus

$$(st)t^{-1}g = sg : X \to Y \overset{\mathrm{Id}_Y}{\cong} Y$$

i.e. $t^{-1}g$ is equivalent to $\Phi(sg)$ and we are done.
Injectivity: Let $f, g : X \to I$ be parallel maps in $\mathcal{K}(\mathcal{A})$ that are identified in $\mathcal{D}(\mathcal{A})$, i.e. by the previous remark second point there exists $s : Z \to I$ in S such that

$$sf = sg$$

But then again, by the previous lemma, applied to

$$X \xrightarrow{f} I \xleftarrow{s} Z$$

there exists a section $t : Z \to I$ such that $ts = \mathrm{Id}_I$ and we conclude that

$$f = tsf = tsg = g$$

in $\mathcal{K}(\mathcal{A})$ as desired. ☐

Finally, assembling all our tools, we reach our second goal :

Theorem 5.51. *1) Let $\mathcal{A}$ be a category with enough injectives, then $\mathcal{D}^+(\mathcal{A})$ always exists, because it is equivalent to the full subcategory $\mathcal{K}^+(I)$ of $\mathcal{K}^+(\mathcal{A})$ whose objects are bounded below co-chain complexes of injectives, i.e.*

$$\mathcal{D}^+(\mathcal{A}) \approx \mathcal{K}^+(I) \ .$$

2)Dually, if $\mathcal{A}$ is a category with enough projectives, then $\mathcal{D}^-(\mathcal{A})$ is equivalent to the full subcategory $\mathcal{K}^-(P)$ of $\mathcal{K}^-(\mathcal{A})$ whose objects are bounded above chain complexes of projectives, i.e.

$$\mathcal{D}^-(\mathcal{A}) \approx \mathcal{K}^-(P) \ .$$

Proof. By duality, we only prove 1).
First, recall that any cochain complex X in $\mathrm{Ch}^+(\mathcal{A})$ admits a Cartan-Eilenerg resolution $X \to I$ with $\mathrm{Tot}(I)$ in $\mathcal{K}^+(I)$. Moreover, since it is bounded below, this is a quasi-isomorphism and we deduce that

$$X \to \mathrm{Tot}(I)$$

is a quasi-isomorphism $\forall X$ in $\mathrm{Ch}^+(\mathcal{A})$. Thus, by Lemma 02 (3.), we deduce that $\mathcal{K}^+(I)$ is a localizing subcategory of $\mathcal{K}^+(\mathcal{A})$, and hence by Lemma 0.3. (1.)

$$Q^{-1}\mathcal{K}^+(I) \cong Q^{-1}\mathcal{K}^+(\mathcal{A}) = \mathcal{D}^+(\mathcal{A})$$

Therefore, now we only need to prove that

$$\mathcal{K}^+(I) \cong Q^{-1}\mathcal{K}^+(I)$$

to conclude. But by Lemma 0.3. (2.), it is enough to show that $Q \cap \mathcal{K}^+(I)$ consists of isomorphisms :
Let X, Y be bounded below cochain complexes of injectives and $t : Y \to X$ a quasi-isomorphism. Then by Lemma 0.10, there exists a map $s : X \to Y$ in Q such that

$$st = \mathrm{Id}_Y$$

However, since s is then also a quasi-isomorphism, again by Lemma 0.10, there exists a section $u : Y \to X$ such that

$$us = \mathrm{Id}_X$$

Putting everything together leads to

$$u = ust = t$$

i.e. t is an isomorphism with inverse s which concludes. ☐

This means that, whenever $\mathcal{A}$ has enough injectives (resp. projectives), and we can reduce our work to complexes of injectives (resp. projectives), then we can always bring it down to the homotopy category $\mathcal{K}^+(\mathcal{A})$.

Remark 5.52. 1) Every short exact sequence

$$0 \to A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \to 0$$

of co-chain complexes will fit into an exact triangle in $\mathcal{D}(\mathcal{A})$ that is isomorphic to the strict triangle on u. Indeed, denote

$$\varphi : \mathrm{Cone}(u) \to C^\bullet$$

the usual quasi-isomorphism. Then we have the following desired diagram

$$\begin{tikzcd}
A^\bullet \arrow[r, "u"] & B^\bullet \arrow[r, "v"] & C^\bullet \arrow[r, dashed, "\delta\varphi^{-1}"] & A^\bullet[-1] \\
A^\bullet \arrow[u, equal] \arrow[r, "u"] & B^\bullet \arrow[u, equal] \arrow[r] & \mathrm{Cone}(u) \arrow[u, "\varphi"] \arrow[r, "\delta"] & A^\bullet[-1] \arrow[u, equal]
\end{tikzcd}$$

which leads the claim.
2) By construction of $\mathcal{D}(\mathcal{A})$, we have two useful criteria for "being 0 in $\mathcal{D}(\mathcal{A})$" :

- A chain complex X is isomorphic to 0 in $\mathcal{D}(\mathcal{A})$ if and only if it is exact.
- A morphism $f : X \to Y$ in $\mathrm{Ch}(\mathcal{A})$ becomes the zero map in $\mathcal{D}(\mathcal{A})$ if and only if there is a quasi-isomorphism $s : Y \to Y'$ such that sf is null homotopic.

5.8 Exercises Week 13

(i) Suppose that $F : \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathcal{C})$ is a morphism of triangulated categories and $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ (see Exercise sheet 12 for definition and properties). If $\mathcal{A}$ has enough injectives, show that the restriction of $\mathbf{R}^+F$ to $\mathbf{D}^+_{\mathcal{B}}\mathcal{A}$ is the derived functor $\mathbf{R}^+_{\mathcal{B}}F$.

(ii) In general we cannot get a unbounded derived functor by taking unbounded projective (injective) resolutions. Here is a counterexample.

  (a) Show that for the $R = \mathbb{Z}/4\mathbb{Z}$-module category, the following complex consists of projective objects and is quasi-isomorphic to the zero complex.

$$P^* : \cdots \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} R \to \cdots$$

  (b) Show that $P^* \otimes_R \mathbb{Z}/2\mathbb{Z}$ is not quasi-isomorphic to zero complex.

This shows that applying an exact functor on quasi-isomorphic unbounded complexes of projective objects can give different result in derived category. Remember that in the bounded case, the result is determined by Comparison Theorem.

(iii) Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories and suppose $\mathcal{A}$ has enough injectives (so the usual derived functors exist). We let the cohomological dimension of F be the first n such that $R^iF = 0\ \forall i > n$. We make a similar definition when $\mathcal{A}$ has enough projectives for the homological dimension.

  (a) Show if F has finite cohomological dimension, then every exact complex of F-acyclic objects is an F-acyclic complex.

  (b) Show if F has finite cohomological dimension, $\mathbf{R}F$ exists on $D(\mathcal{A})$ (Hint: Consider K' the full subcategory of $K(\mathcal{A})$ consisting of complexes of F-acyclic objects in $\mathcal{A}$, and use the Generalized Existence Theorem).

(iv) Consider the derived functor $\mathbf{L}\,\mathrm{Tot}^{\oplus}(\bullet \otimes B)$ from $D(R\textbf{-mod})$ to $D(\mathbf{Ab})$, where R is a commutative ring and B a cochain complexe of R modules. Show that $A \otimes^L B$ is naturally isomorphic to $\mathbf{L}\,\mathrm{Tot}^{\oplus}(\bullet \otimes B)A$ in $D(\mathbf{Ab})$.

(v) Show that the total tensor product may be refined to a functor

$$\otimes^L_R : D^-(R_1\textbf{-mod-}R) \times D^-(R\textbf{-mod-}R_2) \to D^-(R_1\textbf{-mod-}R_2)$$

in the sense that the diagram

$$\begin{tikzcd}
D^-(R_1\textbf{-mod-}R) \times D^-(R\textbf{-mod-}R_2) \arrow[r, "\otimes^L_R"] \arrow[d] & D^-(R_1\textbf{-mod-}R_2) \arrow[d] \\
D^-(\textbf{mod-}R) \times D^-(R\textbf{-mod}) \arrow[r, "\otimes^L_R"] & D^-(\mathbf{Ab})
\end{tikzcd}$$

commutes where the vertical arrows are the forgetful functors. For R a commutative ring, refine it further to

$$\otimes^L_R : D^-(R\textbf{-mod}) \times D^-(R\textbf{-mod}) \to D^-(R\textbf{-mod})$$

and show there is a natural isomorphism $A \otimes^L_R B \cong B \otimes^L_R A$.

(vi) If $F : K'' \to K(\mathcal{D}), G : K' \to K'', H : K \to K'$ are three consecutive morphisms of triangulated categories, where $K \subset K(\mathcal{A}), K' \subset K(\mathcal{B}), K'' \subset K(\mathcal{C})$ are localizing triangulated subcategories. Assume all necessary derived functors exist and can be composed. Using the notation of the composition theorem, show that as natural transformations from $\mathbf{R}(FGH)$ to $\mathbf{R}F \circ \mathbf{R}G \circ \mathbf{R}H$ we have $\zeta_{G,H} \circ \zeta_{F,GH} = \zeta_{F,G} \circ \zeta_{FG,H}$.

5.9 Derived functors

The motivation behind this chapter is to explore the possibilities for inducing a functor $D(\mathcal{A}) \to D(\mathcal{B})$ from a left exact additive functor F between two abelian categories $\mathcal{A}$ and $\mathcal{B}$. The additive criterion ensures that F lifts to a functor $K(F) : K(\mathcal{A}) \to K(\mathcal{B})$. However, without exactness of F, we cannot ensure that $K(F)$ preserves quasi-isomorphisms, which means that $D(F) : D(\mathcal{A}) \to D(\mathcal{B})$ might not exist.

We recall that we have already defined the right derived functors as follows.
Given an object $A \in \mathcal{A}$, and assuming that $\mathcal{A}$ has enough injectives, we can find an injective resolution $A \to I^\bullet$ of A, and we set

$$R^iF(A) = H^i(F(I^\bullet)).$$

In particular this made the right derived functor a universal cohomological δ-functor, which is a concept we will try to replicate in some sense in the following definition.

For the remainder of this chapter we will use $K^\star(\mathcal{A})$ (resp. $D^\star(\mathcal{A})$) to represent any of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$ (resp. $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, or $D^b(\mathcal{A})$).
We will also write

$$q_\mathcal{A} : K^\star(\mathcal{A}) \to D^\star(\mathcal{A})$$

for the universal functor.

Definition 5.53. Let $F : K^\star(\mathcal{A}) \to K(\mathcal{B})$ be a morphism of triangulated categories. A total right derived functor of F is a morphism of triangulated categories

$$RF : D^\star(\mathcal{A}) \to D(\mathcal{B})$$

along with a natural transformation $\xi : q_\mathcal{B}F \Rightarrow RFq_\mathcal{A}$

$$\begin{tikzcd}
K^\star(\mathcal{A}) \arrow[r, "F"] \arrow[d, "q_\mathcal{A}"'] & K(\mathcal{B}) \arrow[d, "q_\mathcal{B}"] \arrow[dl, Rightarrow, "\xi"'] \\
D^\star(\mathcal{A}) \arrow[r, "RF"'] & D(\mathcal{B})
\end{tikzcd}$$

such that if $G : D^\star(\mathcal{A}) \to D(\mathcal{B})$ is another morphism with another natural transformation $\zeta : q_\mathcal{B}F \Rightarrow Gq_\mathcal{A}$ then there exists a unique natural transformation $\eta : RF \Rightarrow G$ such that the diagram

$$\begin{tikzcd}
q_\mathcal{B}F \arrow[rr, Rightarrow, "\xi"] \arrow[dr, Rightarrow, "\zeta"'] & & RFq_\mathcal{A} \arrow[dl, Rightarrow, dashed, "\eta q_\mathcal{A}"] \\
& Gq_\mathcal{A} &
\end{tikzcd}$$

commutes (in other words, ζ must factorise uniquely through ξ).

This notion is similar to the notion of a universal cohomology δ-functor, where the universal property is the existence and uniqueness of η.

Mirroring this statement gives the definition of a total left derived functor $LF : D^\star(\mathcal{A}) \to D(\mathcal{B})$ with its natural transformation $\xi : LFq_\mathcal{A} \Rightarrow q_\mathcal{B}F$.

Remark 5.54. The total right and left derived functors, when they exist, are guaranteed to be unique up to (natural) isomorphism because of the universal property.

Example 5.55. If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor, then it induces a functor $F : K(\mathcal{A}) \to K(\mathcal{B})$ which preserves quasi-isomorphisms, so it extends to a functor $D(F) : D(\mathcal{A}) \to D(\mathcal{B})$.
This gives us a natural isomorphism $\xi : q_\mathcal{B}F \Rightarrow D(F)q_\mathcal{A}$. The fact that this natural transformation is invertible means the universal property is given by $\eta q_\mathcal{A} = \zeta \circ \xi^{-1}$, making $D(F)$ the right (and left) derived functor of F.

Example 5.56. If $K^+(\mathcal{I})$ is the full triangulated subcategory of $K^+(\mathcal{A})$ of bounded below chain complexes of injective objects, then we have seen that

$$q_{\mathcal{I}} : K^+(\mathcal{I}) \to D^+(\mathcal{I})$$

is an isomorphism.
If $F : K^+(\mathcal{I}) \to K^+(\mathcal{B})$ is a morphism of triangulated categories, we then have that $q_{\mathcal{B}} F q_{\mathcal{I}}^{-1}$ is the right (and left) derived functor of F, because

$$q_{\mathcal{B}} F \cong (q_{\mathcal{B}} F q_{\mathcal{I}}^{-1}) q_{\mathcal{I}}$$

so the universal property is satisfied as in the previous example.

In the following theorem we give a more general condition for the total right derived functor to exist.

Theorem 5.57. *(Existence Theorem) : Given $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ a morphism of triangulated categories, assume that $\mathcal{A}$ has enough injectives. Then the total right derived functor R^+F exists on $D^+(\mathcal{A})$. For any bounded below complex of injectives $I^\bullet$, we have*

$$R^+F(I^\bullet) \cong q_{\mathcal{B}} F(I^\bullet).$$

Proof. We have seen previously that there is an equivalence of categories

$$\begin{tikzcd} K^+(\mathcal{I}) \arrow[r, shift left, "T"] & D^+(\mathcal{A}). \arrow[l, shift left, "U"] \end{tikzcd}$$

where T is the inclusion of $K^+(\mathcal{I})$ in $K^+(\mathcal{A})$ composed with the localisation $K^+(\mathcal{A}) \to D^+(\mathcal{A})$.
This means we have a natural transformation $\eta : 1_{D^+(\mathcal{A})} \Rightarrow TU$.
We define R^+F as the composition of the arrows

$$D^+(\mathcal{A}) \xrightarrow{U} K^+(\mathcal{I}) \xrightarrow{F|_{K^+(\mathcal{I})}} K^+(\mathcal{B}) \xrightarrow{q_{\mathcal{B}}} D^+(\mathcal{B})$$

which allows us to use η to define ξ.
For each object X in $K^+(\mathcal{A})$ we have a morphism $\eta_{q_{\mathcal{A}}(X)} : q_{\mathcal{A}}(X) \to TUq_{\mathcal{A}}(X)$.
In the previous chapter we have constructed a natural isomorphism

$$\mathrm{Hom}_{D(A)}(X, I) \cong \mathrm{Hom}_{K(A)}(X, I)$$

so from η_X we obtain $f_X : X \to Uq_{\mathcal{A}}(X)$. Setting

$$\xi_X = q_{\mathcal{B}} F(f_X) : q_{\mathcal{B}} F(X) \to q_{\mathcal{B}} FUq_{\mathcal{A}}(X) = R^+Fq_{\mathcal{A}}(X)$$

gives a well defined natural transformation, for which it can be shown that the universal property is satisfied. □

In fact we can generalise this theorem. Since the proof is almost identical we will omit it.

Theorem 5.58. *(Generalised Existence Theorem) : Let K' be a triangulated subcategory of K such that:*

(i) for every $X \in K$ there is a quasi-isomorphism $s : X \to X'$ with $X' \in K'$;

(ii) every exact complex in K' is F-acyclic;

then $RF : D \to D(\mathcal{B})$ exists, and for every complex of F-acyclics $A^\bullet$, we have

$$RF(A^\bullet) \cong qF(A^\bullet).$$

Corollary 5.59. *If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor of abelian categories and $\mathcal{A}$ has enough injectives, then for all i and all X we have $H^i R^+F(X) \cong \mathbb{R}^i F(X)$.*

Proof. We have $\mathbb{R}^iF(X) = H^i(Tot^\pi(F(I)))$ for some appropriately chosen injective Cartan-Eilenberg resolution of X, so the definition of R^+F (and specifically that of the equivalence U) allows us to conclude. $\square$

Definition 5.60. For $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ a morphism of triangulated categories, we say that $X \in K^+(\mathcal{A})$ is F-acyclic if $F(X)$ is acyclic (quasi-isomorphic to 0), which is to say $H^i(F(X)) = 0$ for all i.

Example 5.61. Let K' be a triangulated subcategory of $K(\mathcal{A})$ such that every acyclic complex in K' is F-acyclic. If $s : X \to Y$ is a quasi-isomorphism in K', then cone(s) and $F(\text{cone}(s))$ are acyclic and $F(s)$ is a quasi-isomorphism since

$$F(s)^\star : H^\star(F(X)) \xrightarrow{\sim} H^\star(F(Y))$$

is an isomorphism.

Since quasi-isomorphisms in K' are mapped to quasi-isomorphisms in $K(\mathcal{B})$, the universal property guarantees that there exists a unique $Q^{-1}F : D' \to D(\mathcal{B})$ such that $q_{\mathcal{B}}F|_{K'} = (Q^{-1}F|_{K'})q_{K'}$. This makes $Q^{-1}F|_{K'}$ the right (and left) derived functor of F restricted to K'.

5.10 The total tensor product

In this chapter we will consider a particular example of a left derived functor. We will consider a ring R and write A, B to represent cochain complexes of R-modules.

The functor in question is $F : K(R - \text{mod}) \to K(Ab)$ which sends B to $\text{Tot}^\oplus(A \otimes_R B)$.

Using the existence theorem and the fact that R-mod has enough projectives we have the existence of the left derived functor $L^-F : D^-(R - \text{mod}) \to D(Ab)$, which we denote

$$A \otimes_R^L \bullet.$$

Lemma 5.62. *If A, A', B are bounded above cochain complexes and $d : A \to A'$ is a quasi-isomorphism, then*

$$A \otimes_R^L B \cong A' \otimes_R^L B.$$

Proof. The functor $A \otimes_R^L \bullet$ must send isomorphisms to isomorphisms, but isomorphisms in our derived categories are quasi-isomorphisms in our homotopy categories, so we can replace B with any cochain complex quasi-isomorphic to it.

Therefore we may assume without loss of generality that B is a cochain complex of projectives, because it is quasi-isomorphic to the total complex of its Cartan-Eilenberg resolution.

Therefore we have

$$A \otimes_R^L B \cong \text{Tot}^\oplus(A \otimes_R B)$$

and

$$A' \otimes_R^L B \cong \text{Tot}^\oplus(A' \otimes_R B)$$

because exact and flat sequences are acyclic, so Example 1.9 may be invoked.

Now, since A, A', and B are bounded above, the spectral sequences

$$E_1^{pq}(A) = H^q(A) \otimes_R B^p \Rightarrow H^{p+q}(\text{Tot}^\oplus(A \otimes_R B))$$

and

$$E_1^{pq}(A') = H^q(A') \otimes_R B^p \Rightarrow H^{p+q}(\text{Tot}^\oplus(A' \otimes_R B))$$

converge.

Since d is a quasi-isomorphism, the induced maps

$$H^q(A) \otimes_R B^p \to H^q(A') \otimes_R B^p$$

are isomorphisms for all p and q, so by the comparison theorem the map

$$H^{p+q}(\text{Tot}^\oplus(A \otimes_R B)) \to H^{p+q}(\text{Tot}^\oplus(A' \otimes_R B))$$

is an isomorphism for all p and q, which allows us to conclude, given that we are working in the derived category $D(Ab)$. $\square$

Corollary 5.63. *There is a bifunctor*

$$\bullet \otimes^L_R \bullet : D^-(mod-R) \times D^-(R-mod) \to D^-(Ab)$$

which we call the total tensor product whose cohomology is the hypertor

$$H^{-i}(A \otimes^L_R B) \cong Tor^R_i(A,B).$$

The negative index is being used because we have been working with cochain complexes.

If we have a morphism of rings $f : R \to S$, then this induces a functor

$$f^\star : R - \text{mod} \to S - \text{mod},$$

that sends an R-module A to the S-module $A \otimes_R S$.
As long as S has finite flat dimension, then $f^\star$ has a derived functor

$$Lf^\star : D(R - \text{mod}) \to D(S - \text{mod}),$$

for which we can show that there is an isomorphism

$$Lf^\star(A) \otimes^L_S Lf^\star(B) \xrightarrow{\cong} Lf^\star(A \otimes^L_R B),$$

as long as R and S are commutative.
This argument involves replacing A and B with complexes of flat modules, and shows that the bifunctor $\otimes^L_R$ in fact behaves better than $\otimes_R$ in some respects.

In particular, in what follows we will show that there is an isomorphism

$$A \otimes^L_R (B \otimes^L_S C) \cong (A \otimes^L_R B) \otimes^L_S C,$$

which we will do by considering the functors

$$F(\bullet) = \text{Tot}^{\oplus}(A \otimes_R \bullet)$$

and

$$G(\bullet) = \text{Tot}^{\oplus}(\bullet \otimes_S C)$$

and claiming that $GF \cong FG$.

Theorem 5.64. *(Composition Theorem) : Let $K \subseteq K(A)$ and $K' \subseteq K(B)$ be localising triangulated subcategories, and suppose we have morphisms $G : K \to K'$, $F : K' \to K(C)$. Assume that the right total derived functors RF, RG, RFG exist, with $RF(D) \subseteq D'$. The following then holds.*

(i) There exists a unique $\zeta = \zeta_{F,G} : RFG \Rightarrow RF \circ RG$ making the diagram

$$\begin{tikzcd}
qFG(A) \arrow[r, "\xi_{F,A}"] \arrow[d, "\xi_{FG,A}"'] & (RF)(qG)(A) \arrow[d, "\xi_{G,A}"] \\
R(FG)(qA) \arrow[r, "\zeta_{qA}"'] & (RF)(RG)(qA)
\end{tikzcd}$$

commute.

(ii) Suppose there are triangulated subcategories $K_0 \subseteq K$, $K'_0 \subseteq K'$ satisfying the conditions of the generalised existence theorem, and $G(K_0) \subseteq K'_0$. Then ζ is a natural isomorphism

$$R(FG) \xrightarrow{\cong} RF \circ RG.$$

Proof. (i) We directly apply the universal property of $R(FG)$, making $\xi_{G,A} \circ \xi_{F,A}$ factor through $\xi_{FG,A}$.

(ii) We need to show that $\xi_{F,A}$, $\xi_{G,A}$, $\xi_{FG,A}$ are natural isomorphisms when restricted to K_0 and K_0'. This follows from the existence theorem, because if $A \in K_0$, then

$$RF(A) \cong qF(A),$$

and the same argument goes for G, and FG.

□

Corollary 5.65. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be abelian categories, $\mathcal{A}$, $\mathcal{B}$ having enough injectives, and $G : \mathcal{A} \to \mathcal{B}$, $F : \mathcal{B} \to \mathcal{C}$ be additive functors. If G sends injectives to F-acyclic objects then*

$$\zeta : R^+(FG) \xrightarrow{\cong} (R^+F) \circ (R^+G)$$

is an isomorphism.
Additionally, if acyclic complexes are sent to F-acyclic complexes, and F and G have finite cohomological dimension, then $R(FG)$ exists and

$$\zeta : R(FG) \xrightarrow{\cong} RF \circ RG$$

is an isomorphism.
In both of the previous cases, for all A there is a convergent spectral sequence

$$E_2^{pq} : (R^pF)(\mathbb{R}^qG)(A) \Rightarrow \mathbb{R}^{p+q}(FG)(A).$$

Proof. The spectral sequence comes directly from the hypercohomology spectral sequence

$$E_2^{pq} : (R^pF)(H^q(RG(A))) \Rightarrow (\mathbb{R}^{p+q}F)(RG(A))$$

since on the second page we have

$$H^q(RG(A)) = \mathbb{R}^qG(A),$$

and

$$\mathbb{R}^{p+q}(FG)(A) \cong (\mathbb{R}^{p+q}F)(RG(A)).$$

□

Consider again a map of rings $f : R \to S$, which induces the exact functor

$$f_\star : S - \text{mod} \to R - \text{mod}$$

which sends a module B onto itself (to multiply by an element $r \in R$ we multiply by $f(r)$).
If A is a bounded above complex of R-modules, the composition theorem then implies that there is an isomorphism

$$Lf^\star(A) \otimes_S^L B \xrightarrow{\cong} A \otimes_R^L f_\star(B).$$

Indeed, since we have

$$f^\star(A) \otimes_S B \cong A \otimes_R f_\star(B)$$

when A and B are individual modules (and not complexes), this means that

$$\begin{aligned} Lf^\star(A) \otimes_S^L B &= L(\text{Tot}^\oplus(\bullet \otimes_R B)) \circ L(f^\star)(A) \\ &\cong L\text{Tot}^\oplus(f^\star(\bullet) \otimes_S B)(A) \\ &\cong L\text{Tot}^\oplus(\bullet \otimes_R f_\star(B))(A) \\ &= A \otimes_R^L f_\star(B) \end{aligned}$$

when A and B is are complexes of modules.

5.11 Exercises Week 14

(i) Suppose that $F : \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathcal{C})$ is a morphism of triangulated categories and $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ (see Exercise sheet 12 for definition and properties). If $\mathcal{A}$ has enough injectives, show that the restriction of $\mathbf{R}^+F$ to $\mathbf{D}^+_{\mathcal{B}}\mathcal{A}$ is the derived functor $\mathbf{R}^+_{\mathcal{B}}F$.

(ii) In general we cannot get a unbounded derived functor by taking unbounded projective (injective) resolutions. Here is a counterexample.

  (a) Show that for the $R = \mathbb{Z}/4\mathbb{Z}$-module category, the following complex consists of projective objects and is quasi-isomorphic to the zero complex.

$$P^* : \cdots \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} R \to \cdots$$

  (b) Show that $P^* \otimes_R \mathbb{Z}/2\mathbb{Z}$ is not quasi-isomorphic to zero complex.

This shows that applying an exact functor on quasi-isomorphic unbounded complexes of projective objects can give different result in derived category. Remember that in the bounded case, the result is determined by Comparison Theorem.

(iii) Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories and suppose $\mathcal{A}$ has enough injectives (so the usual derived functors exist). We let the cohomological dimension of F be the first n such that $R^iF = 0 \ \forall i > n$. We make a similar definition when $\mathcal{A}$ has enough projectives for the homological dimension.

  (a) Show if F has finite cohomological dimension, then every exact complex of F-acyclic objects is an F-acyclic complex.

  (b) Show if F has finite cohomological dimension, $\mathbf{R}F$ exists on $D(\mathcal{A})$ (Hint: Consider K' the full subcategory of $K(\mathcal{A})$ consisting of complexes of F-acyclic objects in $\mathcal{A}$, and use the Generalized Existence Theorem).

(iv) Consider the derived functor $\mathbf{L}\,\mathrm{Tot}^{\oplus}(\bullet \otimes B)$ from $D(R\textbf{-mod})$ to $D(\mathbf{Ab})$, where R is a commutative ring and B a cochain complexe of R modules. Show that $A \otimes^L B$ is naturally isomorphic to $\mathbf{L}\,\mathrm{Tot}^{\oplus}(\bullet \otimes B)A$ in $D(\mathbf{Ab})$.

(v) Show that the total tensor product may be refined to a functor

$$\otimes^L_R : D^-(R_1\textbf{-mod-}R) \times D^-(R\textbf{-mod-}R_2) \to D^-(R_1\textbf{-mod-}R_2)$$

in the sense that the diagram

$$\begin{tikzcd}
D^-(R_1\textbf{-mod-}R) \times D^-(R\textbf{-mod-}R_2) \arrow[r, "\otimes^L_R"] \arrow[d] & D^-(R_1\textbf{-mod-}R_2) \arrow[d] \\
D^-(\textbf{mod-}R) \times D^-(R\textbf{-mod}) \arrow[r, "\otimes^L_R"] & D^-(\mathbf{Ab})
\end{tikzcd}$$

commutes where the vertical arrows are the forgetful functors. For R a commutative ring, refine it further to

$$\otimes^L_R : D^-(R\textbf{-mod}) \times D^-(R\textbf{-mod}) \to D^-(R\textbf{-mod})$$

and show there is a natural isomorphism $A \otimes^L_R B \cong B \otimes^L_R A$.

(vi) If $F : K'' \to K(\mathcal{D}), G : K' \to K'', H : K \to K'$ are three consecutive morphisms of triangulated categories, where $K \subset K(\mathcal{A}), K' \subset K(\mathcal{B}), K'' \subset K(\mathcal{C})$ are localizing triangulated subcategories. Assume all necessary derived functors exist and can be composed. Using the notation of the composition theorem, show that as natural transformations from $\mathbf{R}(FGH)$ to $\mathbf{R}F \circ \mathbf{R}G \circ \mathbf{R}H$ we have $\zeta_{G,H} \circ \zeta_{F,GH} = \zeta_{F,G} \circ \zeta_{FG,H}$.

Chapter 6. Exercise Solutions

6.1 Week 1 (by Eric Chen).

1(a). Let $i : \mathbf{Z} \to \mathbf{Q}$ be the natural inclusion. To show that i is epi in the category of rings, consider two distinct morphisms of rings $f, g : \mathbf{Q} \to R$ to some target ring R. Since $f \neq g$, there is some rational number $a/b \in \mathbf{Q}$ with $a, b \in \mathbf{Z}$ such that

$$f(a/b) \neq g(a/b)$$

Suppose towards a contradiction that f and g agree on integers; then multiplying the previous inequality by $f(b) = g(b)$ gives

$$f(b)f(a/b) \neq g(b)g(a/b)$$

But the LHS of this inequality is $f(a)$, while the right hand side is $g(a)$, so we have a contradiction.
Let $i : \mathbf{Q} \to \mathbf{R}$ be the natural inclusion. To show that i is epi in the category of Hausdorff topological spaces, consider two distinct continuous maps $f, g : \mathbf{R} \to X$ to some target space X, and suppose $x \in \mathbf{R}$ is a point for which $f(x) \neq g(x)$. Since X is Hausdorff, we can find disjoint open sets $U, V \subset X$ such that $f(x) \in U$ and $g(x) \in V$. Considering their preimages under $f \circ i$ and $g \circ i$, we see that

$$(f \circ i)^{-1}(U) \cap (g \circ i)^{-1}(V) = f^{-1}(U) \cap g^{-1}(U) \cap \mathbf{Q}$$

is nonempty, since $f^{-1}(U) \cap g^{-1}(U)$ is an open set containing x, and $\mathbf{Q} \subset \mathbf{R}$ is dense (by definition of $\mathbf{R}$, for every real number x and every open set W containing x, there is always a rational number inside W). Thus, for any point $q \in (f \circ i)^{-1}(U) \cap (g \circ i)^{-1}(V)$ we have $f(q) \neq g(q)$, hence $f \circ i \neq g \circ i$.
(b). Let $f : A \to B$ be a morphism in the category of groups. If f is an injective set map, then for any two distinct morphisms $g_1, g_2 : C \to A$ we can consider an element $c \in C$ on which $g_1(c) \neq g_2(c)$, and the injectivity of f would imply

$$f(g_1(c)) \neq f(g_2(c))$$

Thus, f is monic. For the opposite direction, suppose f is monic. To see that it is injective, for an arbitrary nonidentity element $a \in A$, consider the two distinct group homomorphisms

$$g_1 : \mathbf{Z} \to A \text{ defined by } g_1(1) = a \text{ and } g_2 : \mathbf{Z} \to A \text{ the trivial homomorphism}$$

Since $f \circ g_1 \neq f \circ g_2$, we see that there is some integer n for which $f(a)^n \neq \mathrm{id}_B$. In particular, $f(a) \neq \mathrm{id}_B$, so f is injective.

2. To see that $\mathcal{A}^I$ is a category, the only nontrivial thing to check is that morphisms in $\mathcal{A}^I$ form a set. For this, we consider two functors $F, G \in \mathcal{A}^I$; by definition, the morphisms between F and G are natural transformations from F to G, i.e., for every $i \in I$ the data of morphisms

$$\eta_i : F(i) \longrightarrow G(i)$$

in $\mathcal{A}$ satisfying some compatibilities as i varies. But such collections $\{\eta_i\}_{i \in I}$ embeds into the set of functions

$$\mathrm{Fun}(I, \sqcup_{i \in I} \mathrm{Hom}_{\mathcal{A}}(F(i), G(i))$$

so the collection of natural transformations from F to G is itself a set.
To see that the Yoneda embedding $h : I \to \mathbf{Sets}^{I^{\mathrm{op}}}$ defined by

$$i \mapsto h_i := \mathrm{Hom}(\,\cdot\,, i)$$

is fully faithful, we need to show that the function

$$\eta : \mathrm{Hom}_I(i, j) \longrightarrow \mathrm{Hom}_{\mathbf{Sets}^{I^{\mathrm{op}}}}(h_i, h_j)$$

induced by h is injective, for every pair of objects $i, j \in I$. For this, consider $f, g \in \mathrm{Hom}_I(i, j)$ two distinct morphisms; then $\eta(f), \eta(g)$ are natural transformations from h_i to h_j and to show they are not equal it suffices to present an object $k \in I$ such that

$$\eta(f) \neq \eta(g) \text{ as functions } h_i(k) = \mathrm{Hom}_I(k, i) \to h_j(k) = \mathrm{Hom}_I(k, j)$$

For this, we consider $k = i$ and the identity map $\mathrm{id}_i \in \mathrm{Hom}_I(i,i)$. Then

$$\eta(f)(\mathrm{id}_i) = f \neq g = \eta(g)(\mathrm{id}_i)$$

as we wanted.

3. We aim to show the adjunction between $\Delta : \mathcal{A} \to \mathcal{A}^I$ and $\lim_{i\in I} : \mathcal{A}^I \to \mathcal{A}$, i.e., a bijection between the sets

$$\mathrm{Hom}_{\mathcal{A}^I}(\Delta(a), F) = \mathrm{Hom}_{\mathcal{A}}(a, \lim_{i\in I} F_i) \tag{13}$$

natural in a and F. By the universal property of $\lim_{i\in I} F_i$, morphisms from a to $\lim_{i\in I} F_i$ is in natural bijection with the set of morphisms from a to each F_i, compatible with the arrows in I. But this is the same as morphisms from the diagonal functor $\Delta(a)$ to F, hence we have (13).
Dually, we want to see an adjunction

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{colim}_{i\in I} F_i, a) = \mathrm{Hom}_{\mathcal{A}^I}(F, \Delta(a)) \tag{14}$$

and the argument is identical. By the universal property of $\mathrm{colim}_{i\in I} F_i$, morphisms from $\mathrm{colim}_{i\in I} F_i$ to a is in natural bijection with the set of morphisms from each F_i to a, compatible with the arrows in I. But this is the same as morphisms from F to the diagonal functor $\Delta(a)$.

4. We need to produce an adjunction

$$\mathrm{Hom}_{\mathcal{B}}(Lx, y) = \mathrm{Hom}_{\mathcal{A}}(x, Ry) \tag{15}$$

natural in x, y, given the data of natural transformations $\eta : \mathrm{id}_{\mathcal{A}} \to RL$ and $\epsilon : LR \to \mathrm{id}_{\mathcal{B}}$.
Starting from the LHS, we consider an arrow

$$(f : Lx \to y) \in \mathrm{Hom}_{\mathcal{B}}(Lx, y)$$

in $\mathcal{B}$. Applying R to this arrow gives an arrow

$$(R(f) : RLx \to Ry) \in \mathrm{Hom}_{\mathcal{A}}(RLx, Ry)$$

Using η, we have an arrow $\eta_x : x \to RLx$, which we can post-compose with $R(f)$ to obtain an arrow

$$(R(f) \circ \eta_x : x \to Ry) \in \mathrm{Hom}_{\mathcal{A}}(x, Ry)$$

This procedure $f \mapsto R(f) \circ \eta_x$ gives a function from the LHS to the RHS of (15). To see that it is a bijection, we consider the analogous procedure in the reverse direction:

$$\mathrm{Hom}_{\mathcal{A}}(x, Ry) \ni g \longmapsto \epsilon_y \circ L(g) \in \mathrm{Hom}_{\mathcal{B}}(Lx, y)$$

which we claim to be inverse to the first. To see this, we compute

$$\begin{aligned}
\epsilon_y \circ L(R(f) \circ \eta_x) &= \left(Lx \overset{L\eta_x}{\to} LRLx \overset{LRf}{\to} LRy \overset{\epsilon_y}{\to} y \right) \\
&= \left(Lx \overset{L\eta_x}{\to} LRLx \overset{\epsilon_x}{\to} Lx \overset{f}{\to} y \right) \text{ by naturality of } \epsilon \text{ with respect to } f \\
&= Lx \overset{f}{\to} y \text{ by assumption on } \eta
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
R(\epsilon_y \circ L(g)) \circ \eta_x &= \left(x \overset{\eta_x}{\to} RLx \overset{RLg}{\to} RLRy \overset{R\epsilon_y}{\to} Ry \right) \\
&= \left(x \overset{g}{\to} Ry \overset{R\eta_y}{\to} RLRy \overset{R\epsilon_y}{\to} Ry \right) \text{ by naturality of } \eta \text{ with respect to } g \\
&= x \overset{g}{\to} Ry \text{ by assumption on } \epsilon
\end{aligned}$$

6.2 Week 2 (by Qian Yao).

6.2.1 Some properties of abelian category

Recall that in an abelian category (1) every map has kernel and cokernel,(2) monic is kernel of its cokernel, epic is cokernel of its kernel.

(a) To prove f is monic when $0 \to A$ is a kernel of f, take any object C, and take $g_1, g_2 : C \to A$ two morphisms that satisfy $f \circ g_1 = f \circ g_2$. By abelian group structure of Hom and distribution law, $0 = f \circ g_1 - f \circ g_2 = f \circ (g_1 - g_2)$. By the universal property of kernel, there exists a unique k that satisfy $g_1 - g_2 = h \circ k = 0$, because $h = 0$. So $g_1 - g_2 = 0$, $g_1 = g_2$ which proves that f is monic.

$$\begin{tikzcd}
& 0 = ker(f) \arrow[d, "h"] & \\
C \arrow[ur, dashed, "k"] \arrow[r, "g_1 - g_2"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

To prove that f is epi, when $B \to 0$ is a cokernel of f. Similarly take any object C, suppose $g_1, g_2 : B \to C$ two morphisms that satisfy $g_1 \circ f = g_2 \circ f$, then the universal property of cokernel shows that there exists a unique k such that $g_1 - g_2 = k \circ h = 0$. So $g_1 = g_2$ which proves that f is epi.

$$\begin{tikzcd}
& 0 = coker(f) \arrow[dl, dashed, "k"] & \\
C & B \arrow[l, "g_1 - g_2"'] \arrow[u, "h"] & A \arrow[l, "f"']
\end{tikzcd}$$

(b)To prove isomophism, it suffices to construct $g : B \to A$ such that $g \circ f = id_A$. First take $coker, im$ of f by property (1).

$$\begin{tikzcd}
& im(f) = ker(coker(f)) \arrow[d] & \\
A \arrow[r, "f"] & B \arrow[r] & coker(f)
\end{tikzcd}$$

Then since f is monic, it is kernel of $B \to coker(f)$, $im(f)$ is also kernel of $B \to coker(f)$, by universal property of kernel, they are isomorphism, and the diagram commutes.

$$\begin{tikzcd}
& im(f) = ker(coker(f)) \arrow[d] \arrow[dl, "\sim"'] & \\
A \arrow[r, "f"] & B \arrow[r] & coker(f)
\end{tikzcd}$$

Finally, since f is epi and

$$A \to B \to coker(f) = A \leftrightarrow im(f) \to B \to coker(f) = 0$$

$$B \to coker(f) = 0$$

By adding $id_B : B \to B$, and universal property of kernel, there exists a unique h such that the diagram commutes.

$$\begin{tikzcd}
& & im(f) = ker(coker(f)) \arrow[d] \arrow[dll, "i\sim"'] & \\
A \arrow[r, "f"] & B \arrow[r, "id"] \arrow[ur, dashed, "h"] & B \arrow[r] & coker(f)
\end{tikzcd}$$

Take $g = i \circ h$, then $g \circ f = id_A$.

(c)The induced morphism is given by the universal property of kernel.

$$\begin{tikzcd}
& im(f) \arrow[d] & \\
A \arrow[ur, dashed, "g"] \arrow[r, "f"] & B \arrow[r] & coker(f)
\end{tikzcd}$$

To prove epi, take any object C and $h_1, h_2 : im(f) \to C$ such that $h_1 \circ g = h_2 \circ g$, consider the difference $h_1 - h_2$.

$$\begin{tikzcd}
 & im(f) \arrow[r, "h_1-h_2"] \arrow[d] & C \\
A \arrow[ru, "g"] \arrow[r, "f"] & B \arrow[r] & coker(f)
\end{tikzcd}$$

Then take the kernel of $h_1 - h_2$, since $(h_1 - h_2) \circ g = 0$, by the universal property of kernel, there exists a unique $A \to ker(h_1 - h_2)$ such that the diagram commutes.

$$\begin{tikzcd}
ker(h_1-h_2) \arrow[r, "i"] & im(f) \arrow[r, "h_1-h_2"] \arrow[d, "j"] & C \\
A \arrow[u, dashed] \arrow[ru, "g"] \arrow[r, "f"] & B \arrow[r] & coker(f)
\end{tikzcd}$$

Since i, j are both kernels, they are monics (two morphisms $a_1, a_2 : C \to ker(f)$, then the uniqueness of universal property shows that they are the same), and $j \circ i$ is monic (easy to check that the composition of monics is monic by using the definition of monic twice). By property (2), $j \circ i$ is kernel of its cokernel. And because $A \to B \to coker(j \circ i) = 0$, by the universal property of cokernel, there exists a unique $coker(f) \to coker(j \circ i)$ such that the diagram commutes.

$$\begin{tikzcd}
ker(h_1-h_2) \arrow[r, "i"] \arrow[rd, "j\circ i"] & im(f) \arrow[r, "h_1-h_2"] \arrow[d, "j"] & C \\
A \arrow[u] \arrow[ru] \arrow[r, "f"] & B \arrow[r] \arrow[rd] & coker(f) \arrow[d, dashed] \\
 & & coker(j\circ i)
\end{tikzcd}$$

Then $im(f) \to B \to coker(j \circ i) = 0$, since $j \circ i$ is a kernel, there exists a unique i' such that the diagram commutes, that is to say i is an isomorphism.

$$\begin{tikzcd}
ker(h_1-h_2) \arrow[r, shift left, "i"] \arrow[rd, "j\circ i"] & im(f) \arrow[l, shift left, "i'"] \arrow[r, "h_1-h_2"] \arrow[d, "j"] & C \\
A \arrow[u] \arrow[ru] \arrow[r, "f"] & B \arrow[r] \arrow[rd] & coker(f) \arrow[d, dashed] \\
 & & coker(j\circ i)
\end{tikzcd}$$

Finally, take $id_{im(f)}$, $(h_1 - h_2) = (h_1 - h_2) \circ id_{im(f)} = 0$, this proves that g is epi.

6.2.2 Functor Category of Abelian Category is Abelian

First check that $\mathcal{A}^{\mathcal{I}}$ is additive. For every set $\mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(C, D)$, its structure of abelian group is given by taking I an object in $\mathcal{I}$, then $(\eta_I + \varphi_I)(c) := \eta_I(c) + \varphi_I(c)$, and the composition $(\Psi_I \circ \eta_I)(c) = \Psi_I(\eta_I(c))$, then the abelian group structure and distribution law hold by the corresponding properties of $\mathcal{A}$. The zero object is $Z : \mathcal{I} \to \mathcal{A}$, for any I an object of $\mathcal{I}$, $Z(I) = 0_{\mathcal{A}}$. The product is $F \times G(I) = F(I) \times_{\mathcal{A}} G(I)$.
Then check that $\mathcal{A}^{\mathcal{I}}$ is abelian. For any map $\eta \in \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(C, D)$, take the kernel $\epsilon \in \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(E, C)$ and cokernel $\iota \in \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(D, F)$ for any I an object of $\mathcal{I}$ to be: $E(I) := ker(\eta_I : C(I) \to D(I)), \epsilon_I : E(I) \to C(I)$ the corresponding kernel morphism, and $F(I) := coker(\eta_I : C(I) \to D(I)), \iota_I : D(I) \to F(I)$ the corresponding cokernel morphism. They satisfy the universal properties of kernel and cokernel just by the universal properties in $\mathcal{A}$, and similarly this gives the kernel and cokernel of every map. Monic (or epi) η in $\mathcal{A}^{\mathcal{I}}$ is just to say that for any I an object in $\mathcal{I}$, η_I is monic (or epi), so by the corresponding condition of abelian category of $\mathcal{A}$, each η_I is kernel of its cokernel (or cokernel of its kernel), so η satisfies the same condition. So $\mathcal{A}^{\mathcal{I}}$ is an abelian category.

6.2.3 Exactness of Adjoint Functors

To prove that R is left exact, take exact sequence

$$0 \to A \to B \to C$$

For any M an object in $\mathcal{A}$, act the exact sequence by functor $\mathrm{Hom}_{\mathcal{B}}(L(M), \cdot)$ which is left exact, this gives an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}}(L(M), A) \to \mathrm{Hom}_{\mathcal{B}}(L(M), B) \to \mathrm{Hom}_{\mathcal{B}}(L(M), C)$$

Then act by the natural isomorphism τ, this gives an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M, R(A)) \to \mathrm{Hom}_{\mathcal{A}}(M, R(B)) \to \mathrm{Hom}_{\mathcal{A}}(M, R(C))$$

By Yoneda lemma, this shows that $0 \to R(A) \to R(B) \to R(C)$ is exact. So R is left exact.
To prove that L is right exact, just to take everything in the opposite category. Take exact sequence

$$0 \to C^{op} \to B^{op} \to A^{op}$$

For any M an object in $\mathcal{B}^{op}$, act the exact sequence by functor $\mathrm{Hom}_{\mathcal{A}^{op}}(R^{op}M, \cdot)$ which is left exact, this gives an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}^{op}}(R^{op}M, C^{op}) \to \mathrm{Hom}_{\mathcal{A}^{op}}(R^{op}M, B^{op}) \to \mathrm{Hom}_{\mathcal{A}^{op}}(R^{op}M, A^{op})$$

Then act by the natural isomorphism τ^{op}, this gives an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}^{op}}(M, (LC)^{op}) \to \mathrm{Hom}_{\mathcal{B}^{op}}(M, (LB)^{op}) \to \mathrm{Hom}_{\mathcal{B}^{op}}(M, (LA)^{op})$$

By Yoneda lemma, this shows that $0 \to (LC)^{op} \to (LB)^{op} \to (LA)^{op}$ is exact. Take the opposite category again, $LA \to LB \to LC \to 0$ is exact. So L is right exact.

6.2.4 Five Lemma

To prove this result, it is easier to embed the abelian category in some R-module category to apply diagram-chasing. This embedding can be realized by Freyd-Mitchell embedding theorem, as long as the category is a small category.
To reduce to a small category, we try to construct a small abelian category containing all the objects in the diagram. Take $\mathcal{C}_0$ the full subcategory of $\mathcal{A}$ which contains objects $S_0 = \{0, A, B, C, D, E, A', B', C', D', E'\}$. Then construct $\mathcal{C}_{n+1}$ as full subcategory of $\mathcal{A}$ which contains objects

$$S_{n+1} = S_n \cup \{ker(f) | f \in \mathrm{Hom}_{\mathcal{A}}(J, K), J, K \in S_n\} \cup \{coker(f)|$$

$$f \in \mathrm{Hom}_{\mathcal{A}}(J, K), J, K \in S_n\} \cup \{J \times K | J, K \in S_n\}$$

Finally get the limit $\mathcal{C}_\infty$ which contains objects $S_\infty = \cup_{\mathbb{N}} S_n$, and since S_n are all sets, S_∞ is also a set, so $\mathcal{C}_\infty$ is a small category. And because $\mathcal{C}_\infty$ is full subcategory of $\mathcal{A}$, it is $\mathbb{A}$b-category, and by the construction, it satisfies all the conditions of abelian category.
So without loss of generality, assume $\mathcal{A}$ is small, so by Freyd-Mitchell embedding theorem, there exists some ring R, and a fully faithful exact embedding functor $i : \mathcal{A} \to \mathbf{R} - \mathrm{mod}$. By this embedding, the monic ($ker = 0$), epi ($coker = 0$) and exact property is just the same to check in a full subcategory of $\mathbf{R} - \mathrm{mod}$, where monic (or epi) equals injective (or surjective) morphism. For any morphism $f : A \to B$, take its kernel and cokernel, this gives an exact sequence

$$0 \to ker(f) \to A \to B \to coker(f) \to 0$$

and then apply i, which is exact, this gives the following exact sequence

$$0 \to i(ker(f)) \to i(A) \to i(B) \to i(coker(f)) \to 0$$

This shows that we can check kernel, cokernel and exactness in the full subcategory of $\mathbf{R} - \mathrm{mod}$.

Check that c is monic in $\mathbf{R}-\text{mod}$. $\forall x \in C', c(x) = 0$, take the image y of x in D', then $d(y) = 0$. Since d is monic, $y = 0$.

$$\begin{tikzcd}
x \in C' \arrow[r] \arrow[d, "c"] & y = 0 \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd}$$

By the exactness of the first row, there exists $w \in B'$ such that x is the image of w. Take $\bar{w} = b(w)$, then its image in C is zero, so by exactness of second row, there exists $\bar{v} \in A$ such that $\bar{w}$ is the image of $\bar{v}$.

$$\begin{tikzcd}
 & w \arrow[r] \arrow[d] & x \in C' \arrow[r] \arrow[d, "c"] & y = 0 \arrow[d] \\
\bar{v} \arrow[r] & \bar{w} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Since a is epi, the preimage $v \in A'$ of $\bar{v}$ exists, and the image of v in B equals $\bar{w}$, and since b is monic, the image of v in B' is w.

$$\begin{tikzcd}
v \arrow[r] \arrow[d] & w \arrow[r] \arrow[d] & x \in C' \arrow[r] \arrow[d, "c"] & y = 0 \arrow[d] \\
\bar{v} \arrow[r] & \bar{w} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So x is the image of v, which is zero. This proves that c is injective.

Check c is epi in $\mathbf{R}-\text{mod}$. Fix an element $x \in C$, it suffices to show that there is a preimage of x in C'. Take the images of x in D, E, since d is epi, there exists $y' \in D'$ such that $d(y') = y$, and since e is monic, the image of y' in E' is zero, so there exists $x' \in C'$ such that its image in D' is y'.

$$\begin{tikzcd}
x' \arrow[r] & y' \arrow[r] \arrow[d] & 0 \arrow[d] \\
x \arrow[r] & y \arrow[r] & 0
\end{tikzcd}$$

Take $u = x - c(x') \in C$, its image in D is zero by construction, so there exists $w \in B$ such that its image in C is u. Since b is epic , there exists$w' \in B'$ such that $b(w') = w$, take its image u' in C'.

$$\begin{tikzcd}
w' \arrow[r] \arrow[d] & u' & \\
w \arrow[r] & u = x - c(x') \arrow[r] & 0
\end{tikzcd}$$

Finally, the preimage of x in C' is just $u' + x'$.This proves that c is surjective.

When a, b, d, e are isomorphisms, by the previous proof, c is monic and epi, so c is an isomorphism.

6.3 Week 3 (by Brian Briod).

(i) Let us consider a sequence of chain complexes :

$$0 \xrightarrow{i_\bullet} A_\bullet \xrightarrow{\alpha_\bullet} B_\bullet \xrightarrow{\beta_\bullet} C_\bullet \xrightarrow{j_\bullet} 0$$

To show that the sequence is exact at $B_\bullet$ in the abelian category $\text{Ch}(\mathcal{A})$, we need to show that

$$\beta_\bullet \circ \alpha_\bullet = 0_\bullet$$

and that the induced map

$$\text{Image}(\alpha_\bullet) \to \ker(\beta_\bullet)$$

is an isomorphism of chain complexes. But since

$$0 \xrightarrow{i_n} A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\beta_n} C_n \xrightarrow{j_n} 0$$

is exact in $\mathcal{A}$ $\forall n \in \mathbb{Z}$, we have that

$$(\alpha_\bullet \circ \beta_\bullet)_n = \alpha_n \circ \beta_n = 0 \; \forall n \in \mathbb{Z}$$

Thus $\alpha_\bullet \circ \beta_\bullet = 0_\bullet$. Now again by exactness we get that

$$\text{Image}(\alpha_\bullet)_n = \text{Image}(\alpha_n) \cong \ker(\beta_n) = \ker(\beta_\bullet)_n \; \forall n \in \mathbb{Z}$$

hence the induced map

$$\text{Image}(\alpha_\bullet) \to \ker(\beta_\bullet)$$

is an isomorphism.
We check similarly exactness at $A_\bullet$ and $C_\bullet$ and thus we conclude that the sequence of chain complexes is exact.

(ii) Consider the following diagram :

$$\begin{tikzcd}
\dots \arrow[r] & D_n \arrow[r, "h_n"] \arrow[d, "d"] & B_n \arrow[r, "f_n"] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & \dots \\
\dots \arrow[r] & D_{n-1} \arrow[r, "h_{n-1}"] & B_{n-1} \arrow[r, "f_{n-1}"] & C_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

where $f_n \circ h_n = g_n$ $\forall n \in \mathbb{Z}$. Using that $f_\bullet$ and $g_\bullet$ are chain complexes morphisms we have that

$$f_{n-1} \circ d \circ h_n = d \circ f_n \circ h_n = d \circ g_n = g_{n-1} \circ d = f_{n-1} \circ h_{n-1} \circ d$$

but since $f_\bullet$ is monic we conclude that

$$d \circ h_n = h_{n-1} \circ d$$

Thus $h_\bullet$ is a chain complexes morphism such that $f_\bullet \circ h_\bullet = g_\bullet$.

(iii) Without loss of generality we prove for exact rows. First we recall that since $D_{\bullet,\bullet}$ is a bounded double complex, we have

$$(\text{Tot}(D_{\bullet,\bullet}))_n = \bigoplus_{p+q=n} D_{p,q}$$

and maps $d = d^h + d^v : (\text{Tot}(D_{\bullet,\bullet})_n \to (\text{Tot}(D_{\bullet,\bullet}))_{n-1}$. First we remark that

$$d \circ d = (d^h + d^v) \circ (d^h + d^v) = (d^h \circ d^h) + (d^v \circ d^v) + (d^v \circ d^h + d^h \circ d^v) = 0$$

by definition of a double complex.
Now by Freyd-Mitchell Embedding Theorem we only need to prove that

$$\text{Image}(d_n) = \ker(d_{n+1})$$

in the category of R-modules $\forall n \in \mathbb{Z}$. We know that

$$\text{Image}(d^h_n) = \ker(d^h_{n+1}) \; \forall n \in \mathbb{Z}$$

by exactness of the rows. Also by the previous computation

$$\text{Image}(d_n) \subseteq \ker(d_{n+1})$$

Thus we only need to show that

$$\ker(d_{n+1}) \subseteq \text{Image}(d_n)$$

But notice that

$$\alpha \in \ker(d_{n+1}) \implies \alpha \in \ker(d^v_{n+1})$$

but since $\ker(d^h_{n+1}) = \text{Image}(d^h_n)$ there exists α_1 such that $d^h_n(\alpha_1) = \alpha$. Now using the anti-commutativity $d^h \circ d^v = -d^v \circ d^h$, we get

$$d^v_{n+1}(\alpha) = 0 \iff d^v_{n+1} \circ d^h_n(\alpha_1) = 0 \iff -d^h_n \circ d^v_{n+1}(\alpha_1) = 0$$

which implies that

$$-d^v_{n+1}(\alpha_1) \in \ker(d^h_n) = \text{Image}(d^h_{n+1})$$

i.e. again there exists α_2 such that

$$d^h_{n+1}(\alpha_2) = -d^v_{n+1}(\alpha_2)$$

We iterate the argument to construct $\{\alpha_l\}_{l\geq 1}$ such that

$$d^h_{n+l-1}(\alpha_l) = -d^v_{n+l-1}(\alpha_l)\ \forall n \geq 1$$

But since $D_{\bullet,\bullet}$ is bounded there must exists $m \geq 1$ such that

$$d^v_{n+m-1}(\alpha_m) = 0 \implies \alpha_m = 0$$

Now observe that

$$d_n(\sum_{i=1}^{m} \alpha_i) = \sum_{i=1}^{m} d^v(\alpha_i) + \sum_{i=1}^{m} d^v(\alpha_i) = \alpha_m + \sum_{i=1}^{m-1} d^v(\alpha_i) + \alpha + \sum_{i=2}^{m} d^h(\alpha_i)$$

Which means by definition of $\{\alpha_i\}_{1\leq i\leq m}$

$$d(\sum_{i=1}^{m} \alpha_i) = \alpha_m + \sum_{i=1}^{m-1} d^v(\alpha_i) + \alpha - \sum_{i=1}^{m-1} d^v(\alpha_i) = \alpha + \alpha_m = \alpha$$

which shows that $\ker(d) \subset \text{Image}(d)$ and thus concludes.

(iv) (a) We define the double complex $D_{\bullet,\bullet}$ with only two non trivial rows $B_\bullet$ and $C_\bullet$ by :

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & C_n \arrow[u] \arrow[r, "d"] & C_{n-1} \arrow[u] \arrow[r, "d"] & C_{n-2} \arrow[u] \arrow[r] & \dots \\
\dots \arrow[r] & B_n \arrow[u, "f_n"] \arrow[r, "-d"] & B_{n-1} \arrow[u, "f_{n-1}"] \arrow[r, "-d"] & B_{n-2} \arrow[u, "f_{n-2}"] \arrow[r] & \dots \\
\dots \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \dots
\end{tikzcd}$$

where anti-commutativity of the non-trivial squares is given by f being a chain map.

Note that equivalently we could define $D_{\bullet,\bullet}$ using more explicitly the sign trick :

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d"] \arrow[u] & C_{n-1} \arrow[r, "d"] \arrow[u] & C_{n-2} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d"] \arrow[u, "(-1)^n f_n"'] & B_{n-1} \arrow[r, "d"] \arrow[u, "(-1)^{n-1} f_{n-1}"'] & B_{n-2} \arrow[r] \arrow[u, "(-1)^{n-2} f_{n-2}"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

(b) Let $A_{\bullet,\bullet}$ be a double complex with maps $d^h_{p,q} : A_{p,q} \to A_{p-1,q}$ and $d^v_{p,q} : A_{p,q} \to A_{p,q-1}$. Using the sign trick we get a chain complex $\{A_{\bullet,q}\}_{q\in\mathbb{Z}}$ with boundary maps $\varphi_{\bullet,q} : A_{\bullet,q} \to A_{\bullet,q-1}$, defined by

$$\varphi_{p,q} = (-1)^p d^v_{p,q}$$

Now applying again a sign trick to this chain complex $\{A_{\bullet,q}\}_{q\in\mathbb{Z}}$ we define a double complex $A'_{\bullet,\bullet}$ with maps

$$d'^v_{p,q} = (-1)^p \varphi_{p,q} \qquad d'^h_{p,q} = d^h_{p,q}$$

But

$$d'^v_{p,q} = (-1)^p \varphi_{p,q} = (-1)^p (-1)^p d^v_{p,q} = d^v_{p,q}$$

i.e. $A'_{\bullet,\bullet} = A_{\bullet,\bullet}$ and thus the sign trick is a one to one correspondence between objects of $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$ and double complexes.

(c) We can consider $C_\bullet$ (resp. $B[-1]_\bullet$) as a double complex with only one non trivial row in the following way

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d"] \arrow[u] & C_{n-1} \arrow[r, "d"] \arrow[u] & C_{n-2} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & B[-1]_n \arrow[r, "-d"] \arrow[u] & B[-1]_{n-1} \arrow[r, "-d"] \arrow[u] & B[-1]_{n-2} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Now for the exactness of the short sequence, we denote

$$0 \longrightarrow C_\bullet \xrightarrow{\iota_{\bullet,\bullet}} D_{\bullet,\bullet} \xrightarrow{\pi_{\bullet,\bullet}} B[-1]_\bullet \longrightarrow 0$$

Setting the indices such that

$$C_{1,n} = C_n \qquad D_{1,n} = C_n \qquad D_{0,n} = B[-1]_{n+1} \qquad B[-1]_{0,n} = B[-1]_n$$

And $\iota_{\bullet,\bullet}, \pi_{\bullet,\bullet}$ are defined as

$$\iota_{p,q} = \begin{cases} \mathrm{Id}_{C_q} & \text{if } p = 1 \\ \mathrm{Id}_0 & \text{if } p \neq 1 \end{cases}$$

$$\pi_{p,q} = \begin{cases} \mathrm{Id}_{B_q} & \text{if } p = 0 \\ \mathrm{Id}_0 & \text{if } p \neq 0 \end{cases}$$

From point b) this corresponds to a short sequence in $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$ using the sign trick. From exercise 1, we only to check exactness at each non trivial rows i.e.

i.
$$\iota_{1,n} = \mathrm{Id}_{C_n} : C_{1,n} = C_n \to D_{1,n} = C_n$$
is an injective morphism of chain complexes.

ii.
$$\pi_{0,n} = \mathrm{Id}_{B_n} : D_{0,n} = B_n \to B[-1]_{n+1} = B_n$$
is a surjective morphism of chain complexes.

iii.
$$\iota_{1,n} \circ \pi_{1,n} = 0 \qquad \iota_{0,n} \circ \pi_{0,n} = 0$$

i. and ii. are clear from the definitions of $\iota_{1,n}$ and $\pi_{0,n}$.
Finally iii. holds since

$$\iota_{0,n} = 0 \ \forall n \in \mathbb{Z} \text{ and } \pi_{1,n} = 0 \ \forall n \in \mathbb{Z}$$

Note that the mapping cone of $f : D_\bullet := \mathrm{Tot}(D_{\bullet,\bullet})$ is given by :

$$D_\bullet = B[-1]_\bullet \oplus C_\bullet$$

with chain map

$$d_{D,\bullet} = \begin{pmatrix} -d_{B[-1],\bullet} & 0 \\ f[-1]_\bullet & d_{C,\bullet} \end{pmatrix}$$

We check that it is indeed a chain complex :

$$d_{D,n} \circ d_{D,n-1} = \begin{pmatrix} -d_{B[-1],n} & 0 \\ f[-1]_n & d_{C,n} \end{pmatrix} \begin{pmatrix} -d_{B[-1],n-1} & 0 \\ f[-1]_{n-1} & d_{C,n-1} \end{pmatrix}$$

$$= \begin{pmatrix} d_{B[-1],n} \circ d_{B[-1],n-1} & 0 \\ -f[-1]_n \circ d_{B[-1],n-1} + d_{C,n} \circ f[-1]_{n-1} & d_{C,n} \circ d_{C,n-1} \end{pmatrix} = 0$$

since $f_\bullet : B_\bullet \to C_\bullet$ is a chain map.

6.4 Week 4 (by Milo Nicolas Jacques Blum).

(i) Consider the commutative diagram of R-modules

$$\begin{tikzcd} & A' \arrow[r, "i'"] \arrow[d, "f"] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C & \end{tikzcd}$$

We want to show that the sequence

$$\ker(f) \xrightarrow{\alpha} \ker(g) \xrightarrow{\beta} \ker(h) \xrightarrow{\partial} \mathrm{coker}(f) \xrightarrow{\varphi} \mathrm{coker}(g) \xrightarrow{\psi} \mathrm{coker}(h)$$

is exact, where α, β, φ, and ψ are respectively induced by i', p', i and p. These morphisms are well defined by commutativity of the diagram. Let $c \in \ker(h)$, by surjectivity of p', there is $b \in B'$ with

$p'(b) = c$. By commutativity of the diagram, $pg(b) = hp'(b) = 0$, so $g(b) \in \ker(p) = \mathrm{im}(i)$, thus there exists some $a \in A$ such that $i(a) = g(b)$. In order to define $\partial(c) = a$, we have to verify that this is well defined, the only non-canonical choice we made is the choice of the preimage $b \in B'$. Let $b, b' \in B'$ such that $p'(b) = c = p'(b')$, and a, a' such that $i(a) = g(b), i(a') = g(b')$. Then $b - b' \in \ker(p') = \mathrm{im}(i')$, so there exists $\tilde{a} \in A'$ such that $i'(\tilde{a}) = b - b'$. We have that $g(b - b')) = if(\tilde{a})$, so by injectivity of i, $a - a' = f(\tilde{a}) \in \mathrm{im}(f)$, so $\partial : \ker(h) \to \mathrm{coker}(f)$ is well defined.

We start by showing that $\ker(\beta) = \mathrm{im}(\alpha)$. We know that $p'i' = 0$, thus $\beta\alpha = 0$ so we just have to prove that $\ker(\beta) \subset \mathrm{im}(\alpha)$. Let $x \in \ker(\beta) \subset \ker(g)$, in particular, $x \in \ker(p') = \mathrm{im}(i')$, i.e. there exists some $y \in A'$ such that $i'(y) = x$. We know that $if(y) = gi'(y) = g(x) = 0$, and since i is injective, $y \in \ker(f)$. By definition of α, $\alpha(y) = x$, so the sequence is exact at $\ker(g)$.

We will now show that $\ker(\partial) = \mathrm{im}(\beta)$. Let $x \in \ker(g)$, then

$$\partial\beta(x) = \partial p'(x) = i^{-1}gp'^{-1}p'(x) = i^{-1}g(x) = 0,$$

so $\mathrm{im}(\beta) \subset \ker(\partial)$. Let $x \in \ker(\partial) \subset \ker(h)$, by surjectivity of p', there exists some $y \in B'$ such that $p'(y) = x$. Since $pg(y) = hp'(y) = h(x) = 0$, $y \in \ker(p) = \mathrm{im}(i)$, so there is some $z \in A$ such that $i(z) = g(y)$. By definition of ∂, $z + \mathrm{im}(f) = \partial(x) + \mathrm{im}(f) = 0 + \mathrm{im}(f)$, that means that $z \in \mathrm{im}(f)$, so there exists $a \in A'$ such that $f(a) = z$. Moreover, $g(y - i'(a)) = g(y) - if(a) = g(y) - i(z) = 0$, thus $i'(a) - y \in \ker(g)$ and $\beta(y - i'(a)) = \beta(y) - p'i'(a) = \beta(y) = x$. Therefore, $x \in \mathrm{im}(\beta)$, and we conclude that the sequence is exact at $\ker(h)$.

We prove that $\ker(\varphi) = \mathrm{im}(\partial)$. Let $x \in \ker(h)$, then $\varphi\partial(x) = gp^{-1}(x) = 0 + \mathrm{im}(g)$ i.e. $\mathrm{im}(\partial) \subset \ker(\varphi)$. Now assume $x \in \ker(\varphi)$, and let $\tilde{x} \in A$ be a residue of x, then $i(\tilde{x}) \in \mathrm{im}(g)$, so there is some $y \in B'$ such that $g(y) = i(\tilde{x})$. We see that $hp'(y) = pg(y) = pi(\tilde{x}) = 0$, thus $p'(y) \in \ker(h)$, and by definition $\partial p'(y) = x$, so $x \in \mathrm{im}(\partial)$.

Finally, we prove that the sequence is exact at $\mathrm{coker}(g)$. By exactness of the second row, $\psi\varphi = 0$. Let $x \in \ker\psi$ and $\tilde{x} \in B$ a residue of x, then $p(\tilde{x}) \in \mathrm{im}(h)$, so there is some $y \in C'$ such that $h(y) = p(\tilde{x})$. By surjectivity of p', there also is some $z \in B'$ such that $p'(z) = y$. We can see that $p(\tilde{x}) = h(y) = hp'(z) = pg(z)$, so $\tilde{x} - g(z) \in \ker(p) = \mathrm{im}(i)$. Let $a \in A$ be such that $i(a) = \tilde{x} - g(z)$, then $\varphi(a) = x - g(z) = x + \mathrm{im}(g)$, thus $x \in \mathrm{im}(\varphi)$. We conclude that the sequence is indeed exact.

Moreover, if $i' : A' \to B'$ is injective, its restriction to $\ker(f)$ is also injective. If $p : B \to C$ is surjective, the quotient map $\mathrm{coker}(g) \to \mathrm{coker}(h)$ will also be surjective by definition.

(ii) (a) By the long exact sequence theorem , there is a long exact sequence

$$\cdots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

Recall that a complex A is exact if and only if $H_n(A) = 0$ for every n. If two of the three complexes are exact, the sequence is of the form

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_n(I) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where $I \in \{A, B, C\}$ depending on our assumption. By exactness of the long sequence, $H_n(I) = 0$ for every n, which implies that the complex I is exact.

(b) The long exact sequence induced by the short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow C \xrightarrow{\alpha} \mathrm{im}(f) \longrightarrow 0$$

shows that α is a quasi-isomorphism, and the long exact sequence induced by the short exact sequence

$$0 \longrightarrow \mathrm{im}(f) \xrightarrow{\beta} D \longrightarrow \mathrm{coker}(f) \longrightarrow 0$$

shows that β is a quasi-isomorphism. Thus $f = \beta \circ \alpha$ is a quasi-isomorphism.

The converse is false. Indeed consider the following morphism of chain maps

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "id"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "id"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

The rows are exact, so this is a quasi-isomorphism. The complex of the kernel of this morphism is given by

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \ ,$$

which is not acyclic.

(iii) We can easily see that the kernel and the image of the morphism $\mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z}$ are both $\mathbb{Z}/2\mathbb{Z}$, thus the homology groups are 0 and the complex is acyclic. Let $\varphi : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, the image of $\cdot 2$ is $\{0, 2\}$, $\varphi(2)$ has to be even, and $\cdot 2$ sends even number to 0, so $\cdot 2 \circ \varphi \circ \cdot 2 = 0$, thus this sequence does not split.

(iv) We show that a chain map $\{s_n : C_{n-1} \to D_n\}$ makes the following diagram commutes i.e. f extends to a map $(-s, f) : \text{cone}(C) \to D$

$$\begin{tikzcd}
\text{cone}(C)_{n+1} \arrow[r, "\delta"] \arrow[d, "(-s,f)"] & \text{cone}(C)_n \arrow[d, "(-s,f)"] \\
D_{n+1} \arrow[r, "d"] & D_n
\end{tikzcd}$$

if and only if it satisfies $f = ds + sd$ i.e. f is null homotopic. We recall that the differential of cone(C) is given by $\delta(c_n, c_{n+1}) = (-d(c_n), d(c_{n+1} - c_n))$. Let $c_n \in C_n, c_{n+1} \in C_{n+1}$, then

$$\begin{aligned}
f(c_n) - ds(c_n) - sd(c_n) &= df(c_{n+1}) - fd(c_{n+1}) + f(c_n) - ds(c_n) - sd(c_n) \\
&= d(f(c_{n+1}) - s(c_n)) - (fd(c_{n+1}) - f(c_n) + sd(c_n)) \\
&= (d \circ (-s, f))(c_n, c_{n+1}) - ((-s, f) \circ \delta)(c_n, c_{n+1}).
\end{aligned}$$

6.5 Week 5 (by Deborah Righi).

Let us consider $\mathcal{A}$, $\mathcal{B}$ two abelian categories.

(i) Let $\mathcal{S}$ be the category of short exact sequences in $\mathcal{A}$, and δ a homological δ-functor with additive functors $\{T_i : \mathcal{A} \to \mathcal{B}\}_{i \geq 0}$.
For $i \geq 0$ we call F_i the functor $\mathcal{S} \to \mathcal{B}$ which sends the short exact sequence

$$0 \to A \to B \to C \to 0$$

onto the object $T_i(C)$, and the morphism $f_\bullet$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f_2"] & B \arrow[r] \arrow[d, "f_1"] & C \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

onto the morphism $T_i(f_0)$ in $\mathcal{B}$.
Similarly we define the functor $G_i : \mathcal{S} \to \mathcal{B}$ which sends the short exact sequence

$$0 \to A \to B \to C \to 0$$

onto the object $T_i(A)$, and the morphism $f_\bullet$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f_2"] & B \arrow[r] \arrow[d, "f_1"] & C \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

onto the morphism $T_i(f_2)$ in $\mathcal{B}$.
These are both well defined functors since each T_i is a functor.
Since for each $i \geq 0$, δ_i assigns to each object

$$0 \to A \to B \to C \to 0$$

in $\mathcal{S}$ a morphism $T_i(C) \to T_{i-1}(A)$, for δ_i to be a natural transformation $F_i \Rightarrow G_{i-1}$ it remains to show naturality, which is to say that for all morphisms of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f_2"] & B \arrow[r] \arrow[d, "f_1"] & C \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
T_i(C) \arrow[r, "\delta_i(0\to A\to B\to C\to 0)"] \arrow[d, "T_i(f_2)=F_i(f_\bullet)"'] & T_{i-1}(A) \arrow[d, "T_{i-1}(f_0)=G_{i-1}(f_\bullet)"] \\
T_i(C') \arrow[r, "\delta_i(0\to A'\to B'\to C'\to 0)"'] & T_{i-1}(A')
\end{tikzcd}$$

commutes. This is the case because δ is a homological δ-functor which allows us to conclude.

(ii) Suppose first that $P_\bullet$ is a projective object in $Ch(\mathcal{A})$.
We then have that $P_\bullet[-1]$ is also projective because for any diagram

$$\begin{tikzcd}
 & P_\bullet[-1] \arrow[d] \\
A_\bullet \arrow[r, two heads] & B_\bullet
\end{tikzcd}$$

the projectiveness of $P_\bullet$ means we have a morphism which makes the induced diagram

$$\begin{tikzcd}
 & P_\bullet \arrow[d] \arrow[dl, dashed] \\
A_\bullet[+1] \arrow[r, two heads] & B_\bullet[+1]
\end{tikzcd}$$

commute. Taking the induced morphism $P_\bullet[-1] \to A_\bullet$ makes the diagram

$$\begin{tikzcd}
 & P_\bullet[-1] \arrow[d] \arrow[dl, dashed] \\
A_\bullet \arrow[r, two heads] & B_\bullet
\end{tikzcd}$$

commute.

We can then consider the diagram

$$\begin{tikzcd}
 & P_\bullet[-1] \arrow[d, "Id_{P_\bullet[-1]}"] \\
Cone(P_\bullet) \arrow[r, two heads, "\pi^1_\bullet"] & P_\bullet[-1]
\end{tikzcd}$$

where the morphism $\pi^1_\bullet$ is projection onto the first element

$$\pi^1_n : Cone(P_\bullet)_n = P_{n-1} \oplus P_n \to P_{n-1} = P_\bullet[-1]_n,$$

and similarly $\pi^2_\bullet$ as used below is projection onto the second element.
The projectiveness of $P_\bullet[-1]$ means we have a morphism of chain complexes

$$\psi_\bullet : P_\bullet[-1] \to Cone(P_\bullet)$$

such that $\pi^1_\bullet \circ \psi_\bullet = Id_{P_\bullet[-1]}$.

This identity and $\psi_\bullet$ being a chain morphism yield the following equalities for all $n \in \mathbb{Z}$:

$$\begin{aligned} \pi^1_n \circ \psi_n &= Id_{P_{n-1}} \\ \psi_n \circ d^P_n &= d^{Cone(P)}_{n+1} \circ \psi_{n+1}. \end{aligned}$$

Using the definition of the chain maps for the cone of $P_\bullet$ and plugging in the first equation gives

$$\begin{aligned} \pi^2_n \circ \psi_n \circ d^P_n &= \pi^2_n \circ d^{Cone(P)}_{n+1} \circ \psi_{n+1} \\ &= \pi^2_n \circ \begin{bmatrix} -d^P_n & 0 \\ Id_{P_n} & d^P_{n+1} \end{bmatrix} \circ \psi_{n+1} \\ &= \begin{bmatrix} Id_{P_n} & d^P_{n+1} \end{bmatrix} \circ \psi_{n+1} \\ &= \pi^1_{n+1} \circ \psi_{n+1} + d^P_{n+1} \circ \pi^2_{n+1} \circ \psi_{n+1} \\ &= Id_{P_n} + d^P_{n+1} \circ \pi^2_{n+1} \circ \psi_{n+1} \end{aligned}$$

so

$$Id_{P_n} = (\pi^2_n \circ \psi_n) \circ d^P_n - d^P_{n+1} \circ (\pi^2_{n+1} \circ \psi_{n+1})$$

which means that the identity is nullhomotopic, and $P_\bullet$ is split exact.

To show that each P_n is projective we fix $n \in \mathbb{Z}$ and consider any diagram

$$\begin{tikzcd} & P_n \arrow[d, "f"] \\ A \arrow[r, "\pi"] & B \end{tikzcd}$$

where π is epi, and then construct the morphism of chain complexes

$$\begin{tikzcd} {} \arrow[r, dotted] & 0 \arrow[r] \arrow[d] & A \arrow[r, "Id_A"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r, dotted] \arrow[d] & {} \\ {} \arrow[r, dotted] & 0 \arrow[r] & B \arrow[r, "Id_B"'] & B \arrow[r] & 0 \arrow[r, dotted] & {} \end{tikzcd}$$

where the chain complexes on top and bottom have non-zero elements only in the $(n+1)$-th and n-th positions, noting that it is epi since π is. We also have a morphism of chain complexes

$$\begin{tikzcd} {} \arrow[r, dotted] & P_{n+2} \arrow[r, "d^P_{n+2}"] \arrow[d] & P_{n+1} \arrow[r, "d^P_{n+1}"] \arrow[d, "f \circ d^P_{n+1}"] & P_n \arrow[r, "d^P_n"] \arrow[d, "f"] & P_{n-2} \arrow[r, dotted] \arrow[d] & {} \\ {} \arrow[r, dotted] & 0 \arrow[r] & B \arrow[r, "Id_B"'] & B \arrow[r] & 0 \arrow[r, dotted] & {} \end{tikzcd}$$

which allows us to use the projectiveness of $P_\bullet$ to obtain a morphism of chain complexes $g_\bullet$ such that the diagram

$$\begin{tikzcd} {} \arrow[r, dotted] & P_{n+2} \arrow[r, "d^P_{n+2}"] \arrow[d] & P_{n+1} \arrow[r, "d^P_{n+1}"] \arrow[d, "g_{n+1}"] \arrow[dd, bend right=60, "f \circ d^P_{n+1}"'] & P_n \arrow[r, "d^P_n"] \arrow[d, "g_n"] \arrow[dd, bend right=60, "f"'] & P_{n-2} \arrow[r, dotted] \arrow[d] & {} \\ {} \arrow[r, dotted] & 0 \arrow[r] \arrow[d] & A \arrow[r, "Id_A"'] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r, dotted] \arrow[d] & {} \\ {} \arrow[r, dotted] & 0 \arrow[r] & B \arrow[r, "Id_B"] & B \arrow[r] & 0 \arrow[r, dotted] & {} \end{tikzcd}$$

commutes. This directly implies the commutativity of the diagram

$$\begin{tikzcd}
 & P_n \arrow[dl, "g_n"'] \arrow[d, "f"] \\
A \arrow[r, "\pi"'] & B
\end{tikzcd}$$

and the arbitrary choice of $n \in \mathbb{Z}$ allows us to conclude.

We have now shown that

$$P_\bullet \text{ is a projective object in } Ch(\mathcal{A}) \implies P_\bullet \text{ is split exact and each } P_n \text{ is projective in } \mathcal{A}$$

and go about about proving the converse.

Let $P_\bullet$ be a chain complex in $Ch(\mathcal{A})$ which is split exact and such that each P_n is projective in $\mathcal{A}$.

Option 1:
$P_\bullet$ being split exact means we have morphisms $\psi_n : P_n \to P_{n+1}$ such that

$$\psi_{n-1} \circ d_n^P + d_{n+1}^P \circ \psi_n = Id_{P_n}$$

for all $n \in \mathbb{Z}$.
Let us consider two chain morphisms $f_\bullet$ and $\pi_\bullet$

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & A_1 \arrow[r, "d_1^A"] \arrow[d, "\pi_1"] & A_0 \arrow[r, "d_0^A"] \arrow[d, "\pi_0"] & A_{-1} \arrow[r, dotted] \arrow[d, "\pi_{-1}"] & \cdots \\
\cdots \arrow[r, dotted] & B_1 \arrow[r, "d_1^B"'] & B_0 \arrow[r, "d_0^B"'] & B_{-1} \arrow[r, dotted] & \cdots \\
\cdots \arrow[r, dotted] & P_1 \arrow[r, "d_1^P"'] \arrow[u, "f_1"] & P_0 \arrow[r, "d_0^P"'] \arrow[u, "f_0"] & P_{-1} \arrow[r, dotted] \arrow[u, "f_{-1}"] & \cdots
\end{tikzcd}$$

where $\pi_\bullet$ is epi.
The projectiveness of each P_n means we have morphisms $h_n : P_n \to A_n$ which make the diagram

$$\begin{tikzcd}
 & P_n \arrow[dl, "h_n"'] \arrow[d, "f_n"] \\
A_n \arrow[r, "\pi_n"'] & B_n
\end{tikzcd}$$

commute for each $n \in \mathbb{Z}$.
We claim that setting

$$g_n \coloneqq d_{n+1}^A \circ h_{n+1} \circ \psi_n + h_n \circ \psi_{n-1} \circ d_n^P : P_n \to A_n$$

for each $n \in \mathbb{Z}$ will give the identity

$$f_\bullet = \pi_\bullet \circ g_\bullet.$$

This is easily verified since for all $n \in \mathbb{Z}$ we have

$$\begin{aligned}
\pi_n \circ g_n &= \pi_n \circ (d_{n+1}^A \circ h_{n+1} \circ \psi_n + h_n \circ \psi_{n-1} \circ d_n^P) \\
&= d_{n+1}^B \circ \pi_{n+1} \circ h_{n+1} \circ \psi_n + f_n \circ \psi_{n-1} \circ d_n^P \\
&= d_{n+1}^B \circ f_{n+1} \circ \psi_n + f_n \circ \psi_{n-1} \circ d_n^P \\
&= f_n \circ (\psi_{n-1} \circ d_n^P + d_{n+1}^P \circ \psi_n) \\
&= f_n \circ Id_{P_n} \\
&= f_n
\end{aligned}$$

by using the fact that ψ_n, ψ_{n-1} are homotopy maps.

Additionally, $g_\bullet$ is a well-defined chain morphism because

$$\begin{aligned} d_n^A \circ g_n &= d_n^A \circ (d_{n+1}^A \circ h_{n+1} \circ \psi_n + h_n \circ \psi_{n-1} \circ d_n^P) \\ &= d_n^A \circ h_n \circ \psi_{n-1} \circ d_n^P \\ &= (d_n^A \circ h_n \circ \psi_{n-1} + h_{n-1} \circ \psi_{n-2} \circ d_{n-1}^P) \circ d_n^P \\ &= g_{n-1} \circ d_n^P \end{aligned}$$

for all $n \in \mathbb{Z}$.

Since the chain complex $A_\bullet$ was chosen arbitrarily we conclude that $P_\bullet$ is projective.

Option 2:
$P_\bullet$ being split exact means we have morphisms $\psi_n : P_n \to P_{n+1}$ such that

$$\psi_{n-1} \circ d_n^P + d_{n+1}^P \circ \psi_n = Id_{P_n}$$

for all $n \in \mathbb{Z}$.
The existence of such morphisms means that the chain complex is exact and also allows us to construct the isomorphism

$$\begin{bmatrix} d_{n+1}^P \circ \psi_n \\ d_n^P \end{bmatrix} : P_n \to \ker(d_n^P) \oplus \operatorname{im}(d_n^P)$$

with inverse

$$\begin{bmatrix} \iota_n & \psi_{n-1} \end{bmatrix} : \ker(d_n^P) \oplus \operatorname{im}(d_n^P) \to P_n$$

for all $n \in \mathbb{Z}$.
We would like to show that

$$\begin{tikzcd}
{} \arrow[r, dotted] & P_{n+1} \arrow[r, "d_{n+1}^P"] \arrow[d, "{\begin{bmatrix} d_{n+2}^P \circ \psi_{n+1} \\ d_{n+1}^P \end{bmatrix}}"] & P_n \arrow[r, "d_n^P"] \arrow[d, "{\begin{bmatrix} d_{n+1}^P \circ \psi_n \\ d_n^P \end{bmatrix}}"] & P_{n-1} \arrow[r, dotted] \arrow[d, "{\begin{bmatrix} d_n^P \circ \psi_{n-1} \\ d_{n-1}^P \end{bmatrix}}"] & {} \\
{} \arrow[r, dotted] & \ker(d_{n+1}^P) \oplus \operatorname{im}(d_{n+1}^P) \arrow[r] & \ker(d_n^P) \oplus \operatorname{im}(d_n^P) \arrow[r] & \ker(d_{n-1}^P) \oplus \operatorname{im}(d_{n-1}^P) \arrow[r, dotted] & {}
\end{tikzcd}$$

is an isomorphism of chain complexes, where the differentials of the chain on the bottom are all of the form $\begin{bmatrix} 0 & \varphi_n \\ 0 & 0 \end{bmatrix}$, with φ_n being isomorphisms given by the exactness of the chain complex $P_\bullet$.
This is the case because by choosing the φ_n appropriately, we can use Freyd-Mitchell's embedding theorem and suppose that φ_n is the identity, which then means

$$\begin{aligned} \begin{bmatrix} d_n^P \circ \psi_{n-1} \\ d_{n-1}^P \end{bmatrix} \circ d_n^P &= \begin{bmatrix} d_n^P \circ \psi_{n-1} \circ d_n^P \\ d_{n-1}^P \circ d_n^P \end{bmatrix} \\ &= \begin{bmatrix} d_n^P \circ (\mathrm{Id}_{P_n} - d_{n+1}^P \circ \psi_n) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} d_n^P \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \varphi_n \circ d_n^P \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \varphi_n \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} d_{n+1}^P \circ \psi_n \\ d_n^P \end{bmatrix}. \end{aligned}$$

Finally, there is an isomorphism between the chain complex on the bottom row and the direct sum over every $n \in \mathbb{Z}$ of the chain complexes $P(n)$

$$\begin{tikzcd}
{} \arrow[r, dotted] & 0 \arrow[r] & \operatorname{im}(d_n^P) \arrow[r, "\varphi_n"] & \ker(d_{n-1}^P) \arrow[r] & 0 \arrow[r, dotted] & {} \\
& & \text{n-th position in the chain complex} \arrow[u, bend left]
\end{tikzcd}$$

because the former satisfies the universal properties of the product and coproduct.

This means that it is enough to show that the direct sum is projective, for which it is enough to show that each individual element in the direct sum is projective.

To this end, fix $n \in \mathbb{Z}$ and suppose we have morphisms of chain complexes

$$\begin{tikzcd}
& P(n) \arrow[d, "f_\bullet"] \\
A \arrow[r, "\pi_\bullet"'] & B
\end{tikzcd}$$

where $\pi_\bullet$ is epi.

Since $P_n \cong \ker(d_n^P) \oplus \operatorname{im}(d_n^P)$ is projective, $\operatorname{im}(d_n^P)$ must be as well, so there exists a morphism $g_n : \operatorname{im}(d_n^P) \to A_n$ such that $\pi_n \circ g_n = f_n$. Now setting $g_{n-1} = d_n^A \circ g_n \circ \varphi_n^{-1}$, and completing $g_\bullet : P(n) \to A_\bullet$ everywhere else with zero morphisms we have that $g_\bullet$ is a well defined morphism of chain complexes by definition of g_{n-1}, and satisfies $\pi_\bullet \circ g_\bullet = f_\bullet$. At every index other that $n-1$ this is immediate, and at $n-1$ we have

$$\begin{aligned}
\pi_{n-1} \circ g_{n-1} &= \pi_{n-1} \circ d_n^A \circ g_n \circ \varphi_n^{-1} \\
&= d_n^B \circ \pi_n \circ g_n \circ \varphi_n^{-1} \\
&= d_n^B \circ f_n \circ \varphi_n^{-1} \\
&= f_{n-1}
\end{aligned}$$

because $\pi_\bullet$ and $f_\bullet$ are both morphisms of chain complexes.

We therefore have that each $P(n)$ is projective which allows us to conclude.

(iii) Suppose that $\mathcal{A}$ has enough projectives, and let $A_\bullet$ be a chain complex in $Ch(\mathcal{A})$.
For all $n \in \mathbb{Z}$ we have a projective object P_n in $\mathcal{A}$ and an epi $f_n : P_n \to A_n$.
These projective objects define a chain complex

$$\begin{tikzcd}
{} \arrow[r, dotted] & P_2 \oplus P_1 \arrow[r] & P_1 \oplus P_0 \arrow[r] & P_0 \oplus P_{-1} \arrow[r, dotted] & {}
\end{tikzcd}$$

with differentials $\begin{bmatrix} 0 & Id \\ 0 & 0 \end{bmatrix}$.
Setting $Q_n := P_{n+1} \oplus P_n$ we may call this chain complex $Q_\bullet$, and we claim that setting

$$g_n := \begin{bmatrix} d_{n+1}^A \circ f_{n+1} & f_n \end{bmatrix} : Q_n \to A_n$$

for all $n \in \mathbb{Z}$ defines a chain morphism $g_\bullet : Q_\bullet \to A_\bullet$.

This is the case because

$$\begin{aligned}
d_n^A \circ g_n &= d_n^A \circ \begin{bmatrix} d_{n+1}^A \circ f_{n+1} & f_n \end{bmatrix} \\
&= \begin{bmatrix} d_n^A \circ d_{n+1}^A \circ f_{n+1} & d_n^A \circ f_n \end{bmatrix} \\
&= \begin{bmatrix} 0 & d_n^A \circ f_n \end{bmatrix} \\
&= \begin{bmatrix} 0 & d_n^A \circ f_n \circ Id_{P_n} \end{bmatrix} \\
&= \begin{bmatrix} d_n^A \circ f_n & f_{n-1} \end{bmatrix} \circ \begin{bmatrix} 0 & Id_{P_n} \\ 0 & 0 \end{bmatrix} \\
&= g_{n-1} \circ d_n^Q.
\end{aligned}$$

Moreover, each map g_n is epi, because its second component, f_n, is epi. Using Freyd-Mitchell it is easy to see that any map from a direct sum of two objects to a third object is epi if either of its two components are epi. Therefore $g_\bullet$ is epi in $Ch(\mathcal{A})$.

The maps $\psi_n : Q_n = P_{n+1} \oplus P_n \to P_{n+2} \oplus P_{n+1} = Q_{n+1}$ given by $\begin{bmatrix} 0 & 0 \\ Id & 0 \end{bmatrix}$ are homotopy maps between the identity and 0 on $Q_\bullet$, because

$$d_{n+1}^Q \circ \psi_n = \begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\psi_{n-1} \circ d_n^Q = \begin{bmatrix} 0 & 0 \\ 0 & Id \end{bmatrix}$$

so,

$$Id_{Q_n} = \psi_{n-1} \circ d_n^Q + d_{n+1}^Q \circ \psi_n.$$

Finally, using the result of the previous exercise, it remains to show that each $Q_n = P_{n+1} \oplus P_n$ is projective for us to have defined a projective chain complex $Q_\bullet$ and an epi $g_\bullet : Q_\bullet \to A_\bullet$.

This is immediate since each P_n is projective.

(iv) Suppose we have the following diagram in $\mathcal{A}$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r, dotted] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r, dotted] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the column is exact and the rows are projective resolutions.

We set $P_i := P_i' \oplus P_i''$ and use the canonical injection and projection associated to the direct sum to extend the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r, dotted] & P_2' \arrow[r] \arrow[d, "\iota_2"] & P_1' \arrow[r] \arrow[d, "\iota_1"] & P_0' \arrow[r, "\epsilon'"] \arrow[d, "\iota_0"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & P_2 \arrow[d, "\pi_2"] & P_1 \arrow[d, "\pi_1"] & P_0 \arrow[d, "\pi_0"] & A \arrow[d, "\pi_A"] & \\
{} \arrow[r, dotted] & P_2'' \arrow[r] \arrow[d] & P_1'' \arrow[r] \arrow[d] & P_0'' \arrow[r, "\epsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

noting that the columns are all exact because $\mathcal{A}$ is abelian.
We will now construct the chain maps for $P_\bullet$ by induction such that $\iota_\bullet$ becomes a chain map $P'_\bullet \to P_\bullet$ and $\pi_\bullet$ becomes a chain map $P_\bullet \to P''_\bullet$, and $P_\bullet$ is a projective resolution of A.
In fact we only have to guarantee the first two conditions because by the long exact sequence of homology groups of

$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$

any such chain maps would make the sequence exact.

At step zero we define $\epsilon : P_0 \to A$ by setting $\epsilon = \begin{bmatrix} f_0' & f_0'' \end{bmatrix}$, where $f_0' = \iota_A \circ \epsilon'$, and f_0'' is given by the projectiveness of P_0'', making the diagram

$$\begin{tikzcd}
 & A \arrow[d, "\pi_A"] \\
P_0'' \arrow[ur, dashed, "f_0''"] \arrow[r, "\epsilon''"'] & A''
\end{tikzcd}$$

commute (π_A is epi because the column is exact).
For $n \geq 1$, we write $d_n^P = \begin{bmatrix} f_n' & f_n'' \\ g_n' & g_n'' \end{bmatrix}$.
Considering the diagram

$$\begin{tikzcd}
P_n' \arrow[r, "d_n^{P'}"] \arrow[d, "\iota_n"'] & P_{n-1}' \arrow[d, "\iota_{n-1}"] \\
P_n \arrow[r, "d_n^P"] \arrow[d, "\pi_n"'] & P_{n-1} \arrow[d, "\pi_{n-1}"] \\
P_n'' \arrow[r, "d_n^{P''}"] & P_{n-1}''
\end{tikzcd}$$

we observe that the top square commutes if and only if $f_n' = d_n^{P'}$ and $g_n' = 0$. If this is the case then the bottom square commutes if and only if $g_n'' = d_n^{P''}$. This means that the diagram commutes if and only if $d_n^P = \begin{bmatrix} d_n^{P'} & f_n'' \\ 0 & d_n^{P''} \end{bmatrix}$ for some morphism f_n''.

We define the morphisms f_n'' by induction such that $d_n^P \circ d_{n-1}^P = 0$.

If $n = 1$ then we have $d_n^P \circ d_{n-1}^P = \begin{bmatrix} f_0' \circ d_1^{P'} & f_0' \circ f_1'' + f_0 \circ d_1^{P''} \end{bmatrix}$. Given that $f_0' = \iota_A \circ \epsilon'$ and that ϵ' and $d_1^{P'}$ and consecutive maps in a projective resolution, we need f_n'' such that

$$f_0' \circ f_n'' = -f_0'' \circ d_1^{P''}.$$

We notice that by definition of f''_0 we have

$$\begin{aligned}\pi_A \circ (-f''_0 \circ d_1^{P''}) &= -\pi_A \circ f''_0 \circ d_1^{P''} \\ &= -\epsilon'' \circ d_1^{P''} \\ &= 0\end{aligned}$$

so by exactness of the column in our very first diagram we have a diagram

$$\begin{tikzcd}
P''_1 \arrow[d, dashed, "f''_n"'] \arrow[dr, "-f''_0 \circ d_1^{P''}"] & \\
P'_0 \arrow[r, "f'_0 = \iota_A \circ \epsilon'"'] & \mathrm{im}(\iota)
\end{tikzcd}$$

that is completed by a morphism f''_n by projectiveness of P''_1, since the bottom morphism is epi because ϵ' is.

Suppose now that the morphisms f''_{n-1}, ..., f''_1 have all been constructed, for $n > 1$.

We have $d_n^P \circ d_{n-1}^P = \begin{bmatrix} d_{n-1}^{P'} \circ d_n^{P'} & d_{n-1}^{P'} \circ f''_n + f''_{n-1} \circ d_n^{P''} \\ 0 & d_{n-1}^{P''} \circ d_n^{P''} \end{bmatrix} = \begin{bmatrix} 0 & d_{n-1}^{P'} \circ f''_n + f''_{n-1} \circ d_n^{P''} \\ 0 & 0 \end{bmatrix}$, so we want to find f''_n such that

$$d_{n-1}^{P'} \circ f''_n = -f''_{n-1} \circ d_n^{P''}.$$

By the inductive argument we have that

$$\begin{aligned}d_{n-2}^{P'} \circ (-f''_{n-1} \circ d_n^{P''}) &= -(d_{n-2}^{P'} \circ f''_{n-1}) \circ d_n^{P''} \\ &= -(-f''_{n-2} \circ d_{n-1}^{P''}) \circ d_n^{P''} \\ &= f''_{n-2} \circ 0 \\ &= 0\end{aligned}$$

if $n > 2$, and if $n = 2$ we have

$$\begin{aligned}\iota_A \circ (d_0^{P'} \circ (-f''_1 \circ d_2^{P''})) &= -(\iota_A \circ d_0^{P'} \circ f''_1) \circ d_2^{P''} \\ &= -(-f''_0 \circ d_1^{P''}) \circ d_2^{P''} \\ &= f''_0 \circ 0 \\ &= 0\end{aligned}$$

which means $d_{n-2}^{P'} \circ (-f''_{n-1} \circ d_n^{P''}) = 0$ since ι_A is mono.

Therefore by projectiveness of P''_n we can complete the diagram

$$\begin{tikzcd}
P''_n \arrow[d, dashed, "f''_n"'] \arrow[dr, "-f''_{n-1} \circ d_n^{P''}"] & \\
P'_{n-1} \arrow[r, "d_{n-1}^{P'}"'] & \ker(d_{n-2}^{P'})
\end{tikzcd}$$

with a morphism f''_n such that it commutes.

This way we can define all the chain maps, which allows us to conclude our proof of the Horseshoe Lemma.

6.6 Week 6 (by Zichun Zhou).

(i) Use Baer's Criterion to show that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, and then give an injective resolution of $\mathbb{Z}$.

Baer's Criterion: An R-module M is injective if and only if for every ideal $J \subseteq R$, every R-morphism $J \to M$ can be lifted to a morphism $R \to M$.

Solution: By Baer's Criterion, it suffices to prove the above result for $R = \mathbb{Z}, J = (n)$ and $M = \mathbb{Q}/\mathbb{Z}$. Given $f : n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, one may construct a $\mathbb{Z}$-linear map $\bar{f} : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ by setting $\bar{f}(1) = \frac{1}{n} f(n)$. The linearity can be easily verified as $\mathbb{Z}$ is generated by 1 element and it is clear that $\bar{f} = f$ on $n\mathbb{Z}$. As a result, $\mathbb{Q}/\mathbb{Z}$ is injective and one can apply a similar argument to show that $\mathbb{Q}$ is also injective, hence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

an injective resolution of $\mathbb{Z}$. □

(ii) For A an abelian group, we define its *Pontrjagin dual* as : $A^* = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. Show that when A is finite, we have $(A^*)^* \cong A$, and deduce that there is an equivalence of categories $\mathbf{FAb} \cong \mathbf{FAb^{op}}$, where $\mathbf{FAb}$ is the category of finite abelian groups. However, find an abelian group such that $(A^*)^*$ is not isomorphic to A.

Solution: By classification of finite Abelian groups, we may write A as $A = \bigoplus_{i=1}^{n} \mathbb{Z}_{k_i}$. Since $\mathrm{Hom}_R(-, M)$ commutes with the finite direct sum for R commutative, $A^* = \bigoplus \mathbb{Z}_{k_i}^*$ and it suffices to prove $(\mathbb{Z}_n^*)^* = \mathbb{Z}_n$.
First, we show that $\mathbb{Z}_n^*$ can be viewed as a finite subgroup of $\mathbb{Q}/\mathbb{Z}$, identified with H_n defined as follows:

$$H_n := \{\frac{p}{q} \in \mathbb{Q}/\mathbb{Z} : q|n\}$$

Through direct computation, one can verify that this is indeed a subgroup with respect to addition. Now consider a linear map $\varphi : \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}$. φ is uniquely determined by $\varphi(1)$ and $n\varphi(1) = 0$, thus $\varphi(1) \in H_n$ and one can check that each element of H_n defines a such φ. As a result, we have an isomorphism $\varphi \leftrightarrow \varphi(1)$.
Next we show that $H_n^* = \mathbb{Z}_n$. For each $\frac{p}{q} \in H_n$, one may define an endomorphism

$$k \cdot (-) : k \cdot (\frac{p}{q}) = \frac{kp}{q} \in H_n \subseteq \mathbb{Q}/\mathbb{Z}$$

which is clearly equal to $k \cdot \mathrm{id}$, the sum of k identities in H_n. While $k \cdot (-) = (k \mod n) \cdot (-)$, we may define a homomorphism $F : \mathbb{Z}_n \to H_n^*$ by sending 1 to id.
To construct the inverse, for $\psi \in H^*$, consider a map $G : \psi \to n\psi(\frac{1}{n}) \mod n$. Notice that here $\psi(\frac{1}{n})$ is assumed to be a rational in $\mathbb{Q}$ in $[0, 1)$ instead of an element in $\mathbb{Q}/\mathbb{Z}$. G is well-defined as each equivalent class of $\mathbb{Q}/\mathbb{Z}$ contains exactly one element in $[0, 1)$ and $G(0) = 0$ trivially. For linearity of G, pick $\alpha, \beta \in H^*$, we have

$$G(\alpha + \beta) = n(\alpha + \beta(1/n)) = n\alpha(1/n) + n\beta(1/n) = G(\alpha) + G(\beta)$$

Hence G is well-defined. One can easily check that $G \circ F = \mathrm{id}_{\mathbb{Z}_n}$ and for $F \circ G$, notice that a map $\psi \in H^*$ is uniquely determined by $\psi(1/n)$ and through direct computation one can see that $\varphi(1/n) = F \circ G(1/n)$. This shows that $\mathbb{Z}_n \cong H^*$.

To show that $(-)^*$ defines an equivalence, notice that $\mathrm{Hom}(-, M)$ defines a contravariant functor, hence $(-)^*$ is a functor from $\mathbf{FAb^{op}}$ onto $\mathbf{FAb}$. However, it also defines a functor from $\mathbf{FAb}$ to $\mathbf{FAb^{op}}$: for a map $f : A \to B$, it is sent to $\mathrm{Hom}(f, \mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. By inverting the arrows, we get a map $\mathrm{Hom}(f, \mathbb{Q}/\mathbb{Z})^{\mathbf{op}} : \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z})$ in the opposite category. Thus $(-)^*$ defines an equivalence between $\mathbf{FAb}$ and $\mathbf{FAb}^{\mathbf{op}}$ and $(-)^*$ is equivalent to $(-)^{*\mathbf{op}}$ as $(A^{*\mathbf{op}})^* \cong A$, which also implies $\mathrm{Hom}(A, B) \cong \mathrm{Hom}((A^{*\mathbf{op}})^*, (B^{*\mathbf{op}})^*)$ on the category of abelian groups. The isomorphism between $((-)^*)^{*\mathbf{op}}$ and $\mathrm{id}_{\mathbf{FAb}^{\mathbf{op}}}$ can be proven analogously hence we are finished.

To give a counterexample of non-finite abelian groups such that $(A^*)^* \neq A$, we may consider $\mathbb{Z}$. It is clear that $\mathbb{Z}^* = \mathbb{Q}/\mathbb{Z}$, however $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \neq \mathbb{Z}$, since otherwise $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is generated by id as a $\mathbb{Z}$-module, yet there exists a $\mathbb{Z}$-linear map that sends $\frac{p}{q}$ to $\frac{p}{nq}$, for n an integer. This morphism is not generated by the identity since all maps generated by the identity takes the form $p/q \to n \cdot p/q$ ☐

(iii) Let $F : \mathbf{A} \to \mathbf{B}$ be a right exact functor and$U : \mathbf{B} \to \mathbf{C}$ be an exact functor.If $\mathbf{A}$ has enough projectives, show that we have a natural isomorphism :

$$L_i(UF) \cong U(L_i(F))$$

Solution: Let $X \in \mathbf{A}$ be an object and $P_\bullet \xrightarrow{f_\bullet} X$ be a projective resolution, then we first show that

$$U(L_i(FX)) = U(\ker Ff_i/\mathrm{im}\ Ff_{i+1}) = \ker UFf_i/\mathrm{im}\ UFf_{i+1} = L_i(UFX)$$

It suffices to show that $U(\ker Ff_i) \cong \ker UFf_i$ and $U(\mathrm{im}\ Ff_{i+1}) \cong \mathrm{im}\ UFf_{i+1}$, since then we have $U(\ker Ff_i/\mathrm{im}\ Ff_{i+1}) \cong \ker UFf_i/\mathrm{im}\ UFf_{i+1}$. As the proof is similar we will only prove the case of kernels. Let $f : Y \to Z$ be a morphism in $\mathbf{B}$. Consider the following rows of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & U\ker f \arrow[r] \arrow[d, "g"] & UY \arrow[r] \arrow[d, equal] & UZ \arrow[d, equal] \\
0 \arrow[r] & 0 \arrow[r] & \ker Uf \arrow[r] & UY \arrow[r] & UZ
\end{tikzcd}$$

The first row is exact by exactness of U and the second row is exact by definition of kernel. The map g is induced by universal property of kernels and one can verify that the diagram is commutative. By 5-lemma we have $\ker Uf = U\ker f$. Now we may assume that $f = f_i$ and replace Y, Z by FP_{i+1}, FP_i. This finishes the proof.

To show that there is a natural transformation between the two functors,given $P_\bullet \xrightarrow{a_\bullet} X, Q_\bullet \xrightarrow{b_\bullet} Y$ and a map $f : X \to Y$(notice that this induces a morphism between the projective resolutions),one needs to show that the following diagram is commutative:

$$\begin{tikzcd}
U(L_iFX) \arrow[r, "U(L_iFf)"] \arrow[d, "\tau_X"] & U(L_iFY) \arrow[d, "\tau_Y"] \\
L_i(UFX) \arrow[r, "L_i(UFf)"] & L_i(UFY)
\end{tikzcd}$$

Where τ_X, τ_Y are isomorphisms obtained previously.To do this one may consider the following diagram:

$$\begin{tikzcd}
 & U(L_nFX) \arrow[rd, "\tau_X"] \arrow[ddd, dashed, "UL_n(F\varphi)"] & \\
U\ker Ff_n \cong \ker UFf_n \arrow[ru, "q"] \arrow[rr, "i"] \arrow[d, "UF\varphi_n"] & & L_n(UFX) \arrow[d, "L_n(UF\varphi)"] \\
U\ker Fg_n \cong \ker UFg_n \arrow[rr, "j"] \arrow[rd, "p"] & & L_n(UFY) \\
 & U(L_nFY) \arrow[ru, "\tau_Y"] &
\end{tikzcd}$$

The map $UL_n(F\varphi)$ can be viewed as the natural map induced by $UF\varphi_n$ between the kernels and all sequares,triangles are commutative except the pair $(L_n(UF\varphi) \circ \tau_X, \tau_Y \circ U(L_nF\varphi))$. But since q(respectively q) is the natural quotient morphism, it is an epimorphism and to verify that $L_n(UF\varphi) \circ \tau_X = \tau_Y \circ U(L_nF\varphi)$, it suffices to verify that $L_n(UF\varphi) \circ \tau_X \circ q = \tau_Y \circ U(L_nF\varphi) \circ q$. This can be done through diagram chasing:

$$\begin{aligned} L_n(UF\varphi) \circ \tau_X \circ q &= L_n(UF\varphi) \circ i \\ &= j \circ UF\varphi_n \\ &= \tau_Y \circ p \circ UF\varphi_n \\ &= \tau_Y \circ U(L_nF\varphi) \circ q \end{aligned}$$

This finishes the proof. □

(iv) Let R be a commutative ring, and N an R-module. Show or recall that $- \otimes_R N : R - \mathbf{Mod} \to R - \mathbf{Mod}$ is right exact. We denote by $\mathrm{Tor}^i_R(-, N)$ the associated left derived functor. Find a projective resolution of $\mathbb{Z}/n\mathbb{Z}$, and use it to compute $\mathrm{Tor}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for every $i \geq 0$ and $m \in \mathbb{Z}$.

Solution: we show that $- \otimes N$ is right exact. Consider an exact sequence

$$0 \to M' \xrightarrow{i} M \xrightarrow{\pi} M'' \to 0$$

pass this sequence by the functor $- \otimes N$, one gets

$$0 \to M' \otimes N \xrightarrow{i \otimes N} M \otimes N \xrightarrow{\pi \otimes N} M'' \otimes N \to 0$$

To prove the exactness, one may use the following two lemmas:
Lemma 1: For M, N, P arbitrary R-modules, one has $\mathrm{Hom}(M \otimes N, P) = \mathrm{Hom}(M, \mathrm{Hom}(N, P))$.
Proof: One may construct a pair of isomorphisms between the two Hom modules as follows: for $\varphi : M \otimes N \to P$, there is an induced map $F(\varphi) : M \to \mathrm{Hom}(N, P)$,such that $F\varphi(m)(n) = \varphi(m \otimes n)$. Conversely, given a map $\psi : M \to \mathrm{Hom}(N, P)$, there is an induced map $G(\psi) : M \otimes N, P$ such that $G(\psi)(m \otimes n) = \psi(m)(n)$.One can easily check that the two are well-defined R-homomorphisms that are inverse to each other.□

Lemma 2: A sequence $M' \xrightarrow{f} M \xrightarrow{\pi} M'' \to 0$ is exact if and only if for all R-module N the following sequence is exact:

$$0 \to \mathrm{Hom}(M'', N) \xrightarrow{\bar{\pi}} \mathrm{Hom}(M, N) \xrightarrow{\bar{f}} \mathrm{Hom}(M', N)$$

Proof: $\Rightarrow$: The proof is trivial by definition of right exactness.
$\Leftarrow$: *Surjectivity of* π: Suppose that π is not surjective then let $N = M''/\mathrm{im}\ \pi$, and let q be the projection onto N, then clearly $0 \circ \pi = q \circ \pi$, hence $\bar{\pi}$ is not injective, giving us a contradiction;
$\ker \bar{f} = \mathrm{im}\ \bar{\pi}$: We have $\mathrm{im}\ \bar{\pi} \subseteq \ker \bar{f}$ and it suffices to check that $\ker \pi \subseteq \mathrm{im}\ \bar{f}$. To do this, first let $N = M/\mathrm{im}\ f$ and let φ be the natural projection, then $\varphi \circ f = 0 \implies \varphi \in \ker \bar{f} = \mathrm{im}\ \bar{\pi}$. Hence there is a $\psi : M'' \to M/\mathrm{im}\ f$, such that $\psi \circ \pi = \varphi$. As a result, one has $\ker \pi \subseteq \ker \psi = \mathrm{im}\ f$. □

By Lemma 2, to show the right exactness of tensor product, it suffices to show the exactness of the following sequence for every R-module P:

$$\mathrm{Hom}(M'' \otimes N, P) \to \mathrm{Hom}(M \otimes N, P) \to \mathrm{Hom}(M' \otimes N, P) \to 0$$

By Lemma 1, this sequence is equivalent to

$$\mathrm{Hom}(M'', \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M, \mathrm{Hom})(N, P) \to \text{Hom(M',Hom(N,P))} \to 0$$

Yet the last sequence is naturally exact since $\mathrm{Hom}(-, \mathrm{Hom}(N, P))$ is a left exact functor.

For the following part the tensor product $-\otimes-$ is always tensor product of $\mathbb{Z}$-modules unless specified. Consider the following projective resolution of $\mathbb{Z}_n$:

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$$

Pass it by $-\otimes \mathbb{Z}_m$, one gets

$$0 \to \mathbb{Z}_m \xrightarrow{n} \mathbb{Z}_m \to \mathbb{Z}_n \otimes \mathbb{Z}_m \to 0$$

Suppose that m, n are not coprime. Using $M/IM = M \otimes R/I$, one can show that $\mathbb{Z}_n \otimes \mathbb{Z}_m = \mathbb{Z}_n/m\mathbb{Z}_n = \mathbb{Z}/\gcd(m,n)\mathbb{Z}$. As a result, for $i \geq 2$, $\mathrm{Tor}_i(\mathbb{Z}_n, \mathbb{Z}_m) = 0$; By right exactness we have $\mathrm{Tor}_0(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_{\gcd(n,m)}$. Moreover $\mathrm{Tor}_1(\mathbb{Z}_n, \mathbb{Z}_m) = \ker(\cdot n) \otimes \mathbb{Z}_m$ More explicitly, we may write this kernel as $\{x \in \mathbb{Z}_m : nx = 0 \mod m\} = (\mathrm{scm}(m,n)/n)\mathbb{Z}_m = (m/\gcd(m,n))\mathbb{Z}_m \cong \mathbb{Z}/(\gcd(m,n))\mathbb{Z}$.

For m, n coprime, since $\gcd(m,n) = 1$, we have $m\mathbb{Z}_n = \mathbb{Z}_n$, hence $\mathbb{Z}_n \otimes \mathbb{Z}_m = 0$ and the multiplication by n is invertible on $\mathbb{Z}_m$. Thus $\cdot n$ is an isomorphism and $\mathrm{Tor}^i(\mathbb{Z}_n, \mathbb{Z}_m) = 0, \forall i \geq 0$ ☐

6.7 Week 7 (by Virgile Constantin).

(i)

- We will prove $(a) \implies (b) \implies (c) \implies (a)$.

$(a) \implies (b)$: Let A be a left R-module and $P_\bullet \to A$ a projective resolution of A. By definition the complex

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

is exact, so by flatness of B the tensored complex

$$(P_\bullet \otimes B) = \cdots \to P_2 \otimes B \to P_1 \otimes B \to P_0 \otimes B$$

is exact as well. It follows that the homology groups of this complex are zero and so $\mathrm{Tor}_n^R(A,B) = H_n(P_\bullet \otimes B) = 0$ for every $n > 0$.

$(b) \implies (c)$: immediate.

$(c) \implies (a)$: Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules and consider its associated long exact sequence ($\mathrm{Tor}_\bullet(-,B)$ is a homological δ-functor)

$$\cdots \to \mathrm{Tor}_1(M_3, B) \to M_1 \otimes B \to M_2 \otimes B \to M_3 \otimes B \to 0$$

where we used that $\mathrm{Tor}_0(A,B) = A \otimes B$ for all A. Since $\mathrm{Tor}_1(M_3,B) = 0$ by hypothesis, the functor $-\otimes B$ is exact, as desired.

- Using that $\mathrm{Tor}_\bullet(M,-) \cong L_\bullet(M\otimes -)(-)$ and the long exact sequence for this homological δ-functor associated to the mentioned short exact sequence, we obtain directly from the first part of the exercise that $0 \to \mathrm{Tor}_1(M,B) \to 0$ is an exact sequence for all R-module M, i.e. $\mathrm{Tor}_1(M,B) = 0$ for all M. We conclude by the first part.

(ii) Let $0 \to M \to P \to A \to 0$ be a short exact sequence where P is $\mathcal{F}$-acyclic.

(a) Using the long exact sequence for $L_\bullet \mathcal{F}$ associated to the above short exact sequence, using that $L_n\mathcal{F}(P) = 0$ for all $n > 0$ and that $L_0\mathcal{F}(B) = \mathcal{F}(B)$, yields directly the statement.

(b) Consider the exact sequence

$$0 \to M_m \xrightarrow{f_{m+1}} P_m \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \xrightarrow{f_{-1}} 0 \tag{16}$$

and split it into short exact sequences

$$0 \to K_j \to P_j \xrightarrow{f_j} K_{j-1} \to 0 \tag{17}$$

for all $0 \leq j \leq m$, where $K_j = \ker f_j = \mathrm{im} f_{j+1}$. Note that $K_{-1} = A$ and $K_m = M_m$. Using this and the previous point, we obtain by induction that

$$\begin{aligned} L_i\mathcal{F}(A) &\cong L_{i-0-1}\mathcal{F}(K_0) \cong L_{i-1-1}\mathcal{F}(K_1) \cong \dots \\ &\cong L_{i-m-1}\mathcal{F}(K_m) = L_{i-m-1}\mathcal{F}(M_m) \end{aligned}$$

for $i \geq m+2$, as desired. When $i = m+1$ we use the same sequence of isomorphisms, but use the previous point in the last step to get:

$$L_{m+1}\mathcal{F}(A) \cong \dots \cong L_1\mathcal{F}(K_{m-1}) = \ker(\mathcal{F}(M_m) \to \mathcal{F}(P_m)).$$

(c) Let $P_\bullet \to A$ be an $\mathcal{F}$-acyclic resolution of A. Split this exact sequence in short exact sequences of the form (17) for all $j \geq 0$ (with the same notations). For $i \geq 1$, let $m = i-2$ and set $M_m = M_{i-2} = K_{i-2} = \ker f_{i-2}$ and consider the exact sequence (16). Applying the previous point yields that

$$L_i\mathcal{F}(A) \cong L_1\mathcal{F}(K_{i-2}) = \ker(\mathcal{F}K_{i-1} \to \mathcal{F}P_{i-1}) \tag{18}$$

Since $\mathcal{F}$ is right exact it preserves cokernels, hence we obtain that

$$\begin{aligned} \mathcal{F}K_{i-1} &= \mathcal{F}(\mathrm{im} f_i) \cong \mathcal{F}(P_i/\ker f_i) = \mathcal{F}(P_i/\mathrm{im} f_{i+1}) \\ &= \mathcal{F}(\mathrm{coker} f_{i+1}) \cong \mathrm{coker}(\mathcal{F}f_{i+1}) \end{aligned}$$

Continuing the sequence (18) of isomorphisms, it follows that

$$L_i\mathcal{F}(A) \cong \ker\left(\mathcal{F}P_i/\mathrm{im}(\mathcal{F}f_{i+1}) \to \mathcal{F}P_{i-1}\right) \cong H_i(\mathcal{F}P_\bullet)$$

as desired.

(iii) We use induction to prove that $\mathrm{Tor}_n(A, B) \cong H_n(F \otimes B)$ for all n. For $n = 0$, we have that $F_1 \otimes B \to F_0 \otimes B \to A \otimes B \to 0$ is exact since $\otimes B$ is right exact. Therefore we have that

$$\mathrm{Tor}_0(A, B) = A \otimes B \cong (F_0 \otimes B)/Im(F_1 \otimes B) = H_0(F \otimes B)$$

For $n = 1$, let $K_0 := Ker(F_0 \to A)$ and consider the following exact sequences

$$0 \to K_0 \to F_0 \to A \to 0 \tag{19}$$

$$F_2 \to F_1 \to K_0 \to 0 \tag{20}$$

Using the long exact sequence for $L_\bullet(- \otimes B)$ on the short exact sequence (19) and using that $\mathrm{Tor}_1(F_0, B) = 0$ by the first exercise, we have that

$$\mathrm{Tor}_1(A, B) \cong Ker(K_0 \otimes B \to F_0 \otimes B).$$

Using that the image of (20) under the functor $- \otimes B$ is still exact, the right hand side of the above equation is

$$Ker(F_1 \otimes B/Im(F_2 \otimes B) \to F_0 \otimes B) = H_1(F_\bullet \otimes B)$$

as desired. For $n \geq 2$ notice that $F_{\bullet+1} \to K_0$ is a flat resolution of K_0. Hence by induction and using the dimension shifting exercise on (19) (this makes sense since F_0 is flat, thus $(- \otimes B)$-acyclic by the first exercise) we find that

$$\begin{aligned} \mathrm{Tor}_n(A, B) &= L_n(- \otimes_R B)(A) \cong L_{n-1}(- \otimes_R B)(K_0) \\ &= \mathrm{Tor}_{n-1}(K_0, B) = H_{n-1}(F_{\bullet+1} \otimes B) \\ &= H_n(F_\bullet \otimes B) \end{aligned}$$

(iv) (a) Since $C_{p,q} = 0$ whenever $q < 0$ we have isomorphisms at each level

$$\mathrm{Tot}^{\prod}(C)_n = \prod_{q \geq 0} C_{n-q,q} \cong \prod_{q \geq 0} \mathbb{Z}/4\mathbb{Z}$$

Through these isomorphisms, the differentials at each level are all the same and are given by

$$\left(a_q\right)_{q\geq 0} \mapsto \left(2a_q + 2a_{q+1}\right)_{q \geq 0}.$$

Let us show that $im(d) = \prod_{q\geq 0} 2\mathbb{Z}/4\mathbb{Z}$. Clearly the left hand side is included in the right hand side. Now let $(a_q)_{q\geq 0} \in \prod_{q\geq 0} 2\mathbb{Z}/4\mathbb{Z}$. We can build $(b_q)_{q\geq 0}$ inductively with ones and zeros by setting $b_0 = \frac{a_0}{2}$ and

$$b_q = \begin{cases} 0 & \text{if } a_{q-1} = b_{q-1} = 0 \text{ or } (a_{q-1} = 2 \wedge b_{q-1} = 1); \\ 1 & \text{if } (a_{q-1} = 0 \wedge b_{q-1} = 1) \text{ or } (a_{q-1} = 2 \wedge b_{q-1} = 0) \end{cases}$$

This element is mapped to $(a_q)_{q\geq 0}$ by construction which shows the hint.
Moreover we can easily observe that $(a_q)_{q\geq 0}$ is a cycle if and only if it is a boundary ($a_q \in \{0, 2\}$ for all q) or if it is of the form $a_q \in \{1, 3\}$ for all q. Since any two cycles that both fall in the second case differ by a boundary (an element of $\prod 2\mathbb{Z}/4\mathbb{Z}$), we obtain that the quotient $\ker(d)/im(d) = H_0(Tot^{\prod}(C))$ has two classes (generated by a boundary and by $(1, 1, \ldots)$ for example), i.e. it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(b) Since the rows of C are exact, the acyclic assembly lemma shows that that $Tot^{\oplus}(C)$ is acyclic. Another way of seeing this directly is by observing that the cycles in $Tot^{\prod}(C)$ are either boundaries, either sequences of elements in $\{1, 3\}$. Since the latter are not elements of the direct product total complex $Tor^{\oplus}(C)$, the boundaries are equal to the cycles.

(c) The same formulas hold in this case, except that the products are indexed by $q \in \mathbb{Z}$ instead of positive integers. Again the element $x = (\ldots, 1, 1, 1, \ldots)$ is a cycle, but can't be a boundary since those are precisely $\prod_{\mathbb{Z}} 2\mathbb{Z}/4\mathbb{Z}$. Hence the total complex of D is not acyclic. Moreover there is a chain map $Tot^{\prod}(D) \to Tot^{\prod}(C)$ given at each level by $(a_q)_{q\in\mathbb{Z}} \mapsto (a_q)_{q\geq 0}$. It induces a surjective map on the 0-th homology groups

$$H_0(Tot^{\prod}(D)) \to H_0(Tot^{\prod}(C)) \cong \mathbb{Z}/2\mathbb{Z}.$$

Indeed, the class of the cycle $(1)_{q\in\mathbb{Z}}$ is mapped to the class of the cycle $(1)_{q\geq 0}$, which corresponds to the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$.
Lastly, the element $(a_q)_{q\in\mathbb{Z}}$ defined by $a_q = 0$ for all $q \neq 0$ and $a_0 = 2$ is a cycle, but can't be a boundary. The only preimage of such an element in $Tot^{\prod}(D)$ are of the form $(\ldots, b_3, b_2, b_1, 0, 0, 0, \ldots)$ or $(\ldots, 0, 0, 0, b_0, b_{-1}.b_{-2}, \ldots)$ for $b_i \in \{1, 3\}$. Those possible elements are not in $Tot^{\oplus}(D)$, so (a_q) is not a boundary and the homology groups are non trivial.

6.8 Week 8 (by Runchi Tan).

(1)(a)

Proof. Through out the exercise, we use $d^p_{x,y}$ to denote a horizontal/vertical differential leaving $E^p_{x,y}$ on page p. And we'll apply Freyd-Mitchell Embedding, making each $E_{\bullet\bullet}$ an R-module to perform an element chasing argument. On page 0 we have the following diagram

$$\begin{tikzcd}
E^0_{0,2} \arrow[d] & E^0_{1,2} \arrow[l] \arrow[d] & & E^1_{0,2} \arrow[d] & E^1_{1,2} \arrow[l] \arrow[d] \\
E^0_{0,1} \arrow[d, "d^{0,v}_{0,1}"] & E^0_{1,1} \arrow[l, "d^{0,h}_{1,1}"'] \arrow[d, "d^{0,v}_{1,1}"] & & E^1_{0,1} = \operatorname{Ker} d^{0,v}_{0,1} / \operatorname{Im} d^{0,v}_{0,2} \arrow[d] & E^1_{1,1} \arrow[l] \arrow[d] \\
E^0_{0,0} & E^0_{1,0} \arrow[l, "d^{0,h}_{1,0}"'] & & E^1_{0,0} & E^1_{1,0} \arrow[l]
\end{tikzcd}$$

For example, pick an element $\alpha := \alpha^2_{x,y} \in E^2_{x,y}$, we can express it as

$$\alpha = \alpha^1_{x,y} + \operatorname{Im} d^{1,h}_{x+1,y}$$

where $\alpha^1_{x,y} \in \operatorname{Ker}(d^{1,h}_{x,y})$. Recall that d^h is induced taking homology as $d^h = H_\bullet(d^{0,h}_{x,y})$, then

$$0 = d^{1,h}_{x,y}(\alpha^1_{x,y}) \ \Rightarrow \ d^{0,h}_{x,y}(\alpha^0_{x,y}) \in \operatorname{Im} d^{0,v}_{x,y+1}$$

where $\alpha^1_{x,y} = \alpha^0_{x,y} + \operatorname{Im} d^{0,v}_{x,y+1}$ for $\alpha^1_{x,y} \in E^0_{x,y}$. While $d^{0,h}_{x,y}(\alpha^0_{x,y}) \subset E^0_{x-1,y+1}$, we can find some element $\beta^0_{x-1,y+1}$ such that

$$d^{0,v}_{x-1,y+1}(\beta^0_{x-1,y+1}) = d^{0,h}_{x,y}(\alpha^0_{x,y})$$

We can define $a := \alpha^0_{x,y}$ and $b := \beta^0_{x-1,y+1}$. They satisfy the equation by construction. In summary, we've defined a map

$$\psi : E^2_{x,y} \to \{(a,b) \in E_{x-1,y+1} \times E_{x,y} \mid ...\}$$

$$\begin{tikzcd}
a \in E^0_{x-1,y+1} \arrow[d, "d^{0,v}_{x-1,y+1}"'] & E^0_{x,y+1} \arrow[l] \arrow[d] \\
E^0_{x-1,y} & E^0_{x,y} \ni b \arrow[l, "d^{0,h}_{x,y}"]
\end{tikzcd}$$

Notation: the three requirement given in the prompt (a) as $\sim_1, \sim_2, \sim_3$.
We wish to construct a map from

$$\varphi : \{(a,b) \in E_{p-1,q+1} \times E_{p,q} \mid ...\}/\langle \sim_1,\ \sim_2,\ \sim_3 \rangle \to E^2_{p,q}.$$

We proceed by picking up an element (a,b) such that $0 = d^v b = d^h b + d^v a$. More precisely, since $d^v b = d^{0,v}_{x,y}(b) = 0$, then $b \in \operatorname{Ker} d^v$ and $b^1_{x,y} := b + \operatorname{Im} d^{0,v}_{x,y+1} \in E^1_{x,y}$. And we notice that $b^1_{x,y} \in \operatorname{Ker} d^{1,h}_{x,y}$, because

$$d^{0,h}_{x,y}(b) = -d^{0,v}_{x-1,y+1}(a) \in \operatorname{Im}(d^{0,v}_{x-1,y+1})$$
$$\Rightarrow \ d^{1,h}_{x,y}(b^1_{x,y}) = d^{1,h}_{x,y}(b + \operatorname{Im} d^{0,v}_{x,y+1}) = 0 \in E^1_{x-1,y}.$$

Hence $b^1_{x,y} \in \operatorname{Ker}(d^{1,h}_{x,y})$, then $b^1_{x,y} + \operatorname{Im}(d^{1,h}_{x+1,y}) \in E^2_{x,y}$. Hence we can simply define φ by

$$E^0_{x-1,y+1} \times E^0_{x,y} \ni (a,b) \mapsto (b + \operatorname{Im}(d^{0,v}_{x,y+1})) + \operatorname{Im}(d^{1,h}_{x+1,y}) \in E^2_{x,y}.$$

According to First Isomorphism Theorem, we've constructed a map φ. It's surjective by the first part of the proof. It remains to compute the kernel. Through explicity computations, we can show elements in $\sim_1, \sim_2, \sim_3$ will be 0 for each element will either lie in $\operatorname{Im}(d^{0,v}_{x,y+1})$ or $\operatorname{Im}(d^{1,h}_{x+1,y})$. Hence φ induces a map $\bar{\varphi}$ mapping

$$(\varphi : \{(a,b) \in E_{p-1,q+1} \times E_{p,q} \mid ...\}/ \sim_1, \sim_2, \sim_3) \to E^2_{p,q}.$$

Denote π as the natural projection from $\varphi : \{(a,b) \in E_{p-1,q+1} \times E_{p,q} \mid ...\} \to (\varphi : \{(a,b) \in E_{p-1,q+1} \times E_{p,q} \mid ...\}/ \sim_1, \sim_2, \sim_3)$, then we can check

$$\varphi\pi\psi = \mathrm{id},\ \pi\psi\varphi = \mathrm{id}\,.$$

And this finishes the proof of identification. □

(b)

Proof. Here I assume the pair $(a,b) \in E^0_{p-1,q+1} \times E^0_{p,q}$ has properties given by previous part. Since $E_{\bullet\bullet}$ is in first quadrant, then on degree $n := p + q$ we have $\operatorname{Tot}(E_{\bullet\bullet})$ is a finite direct sum

$$\bigoplus_{x+y=p+q} E_{x,y}.$$

$$\begin{tikzcd}
E^0_{a-3,b+2} \arrow[d] & E^0_{a-2,b+2} \ni e_{a-2,b+2} \arrow[l, "d^{0,h}_{a-2,b+2}"'] \arrow[d, "d^{0,v}_{a-2,b+2}"] \\
E^0_{a-3,b+1} & E^0_{a-2,b+1} \arrow[l]
\end{tikzcd}$$

On $E_{a-1,b+1}$ we pick a, on $E_{a,b}$ we pick b. Because $d^h(a) = 0$, then for $E_{a-2,b+2}$ we can pick an element in $e_{p-2,q+2} \in \mathrm{Ker}(d^{0,v}_{a-2,b+2}) \cap \mathrm{Ker}(d^{0,h}_{a2,b+2})$. Inductively, we can choose a sequence of elements in $E^0_{x,y}$ such that $x + y = p + q$ and $x \leq a - 1$. Similarly we can choose some element in another direction where $x \geq a$ and $x + y = p + q$. Denote this sequence of elements as $l = (e_{0,p+q}, ..., a, b, ..., e_{p+q,0})$, by construction we have

$$d^{\mathrm{Tot}}_n(l) = 0(e_{0,p+q}), \cdots, d^v(a) + d^h(b), ..., 0(e_{p+q,0}) = 0 \in \bigoplus_{x+y=p+q} E^0_{x,y}.$$

So $l + \mathrm{Im}(d^{\mathrm{Tot}}_{n-1}) \in H_{p+q}(T)$, and then (a, b) can determine an element of $H_{p+q}(T)$. ☐

(c)

Proof. We can check $(0, d^h(a)) \in E^2_{p-2,q+1}$ by verifying that

$$\begin{aligned} d^v(0) + d^h d^h(a) &= 0, \\ d^v(d^h(a)) = &- d^h d^v(b) = -d^h(0) = 0. \end{aligned}$$

It remains to check that all elements given by $\sim_1, \sim_2, \sim_3$ will be mapped to 0 under d defined in prompt.

- For $(a, 0)$, notice that $d^v a + d^h 0 = 0$ implies $d^v a = 0$ while $a \in E_{p-1,q+1}$, then $\sim_3$ implies that $(0, d^h(a)) = 0 \in E^2_{p-2,q+1}$.
- For $(d^h x, d^v x)$ where $x \in E_{p,q+1}$. Recall that vertical chains form complex and the square is anti-commutative, then we have the following
$$d^v(d^v(x)) = 0, \ d^v d^h(x) + d^h d^v(x) = 0.$$
- For $(0, d^h c)$ where $d^v c = 0$ for $c \in E_{p+1,q}$, we have $d(0, d^h c) = (0, 0)$.

Therefore the formula is well-defined.

☐

(2)

Proof. As usual we'll apply Freyd-Mitchell to enable us to interpret $E^{\bullet}_{\bullet\bullet}$ as some object in the category of R-module so that we can perform diagram-chasing arguments. By the result of the first exercise, we can identify an element of $E^2_{0,0}$ as a pair of element

$$(a, b) \in E_{-1,1} \times E_{0,0}$$

such that $a = 0$ and $d^v(b) = d^h(b) = 0$. Therefore, the element $(0, b) \in \mathrm{Ker}(d^{T_{\bullet,\bullet}}_0)$. Any two representative of the element, $(0, b_1) = (0, b_2)$, will differ by three equivalent relations. We can easily check their difference will be mapped to $0 \in H_0(T_{\bullet,\bullet})$. Therefore we've obtained a well-defined map

$$E^2_{0,0} \to H_0(T).$$

Conversely, an element of $H_0(T)$ can be expressed by $(a) + \mathrm{Im}\, d^{T_{\bullet,\bullet}}_1$ where $a \in E_{0,0}$ is arbitrary. This is because we're working in the first quadrant and $\mathrm{Ker} d^{T_{\bullet,\bullet}}_0 = E_{0,0}$. And we can define a map conversely by

$$(a) + \mathrm{Im}\, d^{T_{\bullet,\bullet}}_1 \mapsto (0, a) \in E^2_{0,0}.$$

We'll verify well-definedness: assume $a_1 - a_2 \in \operatorname{Im} d_1^{T_{\bullet,\bullet}}$, then $(0, a_1) - (0, a_2) = (0, a_1 - a_2) = d^v(x_1) + d^h(x_2)$. We can express $a_1 - a_2$ as a sum of $d^v(a_3) + d^h(a_4)$ where $a_3 \in E_{0,1}$ and $a_4 \in E_{1,0}$. Then apply equivalence relations $\sim_3, \sim_4$ in first exercise we know $(0, a_1 - a_2)$ will be mapped to 0 and hence the map is well-defined. We can check either compositions yield identity, therefore

$$E^2_{0,0} = H_0(T).$$

We can obtain the second result by applying Exercise 3 below. $\square$

(3)

Proof.

$$\begin{tikzcd}
\bullet & E^2_{0,1} & E^2_{1,1} & \bullet & \\
\bullet & E^2_{0,0} & E^2_{1,0} & \bullet & \\
& & & & 0 \arrow[uulll, "d^3_{3,-1}"]
\end{tikzcd}$$

Since on page 2, we only have two nontrivial rows, then:

- differential on page 3 that lands in $E^3_{a,b}$ is $d^3_{a+3,b-2} : 0 \to E^3_{a,b}$.
- differential on page 3 that leaves from $E_{a,b}$ is $d^3_{a,b} : E_{a,b} \to 0$.

And this forces $E^4_{a,b} = \mathrm{Ker}(d^3_{\bullet,\bullet})/\mathrm{Image}(d^3_{\bullet+3,\bullet-2}) = E^3_{\bullet,\bullet}$. Therefore the sequence stablize for any page $r \geq 5$. Assumption states that (bounded) spetral sequence converges to H_*, therefore we have a finite filtration

$$0 = F_s H_n \subset ... \subset F_{p-1} H_n \subset F_p H_n \subset ... \subset F_t H_n = H_n$$

such that the following holds:
consider H_p where $a + b = p$ and $b \neq 0, 1$. We have $0 = E^2_{a,b} = F_a H_p / F_{a-1} H_p$. Therefore $F_{a-1} H_p = F_a H_p$ for all $a \neq p-1, p$. While the filtration is of finite length,

$$0 \subset F_{p-1} H_p \subset F_p H_p = H_p.$$

In summary, we have $E^3_{p,0} = H_p / F_{p-1} H_p$ and $E^3_{p-1,1} = F_{p-1} H_p$. So we have a short exact sequence $0 \to E^3_{p-1,1} \to H_p \to E^3_{p,0} \to 0$. We'll define the maps $E^2_{p-1,1} \to H_p$ and $H_p \to E^2_{p,0}$ as the composition of the triangle above and below them respectively.

$$\begin{tikzcd}
& E^3_{p-1,1} = E^2_{p-1,1}/\operatorname{Im}(d^2_{\bullet,\bullet}) \arrow[dr, hook] & & & \\
E^2_{p-1,1} \arrow[ur, two heads] \arrow[rr, dashed] & & H_p \arrow[rr, dashed] \arrow[dr, two heads] & & E^2_{p,0} \\
& & & E^3_{p,0} = \mathrm{Ker}(d^2_{\bullet,\bullet})/0 \arrow[ur, hook] &
\end{tikzcd}$$

And we can verify that

$$\begin{gathered}
\mathrm{Image}(E^2_{p-1,1} \to H_p) = E^3_{p-1,1} \\
= \mathrm{Ker}(H_p \to E^3_{p,0}) = \mathrm{Ker}(H_p \to E^2_{p,0}).
\end{gathered}$$

Hence the map is exact at H_p.

$$\begin{tikzcd}
 & E^3_{p,0} = \operatorname{Ker}(d^2_{p,0})/0 \arrow[d, hook] & & \\
H_p \arrow[ur, two heads] \arrow[r, dashed] & E^2_{p,0} \arrow[r, "d^2_{p,0}"] & E^2_{p-2,1} \arrow[r, dashed] \arrow[d, two heads] & H_{p-1} \\
 & & E^2_{p-2,1}/\operatorname{Im}(d^2_{p,0}) = E^3_{p-2,1} = F_{p-2}H_{p-1} \arrow[ur, hook] &
\end{tikzcd}$$

Again we can obtain similar results for filtration of H_{p-1}. And we can verify that

$$\begin{aligned}
&\mathrm{Image}(H_p \to E^2_{p,0}) = E^3_{p,0} = \mathrm{Ker}(d^2_{p,0})/\mathrm{Image}(d^2_{p+2,-1}) = \mathrm{Ker}(d^2_{p,0})/0 = \mathrm{Ker}(d^2_{p,0}),\\
&\mathrm{Image}(d^2_{p,0}) = \mathrm{Ker}(E^2_{p-2,1} \to E^2_{p-2,1}/\mathrm{Image}(d^2_{p,0})) = \mathrm{Ker}(E^2_{p-2,1} \to H_{p-1}).
\end{aligned}$$

Hence we've check exactness at $E^2_{p,0}, E^2_{p-2,1}$. By induction we can checked exactness at any spot, therefore we've proved the statement as desired.

☐

(4)

Proof.

$$\begin{tikzcd}
\operatorname{Im} d^r_{p+r,q-r+1} \arrow[r] \arrow[d, "d^r_{p+r,q-r+1}|_\bullet{,}\simeq"] & \operatorname{Ker}(d^r_{p,q}) \arrow[r] \arrow[d, "d^r_{p,q}|_\bullet{,}\simeq"] & E^{r+1}_{p,q} \arrow[r] \arrow[d, "f^r"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\operatorname{Im} d'^r_{p+r,q-r+1} \arrow[r] & \operatorname{Ker}(d'^r_{p,q}) \arrow[r] & (E')^{r+1}_{p,q} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Here the isomorphism of first two columns implies the isomorphism of the third column. And apply induction gives us $E^\infty_{p,q} \simeq (E')^\infty_{p,q}$. ☐

6.9 Week 9 (by Benoît Cuenot).

(i) Consider the first quadrant double complex C given by

$$\begin{tikzcd}
0 & C' \arrow[l] \arrow[d, "h"] & B' \arrow[l] \arrow[d, "g"] & A' \arrow[l, "\alpha"'] \arrow[d, "f"] & K' \arrow[l] \arrow[d] \\
K \arrow[from=u] & C \arrow[l] & B \arrow[l, "\beta"'] & A \arrow[l] & 0 \arrow[l]
\end{tikzcd},$$

where the rows are exact, and $K = \mathrm{coker}(\beta), K' = \ker(\alpha)$. We know that there exists a spectral sequence whose 0^{th} page is given by the above diagram (with vertical differentials), and who is converging to the homology of the total complex $H_*(T)$. Notice that since C has exact rows, the total complex of C is acyclic by the Acyclic Assembly Lemma. We now compute the pages $E^1_{\bullet,\bullet}, E^2_{\bullet,\bullet}$ of our spectral sequence. The first page is given by:

$$\begin{tikzcd}
0 & \ker(h) \arrow[l] & \ker(g) \arrow[l] & \ker(f) \arrow[l] & K' \arrow[l] \\
K & \mathrm{coker}(h) \arrow[l] & \mathrm{coker}(g) \arrow[l] & \mathrm{coker}(f) \arrow[l] & 0 \arrow[l]
\end{tikzcd}.$$

Taking homology, we get the second page :

$$\begin{tikzcd}
0 & E^2_{1,1} & E^2_{1,2} & E^2_{1,3} & E^2_{1,4} \\
E^2_{0,0} & E^2_{1,0} & E^2_{2,0} \arrow[ull] & E^2_{3,0} \arrow[ull] & 0 \arrow[ull]
\end{tikzcd}.$$

Now notice that except for $E^2_{1,1}$ and $E^2_{3,0}$, the differentials leaving and arriving at $E^2_{p,q}$ are 0, and this will remain the same for every page after. Since the sequence converges to 0, this means that $E^2_{p,q} = E^\infty_{p,q} = 0$ for (p, q) not $(1, 1)$ or $(3, 0)$. But this implies that the sequences

$$\begin{tikzcd}
\ker(h) & \ker(g) \arrow[l] & \ker(f) \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\operatorname{coker}(h) & \operatorname{coker}(g) \arrow[l] & \operatorname{coker}(f) \arrow[l]
\end{tikzcd}$$

are exact.

We now investigate the map $E^2_{3,0} \to E^2_{1,2}$. Its kernel (resp. cokernel) is given by the object $E^3_{3,0}$ (resp. $E^3_{1,2}$) of page 3. However, we can argue as before that these are the infinity terms, as no non zero differentials are ever going to reach or leave them in the later pages. This implies that $E^2_{3,0} \to E^2_{1,2}$ is an isomorphism. Denote by $\sigma : \ker(g) \to \ker(h)$ and $\tau : \ker(f) \to \ker(g)$ the maps obtained before as differentials of the first page. We have that $\operatorname{coker}(\sigma) \cong E^2_{3,0} \cong E^2_{1,2} \cong \ker(\tau)$, so we can glue our two sequences into the following exact sequence :

$$\begin{tikzcd}
\operatorname{coker}(\sigma) \arrow[drrr, "\varphi"] & \ker(h) \arrow[l, "\eta"'] & \ker(g) \arrow[l, "\sigma"'] & \ker(f) \arrow[l] \\
\operatorname{coker}(h) & \operatorname{coker}(g) \arrow[l] & \operatorname{coker}(f) \arrow[l, "\tau"] & \ker\tau \arrow[l, "\epsilon"]
\end{tikzcd}.$$

Denote by δ the compositum $\epsilon \circ \varphi \circ \eta$. Note that $\ker(\delta) \cong \ker(\eta)$, as ϵ, φ are monic (for example using Freyd Mitchel's embedding theorem), so that $\ker(\delta) = \operatorname{coker}(\sigma)$ by exactness. Similarly, we have that $\operatorname{coker}(\delta) \cong \operatorname{coker}(\epsilon)$ as η, φ are epic, which in turn implies $\operatorname{coker}(\delta) \cong \ker(\tau)$. It follows that the sequence

$$\begin{tikzcd}
\ker(h) \arrow[drr, "\delta"] & \ker(g) \arrow[l] & \ker(f) \arrow[l] \\
\operatorname{coker}(h) & \operatorname{coker}(g) \arrow[l] & \operatorname{coker}(f) \arrow[l]
\end{tikzcd}$$

is exact, which is exactly the Snake Lemma.

(ii) Let $\{E^r_{pq}\}_{p,q} \in \mathbb{Z}$ be a regular upper half-plane sequence with $E^\infty_{pq} \cong \mathbb{Z}/2\mathbb{Z}$ for any p, q with $q \geq 0$. Let H_* be the family of objects defined by $H_n = \mathbb{Z}$. Note that for $p \in \mathbb{Z}$ and $F_p = 2^p\mathbb{Z}$, we have a filtration

$$... \subseteq F_1H_n \subseteq F_0H_n = H_n,$$

which satisfies $F_pH_n/F_{p+1}H_n = 2^p\mathbb{Z}/2^{p+1}\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \cong E^\infty_{pq}$ for any p, q with $q \geq 0$. In other words, $\{E^r_{pq}\}$ weakly converges to H_*. Moreover, we see that this filtration is both Hausdorff and exhaustive

$$\bigcup_p F_pH_n = \bigcup_{p\geq 0} 2^p\mathbb{Z} = \mathbb{Z}, \quad \bigcap_p F_pH_n = \bigcap_{p\geq 0} 2^p\mathbb{Z} = 0,$$

so that $\{E^r_{pq}\}$ approaches H_*.
Now we set $H'_n = \mathbb{Z}_2$, where $\mathbb{Z}_2$ is by definition $\varprojlim \mathbb{Z}/2^p\mathbb{Z}$. Letting $F_p = 2^p\mathbb{Z}_2$ for $p \geq 0$, we find as before that $\{E^r_{pq}\}$ approaches H'_*. Now it is a classical computation in number theory that we have an isomorphism $\mathbb{Z}_2/2^p\mathbb{Z}_2 \cong \mathbb{Z}/2^p\mathbb{Z}$. Alternatively, it is a consequence of Proposition 10.15 in Atiyah-McDonald, Introduction to commutative algebra. It follows that our filtration is also complete :

$$\varprojlim H_n/F_pH_n = \varprojlim \mathbb{Z}_2/2^p\mathbb{Z}_2 \cong \varprojlim \mathbb{Z}/2^p\mathbb{Z} \cong \mathbb{Z}_2.$$

As $\{E^r_{pq}\}$ is regular, we deduce that $\{E^r_{pq}\}$ converges to H_* as wanted.

(iii) Let $n \geq 2$, and suppose the following is a Serre fibration

$$\begin{tikzcd} F \arrow[r, "\iota"] & E \arrow[r, "\pi"] & S^n \end{tikzcd} .$$

We saw in class that there exists a first quadrant spectral sequence whose second page is given by $E^2_{p,q} = H_p(S^n, H_q(F))$ and which converges to $H_{p+q}(E)$. We first compute these $E^2_{p,q}$ more explicitly. First, it is a classical result from topology that the p^{th} homology group of the n-sphere is given by :

$$H_p(S^n) = H_p(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0, n \\ 0 & \text{else.} \end{cases}$$

By the Universal Coefficient Theorem for Homology, we deduce that for $p \geq 1$:

$$H_p(S^n, H_q(F)) \cong H_p(S^n) \otimes H_q(F) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(S^n), H_q(F)) \cong \begin{cases} H_q(F) & p = 0, n \\ 0 & \text{else.} \end{cases} ,$$

where we have used that the Tor term is always 0, as $H_p(S^n)$ is either 0 or $\mathbb{Z}$ (which is a free $\mathbb{Z}$-module, and thus in particular flat). On the other hand, when $p = 0$, we have

$$H_0(S^n, H_q(F)) = H_q(F)^{\pi_0(S^n)} = H_q(F).$$

We have just proven that the p^{th} homology group of S^n with coefficient in any abelian group G is given by G when $p = 0$ or n, and 0 otherwise. We deduce that the only now zero entries of the second page of the Serre spectral sequence are located in the 0^{th} and the n^{th} columns, where $E^2_{0,q} = E^2_{n,q} = H_q(F)$, for $q \geq 0$. Inspecting the shape of the differentials in the next pages, we see that the only differentials whose domain and codomain are both non zero are located on the n^{th} page :

$$d^n_{n,q} : E^n_{n,q} = H_q(F) \to H_{n+q-1}(F) = E^n_{0,n+q-1}$$

Since 0 and n are the only two non zero colums and the differentials on the later pages are always the zero map, we see that

$$\ker(d^n_{n,q}) = E^{n+1}_{n,q} = E^\infty_{n,q}, \quad \mathrm{coker}(d^n_{nq}) = E^{n+1}_{0,n+q-1} = E^\infty_{0,n+q-1}.$$

It follows that the sequence

$$\begin{tikzcd} 0 \arrow[r] & E^\infty_{n,q} \arrow[r] & H_q(F) \arrow[r] & H_{n+q-1}(F) \arrow[r] & E^\infty_{0,n+q-1} \arrow[r] & 0 \end{tikzcd}$$

is exact.
Now since our sequence converges to the homology groups of E, we have isomorphisms :

$$E^\infty_{p,q} \cong F_pH_{p+q}(E)/F_{p-1}H_{p+q}(E),$$

for some filtration

$$... \subseteq F_pH_{p+q}(E) \subseteq F_{p+1}H_{p+q}(E) \subseteq ... \subseteq H_{p+q}(E)$$

of $H_{p+q}(E)$. We will now use the convergence of our spectral sequence to deduce some information on this filtration. Indeed, for $p \neq 0, n$, we know that

$$F_pH_{p+q}(E)/F_{p-1}H_{p+q}(E) \cong E^\infty_{p,q} = E^2_{p,q} = 0.$$

Now by definition of convergence, the filtrations $(F_\bullet)$ are Hausdorff, so that $F_iH_{p+q}(E) \cong F_{-1}H_{p+q}(E)$ for all $i < 0$ implies that all these terms are zero. Since it is also exhaustive, we deduce that $F_iH_{p+q}(E) \cong H_{p+q}(E)$ for every $i \geq n$. Using additionally that all $F_iH_{p+q}(E)$ are isomorphic for $i = 0, ..., n-1$, we deduce that the filtration is of the shape

$$0 = F_{-1}H_{p+q}(E) \subseteq F_1H_{p+q}(E) \cong F_2H_{p+q}(E) ... \cong F_{n-1}H_{p+q}(E) \subseteq F_nH_{p+q}(E) \cong H_{p+q}(E).$$

In particular,

$$E^{\infty}_{n,q-n} \cong H_q(E)/F_0H_q(E), \quad E^{\infty}_{0,q} \cong F_0H_q(E)$$

so that

$$0 \longrightarrow E^{\infty}_{0,q} \longrightarrow H_q(E) \longrightarrow E^{\infty}_{n,q-n} \longrightarrow 0$$

is exact. Combining these two sequences with the“edge map” $\iota_* : H_q(F) = E^2_{0,q} \twoheadrightarrow E^{\infty}_{0,q} \subseteq H_q(E)$ (see Addendum 1 to Theorem 5.3.2 in Weibel), we get the Wang sequence. The construction is summarized by the following diagram, where the Wang sequence is given by the middle row :

$$\begin{tikzcd}
 & & & & 0 & & & & \\
... \arrow[r, "d^n"] & H_q(F) \arrow[r] & E^{\infty}_{0,q} \arrow[r] & 0 \arrow[r] & E^{\infty}_{n,q-n} \arrow[r] \arrow[u] & H_{q-1}(F) \arrow[r, "d^n"] & H_{q-1}(F) \arrow[r] & E^{\infty}_{0,q-1} \arrow[r] & 0 \\
 & H_q(F) \arrow[rrr, "\iota_*"] \arrow[u, equal] & & & H_q(E) \arrow[r] \arrow[u] & H_{q-1}(F) \arrow[r] \arrow[u, equal] & H_{q-1}(F) \arrow[rr, "\iota_*"] \arrow[u, equal] & & ... \\
 & E^2_{0,q} \arrow[rrr, two heads] \arrow[u, equal] & & & E^{\infty}_{0,q} \arrow[u, hook] & & E^2_{0,q-1} \arrow[rr, two heads] \arrow[u, equal] & & ... \\
 & & & & 0 \arrow[u] & & & &
\end{tikzcd}$$

(iv) Let $n \geq 2$. As seen in class, the following is a Serre fibration :

$$\Omega S^n \longrightarrow PS^n \xrightarrow{\pi} S^n.$$

We then have the following long exact sequence (see paragraph before Theorem 5.3.2 in Weibel) :

$$... \longrightarrow \pi_{k+1}(S^n) \longrightarrow \pi_k(\Omega S^n) \longrightarrow \pi_k(PS^n) \longrightarrow \pi_k(S^n) \longrightarrow ...$$

Since PS^n is contractible, we have that $\pi_k(PS^n) = 0$ for every $k > 0$. By exactness, this implies that $\pi_k(\Omega S^n) \cong \pi_{k+1}(S^n)$ for every $k > 0$. On the other hand, we have that

$$H_0(\Omega S^n) \cong \mathbb{Z}^{\pi_0(\Omega S^n)} \cong \mathbb{Z}^{\pi_1(S^n)} = \mathbb{Z}.$$

We will now recall a fundamental theorem of topology due to Hurewicz (Theorem 4.32 in J.Hatcher, Algebraic Topology) :

Theorem 6.1. *Let X be a path-connected space satisfying $\pi_k(X) = 0$ for every $k < n$. Then $H_i(X) = 0$ for every $0 < i < n$, and $\pi_n(X) \cong H_n(X)$.*

Now it is a classical result from topology that the homotopy groups of S^n are zero for k less than n. Combining Hurewicz theorem and the last exercise, we deduce that $\pi_n(S^n) \cong \mathbb{Z}$. This means that $\pi_k(\Omega S^n)$ is 0 for $0 < k < n-1$, and $\pi_{n-1}(\Omega S^n) \cong \mathbb{Z}$. Applying Hurewicz theorem, we have obtained so far that

$$H_k(\Omega S^n) = \begin{cases} \mathbb{Z} & k = 0,\ n-1 \\ 0 & 0 < k < n-1 \end{cases}$$

For higher dimension, we use the Wang sequence. Once again, we have that $H_k(PS^n) = 0$ for every $k > 0$. This implies that for $k \geq n-1$, we have

$$H_k(\Omega S^n) \cong H_{k-(n-1)}(\Omega S^n).$$

We deduce immediately from this that the k^{th} homology group of the loop space is isomorphic to the k_0^{th} one, where k_0 is the unique integer $0 \leq k_0 < n-1$ satisfying $k_0 \equiv k \mod n-1$. By the first part, we deduce that

$$H_k(\Omega S^n) = \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } (n-1) \mid k \\ 0 & \text{else} \end{array} \right. ,$$

which concludes.

6.10 Week 10 (by Héloïse Mansat).

Throughout, we will abreviate *Cartan-Eilenberg resolution* as *CE-resolution.*

(i) (a) Consider the commutative diagram with exact column

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
\cdots \arrow[r] & B_p(P,d^h)_1 \arrow[r] & B_p(P,d^h)_0 \arrow[r] & \mathrm{im}(d^A_{n+1}) = B_p(A) \arrow[d] \\
 & & & \ker(d^A_p) = Z_p(A) \arrow[d] \\
\cdots \arrow[r] & H_p(P,d^h)_1 \arrow[r] & H_p(P,d^h)_0 \arrow[r] & Z_p(A)/B_p(A) = H_p(A) \arrow[d] \\
 & & & 0
\end{tikzcd}$$

where the rows are projective resolutions of $B_p(A)$ and $H_p(A)$. By the Horseshoe Lemma (Exercise 5.4), we can assemble a projective resolution of $Z_p(A)$

$$B_p(P,d^h)_\bullet \oplus H_p(P,d^h)_\bullet \to Z_p(A)$$

where the right hand column lifts to an exact sequence of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & B_p(P,d^h)_\bullet \arrow[r] & B_p(P,d^h)_\bullet \oplus H_p(P,d^h)_\bullet \arrow[r] & H_p(P,d^h)_\bullet \arrow[r] & 0
\end{tikzcd} .$$

On the other hand, for every n, the SES in $\mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] & B_p(P,d^h)_n \arrow[r] & Z_p(P,d^h)_n \arrow[r] & H_p(P,d^h)_n \arrow[r] & 0
\end{tikzcd}$$

is split, because $H_p(P,d^h)_n$ is projective. Therefore, $Z_p(P,d^h)_n \cong B_p(P,d^h)_n \oplus H_p(P,d^h)_n$. We conclude that

$$Z_p(P,d^h)_\bullet \to Z_p(A)$$

is a projective resolution.

Similarly, since $Z_p(P,d^h)_\bullet \to Z_p(A)$ is a projective resolution for every p, we apply the horseshoe lemma to the commutative diagram with exact columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
\cdots \arrow[r] & Z_p(P,d^h)_1 \arrow[r] & Z_p(P,d^h)_0 \arrow[r] & \ker(d_p^A)=Z_p(A) \arrow[d, "i"] \\
 & & & A_p \arrow[d, "d_p^A"] \\
\cdots \arrow[r] & B_{p-1}(P,d^h)_1 \arrow[r] & B_{p-1}(P,d^h)_0 \arrow[r] & \operatorname{im}(d_p)=B_{p-1}(A) \arrow[d] \\
 & & & 0
\end{tikzcd}$$

A similar reasoning allows us to conclude.

(b) Provided that $\mathcal{A}$ is a small abelian category, we may assume by *Freyd-Mitchell's embedding theorem* that $\mathcal{A}$ is the category of R-modules for some ring R. We apply a similar reasoning as in the proof of theorem 2.7.2.

Consider $A_\bullet = A_{\bullet,0}$ viewed as double complex concentrated in degree 0. We start with the CE-resolution $P_{\bullet,\bullet} \to A_\bullet$ and consider the augmented double complex C' by adding the shifted double complex $A[-1]_\bullet$ in the row $q=-1$.

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d, "d^h"'] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_{0,0} \arrow[d, "\epsilon_0"'] & \cdots \arrow[l] & P_{p-1,0} \arrow[l] \arrow[d, "\epsilon_{p+1}"'] & P_{p,0} \arrow[l, "d^r"'] \arrow[d, "\epsilon_p"] & P_{p+1,0} \arrow[l] \arrow[d, "\epsilon_{p-1}"] & \cdots \arrow[l] \\
A_0 & \ldots \arrow[l] & A_{p-1} \arrow[l] & A_p \arrow[l] & A_{p+1} \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

We notice that $\epsilon : \operatorname{Tot}^{\oplus}(P_{\bullet,\bullet}) \to A$ has mapping cone

$$\operatorname{Cone}(\epsilon) = \operatorname{Tot}(C').$$

Using corollary 1.5.4, ϵ is a quasi-isomorphism if and only if $\operatorname{Cone}(\epsilon)$ is exact. But by the *Acyclic assembly lemma* (Lemma 2.7.3), since C' is a right half plane complex with exact columns (since there are projective resolutions by part (a)) we conclude.

(ii) We construct the induced map $\tilde{f} : P_{\bullet,\bullet} \to Q_{\bullet,\bullet}$ as follows. We begin by constructing chain maps $\tilde{f} : P_{p,\bullet} \to Q_{p,\bullet}$ between the p-th columns of $P_{\bullet,\bullet}$ and $Q_{\bullet,\bullet}$ then verify that they assemble to a double chain complex map $\tilde{f}$. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z_p(A) \arrow[r, "i"] \arrow[d, "h"'] & A_p \arrow[r, "d_p"] \arrow[d, "f_p"'] & B_{p-1}(A) \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & Z_p(B) \arrow[r, "i'"'] & B_p \arrow[r, "d_p"'] & B_{p-1}(B) \arrow[r] & 0
\end{tikzcd}$$

where $g = f_p|_{Z_p(A)}$ and $h = f_{p-1}|_{B_{p-1}(A)}$. By the *Comparison theorem* (Theorem 2.2.6), the maps h

and g can be lifted to chain complex maps (red arrows in the diagram below).

$$\begin{tikzcd}
0 \arrow[r] & Z_p(P,d^h)_\bullet \arrow[r] \arrow[d, "H_\bullet"'] & P_{p,\bullet} \arrow[r, "d^h_p"] & B_{p-1}(P,d^h) \arrow[r] \arrow[d, "G_\bullet"] & 0 \\
0 \arrow[r] & Z_p(Q,d^h) \arrow[r] & Q_{p,\bullet} \arrow[r, "d^h_p"'] & B_{p-1}(Q,d^h) \arrow[r] & 0
\end{tikzcd}$$

We build a lift F_p of f_p out of $H_\bullet$ and $G_\bullet$. Recall that since $Z_p(P,d^h)_\bullet$ and $B_p(P,d^h)_\bullet$ are projective resolutions, the rows of the commutative diagram above are split exact so that

$$P_{p,\bullet} = Z_p(P,d^h) \oplus B_{p-1}(P,d^h) \text{ and } Q_{p,\bullet} = Z_p(Q,d^h) \oplus B_{p-1}(Q,d^h).$$

Since se work in the q direction, we will alleviate notations by writing

$$Z^P := Z_p(P,d^h) \text{ and } Z^Q := Z_p(Q,d^h)$$

$$B^P := B_{p-1}(P,d^h) \text{ and } B^Q := B_{p-1}(Q,d^h)$$

Claim 6.2. We can construct a maps $\gamma_q : B^P_q \to Z^Q_q$ so that the following maps assemble to a chain map $F_\bullet := \{F_q\}_q$

$$F_q := \begin{pmatrix} H_q & \gamma_q \\ 0 & G_q \end{pmatrix} : Z^P_q \oplus B^P_q \to Z^Q_q \oplus B^Q_q.$$

In other words, for any $(z,b) \in Z^P_q \oplus B^P_q$,

$$F_q(z,b) = (H_q(z) + \gamma_q(b), G_q(b)).$$

Sketch of proof. The proof is similar to the inductive construction of γ_n in Theorem 2.4.6. Simply replace P'_n with Z^P_n, P''_n with B^Q_n and the maps F'_n with H_n, F''_n with G_n. △

Uisng the claim, we get a collection of maps $F_p : P_{p,\bullet} \to Q_{p,\bullet}$. It remains to prove that we can assemble them to a double chain complex map. Recall that we have shown in Exercise 10.1.(a) that $\epsilon_p : P_{p,\bullet} \to A_p$ and $\eta_p : Q_{p,\bullet} \to B_p$ are projective resolutions, this yields the following commutative diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & Z_p(P,d^h)_\bullet \arrow[r] \arrow[d, "H_\bullet"'] & P_{p,\bullet} \arrow[r, "d^h_p"] \arrow[d, "F_p"] & \overbrace{B_{p-1}(P,d^h)_\bullet}^{=Z_{p-1}(P,d^h)_\bullet} \arrow[r] \arrow[d, "G_\bullet"] & P_{p-1,\bullet} \arrow[r] \arrow[d, "F_{p-1}"] & Z_{p-2}(P,d^h)_\bullet \arrow[r] \arrow[d, "H_\bullet"] & \cdots \\
\cdots \arrow[r] & Z_p(Q,d^h) \arrow[r] & Q_{p,\bullet} \arrow[r, "d^h_p"'] & \underbrace{B_{p-1}(Q,d^h)_\bullet}_{=Z_{p-1}(P,d^h)_\bullet} \arrow[r] & Q_{p-1,\bullet} \arrow[r] & Z_{p-2}(Q,d^h)_\bullet \arrow[r] & \cdots
\end{tikzcd}$$

This allows to conclude that $\tilde{f} := \{F_p\}_p : P_{\bullet,\bullet} \to Q_{\bullet,\bullet}$ is a double chain complex map that lifts $f : P \to Q$.

(iii) (a) Suppose that A is concentrated in degree 0. Then the CE-resolution of A is supported on the single column $p = 0$. In particular,

$$\mathbb{L}_i F(A) = H_i(\mathrm{Tot}^{\oplus}(F(P_{\bullet,\bullet}))) = H_i(F(P_{0,\bullet})) = L_i F(A_0).$$

where the last equality follows from the fact that $P_{0,\bullet} \to A_0$ is a projective resolution.

(b) Let $A_\bullet \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$. On the one hand for all i, by definition

$$\mathbb{L}_i F(A_\bullet) = H_i(\mathrm{Tot}^{\oplus}(F(P_{\bullet,\bullet}))$$

where $P_{\bullet,\bullet} \to A_\bullet$ is a CE-resolution of $A_\bullet$. On the other hand,

$$L_i H_0 F(A_\bullet) = H_i^h(H_0^v F(P_{\bullet,\bullet})) \overset{\clubsuit}{=} H_i^h(F H_0^v(P_{\bullet,\bullet})).$$

The equality ♣ follows from the fact that F is right exact and $P_{\bullet,\bullet} \to A_\bullet$ is a resolution. Indeed this implies that $H_0(F(P_{\bullet,\bullet})) \cong F(A_\bullet)$. On the other hand, $H_0(P_{\bullet,\bullet}) \cong A_\bullet$. We conclude by unicity of the cokernel.

Henceforth, we write $P := P_{\bullet,\bullet}$.

Claim 6.3. $\mathrm{Tot}^{\oplus}(P)$ is chain homotopy equivalent to $H_0(P)$.

Proof. Consider the spectral sequence associated to P

$$^{I}E^2_{p,q} = H_p^h H_q^v(P) \implies H_{p+q}(\mathrm{Tot}^{\oplus}(P).$$

Since P is a CE-resolution, it has exact columns and in particular, $H_p^h H_q^v(P) = 0$ for all $q > 0$. Therefore page 2 of the above spectral sequence collapses on the single row $q = 0$ and we conclude that $\forall i \geq 0$

$$H_i^h H_0^v(P) = {}^{I}E^2_{i,0} = {}^{I}E^\infty_{i,0} \cong H_i(\mathrm{Tot}^{\oplus}(P)).$$

This shows that $\mathrm{Tot}^{\oplus}(P)$ is quasi-isomorphic to $H_0(P)$. But since $\mathrm{Tot}^{\oplus}(P)$ and $H_0(P)$ are chain complexes of pointwise projective objects, there are in fact chain homotopy equivalent (see for example Lemma 10.4.6).

△

It follows from the claim and the fact that F is an additive functor (as it is right exact) that $F(H_0(P))$ and $F(\mathrm{Tot}^{\oplus}(P)) = \mathrm{Tot}^{\oplus}(F(P))$ are chain homotopy equivalent. This latter conclusion yields the desired isomorphism

$$\begin{aligned} L_i H_0 F(A) &= H_i(F H_0(P)) \\ &\cong H_i(\mathrm{Tot}^{\oplus}(F(P)) \\ &= \mathbb{L}^i F(A). \end{aligned}$$

(c) Let $A \in \mathrm{Ch}(\mathcal{A})$ and let $P_{\bullet,\bullet}$ be a CE-resolution of A. Consider the shifted complex $A[n]$. Then the double complex obtained by shifting the columns of $P_{\bullet,\bullet}$, denoted $P[n,0]_{\bullet,\bullet}$ is a CE-resolution of $A[n]$.

On the other hand, note that for all k

$$\begin{aligned} \left(\mathrm{Tot}^{\oplus}(P[n,0])\right)_k &= \bigoplus_{i+j=k} P[n,0]_{i,j} \\ &= \bigoplus_{i+j=k} P_{i+n,j} \\ &= \bigoplus_{i'+j=k+n} P_{i',j} \\ &= \left((\mathrm{Tot}^{\oplus} P)[n]\right)_k. \end{aligned}$$

We conclude by dimension shifting

$$\mathbb{L}_i F(A[n]) = H_i\left(\mathrm{Tot}^{\oplus} F(P[n,0])\right) = H_i\left((\mathrm{Tot}^{\oplus} P)[n]\right) = H_{i+n}\left(\mathrm{Tot}^{\oplus} P\right) = \mathbb{L}_{i+n} F(A).$$

(iv) Consider the short short exact sequence associated to the cone A

$$0 \longrightarrow A_0 \longrightarrow A \longrightarrow A_1[-1] \longrightarrow 0 \ .$$

Using Lemma 5.7.5, there is a long-exact sequence

$$\cdots \longrightarrow \mathbb{L}_{i+1}F(A_1[-1]) \longrightarrow \mathbb{L}_iF(A_0) \longrightarrow \mathbb{L}_iF(A) \longrightarrow \mathbb{L}_iF(A_1[-1]) \longrightarrow \cdots$$

But, by Exercises 10.3.c then 10.3.a, for any integer k,

$$\mathbb{L}_{k+1}F(A_1[-1]) = \mathbb{L}_kF(A_1) = L_kF(A_1).$$

Similarly

$$\mathbb{L}_kF(A_0) = L_kF(A_0).$$

Substituting these in the above long exact sequence yields

$$\cdots \longrightarrow \mathbb{L}_{i+1}F(A) \longrightarrow L_iF(A_1) \longrightarrow L_iF(A_0) \longrightarrow \mathbb{L}_iF(A) \longrightarrow \cdots$$

(v) For the morphism of rings $f : X \to Y$ and the $\mathcal{O}_X$-module $\mathcal{F}$, consider the associated Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^qf_*\mathcal{F}) \implies H^{p+q}(X, \mathcal{F}).$$

(a) The condition that $R^qf_*\mathcal{F} = 0$ for every $q > 0$ implies that page 2 of this spectral sequence is supported in the single row $q = 0$.

$$\begin{array}{c|cccc}
 & \vdots & \vdots & \vdots & \\
1 & 0 & 0 & 0 & \\
0 & H^0(Y, R^0f_*\mathcal{F}) & H^1(Y, R^0f_*\mathcal{F}) & H^2(Y, R^0f_*\mathcal{F}) & \cdots \\
\hline
 & 0 & 1 & 2 &
\end{array}$$

Using bounded convergence, we conclude that for every $p \geq 0$

$$H^p(X, \mathcal{F}) \cong E_\infty^{p,0} = E_2^{p,0} = H^p(Y, f_*\mathcal{F}).$$

Remark 6.4. In algebraic geometry, the condition that $R^qf_*\mathcal{F} = 0$ holds for example when f is affine.

(b)

Remark 6.5. We correct the assumptions in the statement of the exercise. We assume that $H^p(Y, R^qf_*\mathcal{F}) = 0$ for all $p > 0$. In algebraic geometry, these condition is satisfied for Y an affine scheme for example.

The condition that $H^p(Y, R^qf_*\mathcal{F}) = 0$ for every $p > 0$ implies that page 2 of the Leray spectral sequence is supported on the single column $p = 0$.

$$\begin{array}{c|cccc}
 & \vdots & \vdots & \vdots & \\
2 & H^0(Y, R^2 f_*\mathcal{F}) & \vdots & \vdots & \\
1 & H^0(Y, R^1 f_*\mathcal{F}) & 0 & 0 & \\
0 & H^0(Y, R^0 f_*\mathcal{F}) & 0 & 0 & \dots \\
\hline
 & 0 & 1 & 2 &
\end{array}$$

Since at each pages, the differentials going in and out of the complexes in the single column have either 0 source or target, we conclude that for every $q \geq 0$, by bounded convergence

$$H^q(X, \mathcal{F}) \cong E_\infty^{0,q} = E_2^{0,q} = H^0(Y, R^q f_* \mathcal{F}).$$

$$\begin{array}{c|cccc}
 & \vdots & \vdots & \vdots & \\
2 & H^0(Y, R^2 f_*\mathcal{F}) & \vdots & \vdots & \\
1 & H^0(Y, R^1 f_*\mathcal{F}) & 0 & 0 & \\
0 & H^0(Y, R^0 f_*\mathcal{F}) & H^1(Y, R^0 f_*\mathcal{F}) & H^2(Y, R^q f_*\mathcal{F}) & \dots \\
\hline
 & 0 & 1 & 2 &
\end{array}$$

where $Q_{\bullet,\bullet} \to A$ is a projective resolution of A.
Consider the spectral sequences E, E' associated to $F(P_{\bullet,\bullet})$ and $F(Q_{\bullet,\bullet})$ respectively:

$${}^{I}E^{2}_{p,q} = H^h_p H^v_q(F(P)) \implies H_{p+q}\left(\mathrm{Tot}^{\oplus} F(P)\right)$$

$${}^{I}E'^{2}_{p,q} = H^h_p H^v_q(F(Q)) \implies H_{p+q}\left(\mathrm{Tot}^{\oplus} F(Q)\right).$$

We construct a morphism of spectral sequences $f : E \to E'$ as follows. On page zero, a morphism $f^0_{p,q} : F(P_{p,q}) \to F(Q_{p,q})$ is given by lifting the identity $\mathrm{id}_{A_p} : A_p \to A_p$ which produces a morphism $f_{p,\bullet} : P_{p,\bullet} \to Q_{p,\bullet}$. We then set, $f^0_{p,q} := F(f_{p,q})$.
On the next pages, functoriality of $H(\_)$ yields morphisms from the ones contructed on page zero.
As P is a CE-resolution, the columns of P are projective resolutions of A. Since the columns of Q are also projective resolutions of A Lemma 2.4.1 implies that $H^v_q(F(P)) \cong H^v_q(F(Q))$. In particular, on page 1 we get an isomorphism

$$f^r_{p,q} : E^1_{p,q} \to E'^1_{p,q}$$

for all p, q.

Therefore, by the Comparison Theorem (Theorem 5.2.12)

$$h : H_i(\mathrm{Tot}^{\oplus}(F(P))) \cong H_i(\mathrm{Tot}^{\oplus}(F(P))).$$

Now consider the following claim.
Consider the following claim.

Claim 6.6.

$$H^v(F(P)) \cong FH^v(P).$$

Proof. $\triangle$

Now consider the spectral sequence associated to the double complex $F(P_{\bullet,\bullet})$

$${}^I E^2_{p,q} \implies H_{p+q}(\mathrm{Tot}^{\oplus}(F(C)).$$

Using the lemma, we get that for any p and $q > 0$,

$$H^h_p H^v_q(F(P)) = H^h_p F H^v_q(P) = H_p F(0) = 0.$$

Therefore, page 2 of the spectral collapses on the single row $q = 0$. This yields
Consider the following claim.

Claim 6.7.

$$H^v(F(P)) \cong FH^v(P).$$

Proof. $\triangle$

Now consider the spectral sequence associated to the double complex $F(P_{\bullet,\bullet})$

$${}^I E^2_{p,q} \implies H_{p+q}(\mathrm{Tot}^{\oplus}(F(C)).$$

Using the lemma, we get that for any p and $q > 0$,

$$H^h_p H^v_q(F(P)) = H^h_p F H^v_q(P) = H_p F(0) = 0.$$

Therefore, page 2 of the spectral collapses on the single row $q = 0$. This yields

$$\begin{aligned} L_i H_0 F(A) &= {}^I E^2_{i,0} \\ &= {}^I E^\infty_{i,0} \\ &\cong H_i(\mathrm{Tot}^{\oplus}(F(C)) \\ &= \mathbb{L}^i F(A). \end{aligned}$$

Proof of Claim 6.2. We construct γ inductively via successive lifts.
Write

$$\begin{gathered} P_\bullet := P_{p,\bullet}. \\ Q_\bullet := Q_{p,\bullet}. \\ \eta := P_{p,0} \to A_p \\ \epsilon' := Z_p(\epsilon) : Z_p(P, d^h)_0 \to Z_p(A) \\ \epsilon'' := B_q(\epsilon) : B_{p-1}(P, d^h)_0 \to B_{p-1}(A) \\ \epsilon := Q_{p,0} \to B_p \\ \eta' := Z_q(\epsilon) : Z_p(Q, d^h)_0 \to Z_p(B) \\ \eta'' := B_q(\epsilon) : B_{p-1}(Q, d^h)_0 \to B_{p-1}(B) \\ f := f_p \end{gathered}$$

Construction of γ_0. In order for F to be a chain map, we need on level 0 that $(\eta F_0 - f\epsilon) = 0$

$$(\eta F_0 f \epsilon) = 0.$$

The only obstacle is on B^P_q coordinate. Cosnider the following restrictions:

$\triangle$

6.11 Week 11 (by William Ballard and Dév Vorburger).

(i) (a) It suffices to show that $d_{\text{cyl}}^{n+1}d_{\text{cyl}}^{n} = 0$ for any integer n. The matrix of this composition is given by

$$\begin{bmatrix} d_B^{n+1} & \text{id}_{B^{n+2}} & 0 \\ 0 & -d_B^{n+2} & 0 \\ 0 & -f^{n+2} & d_C^{n+1} \end{bmatrix} \begin{bmatrix} d_B^{n} & \text{id}_{B^{n+1}} & 0 \\ 0 & -d_B^{n+1} & 0 \\ 0 & -f^{n+1} & d_C^{n} \end{bmatrix}$$
$$= \begin{bmatrix} d_B^{n+1}d_B^n & d_B^{n+1} - d_B^{n+1} & 0 \\ 0 & d_B^{n+2}d_B^{n+1} & 0 \\ 0 & f^{n+2}d_B^{n+1} - d_C^{n+1}f^{n+1} & d_C^{n+1}d_C^n \end{bmatrix}.$$

The matrix in the second line is 0 because B and C are cochain complexes and $f : B \to C$ is a morphism of cochain complexes.

(b) The chain maps $f, g : B \to C$ are chain homotopic if and only if there are maps $\{s^n : B^{n+1} \to C^n\}_{n\in\mathbb{Z}}$ such that

$$d_C^n s^n + s^{n+1}d_B^{n+1} = f^{n+1} - g^{n+1}$$

for all $n \in \mathbb{Z}$. Meanwhile, $(f, s, g) : \text{cyl}(B) \to C$ is a morphism of cochain complexes iff

$$(f^{n+1}, s^{n+1}, g^{n+1})d_{\text{cyl}}^n = d_C^n(f^n, s^n, g^n)$$
$$\begin{bmatrix} f^{n+1} & s^{n+1} & g^{n+1} \end{bmatrix} \begin{bmatrix} d_B^{n} & \text{id}_{B^{n+1}} & 0 \\ 0 & -d_B^{n+1} & 0 \\ 0 & -\text{id}_{B^{n+1}} & d_B^{n} \end{bmatrix} = \begin{bmatrix} d_C^n \end{bmatrix} \begin{bmatrix} f^n & s^n & g^n \end{bmatrix}$$
$$\begin{bmatrix} f^{n+1}d_B^n \\ f^{n+1} - s^{n+1}d_B^{n+1} - g^{n+1} \\ g^{n+1}d_B^n \end{bmatrix}^T = \begin{bmatrix} d_C^n f^n \\ d_C^n s^n \\ d_C^n g^n \end{bmatrix}^T.$$

for all $n \in \mathbb{Z}$. The first and third components of this matrix equation hold because f and g are morphisms of cochain complexes. Thus, there exists a family of maps $\{s^n : B^{n+1} \to C^n\}_{n\in\mathbb{Z}}$ such that $(f, s, g) : \text{cyl}(B) \to C$ is a morphism of cochain complexes iff there exists a family of maps $\{s^n : B^{n+1} \to C^n\}_{n\in\mathbb{Z}}$ such that

$$d_C^n s^n + s^{n+1}d_B^{n+1} = f^{n+1} - g^{n+1},$$

i.e., iff the maps $f, g : B \to C$ are chain homotopic.

(c) We have $\beta\alpha = \text{id}_B$, thus to show that α is a chain homotopy equivalence, it suffices to show that $\alpha\beta$ is chain homotopic to $\text{id}_{\text{cyl}(B)}$. For each integer n, we define $s^n : \text{cyl}(B)^{n+1} \to \text{cyl}(B)^n$ by $s^n(b', b'', b) = (0, b, 0)$, i.e., s^n can be represented by the matrix

$$s^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \text{id}_{B^{n+1}} \\ 0 & 0 & 0 \end{bmatrix}.$$

We just have to confirm that $d_{\text{cyl}(B)}^n s^n + s^{n+1}d_{\text{cyl}(B)}^{n+1} = \alpha^{n+1}\beta^{n+1} - \text{id}_{\text{cyl}(B)^{n+1}}$ for any integer n. Since $\alpha\beta(b', b'', b) = \alpha(b' + b) = (b' + b, 0, 0)$, we can represent $\alpha^{n+1}\beta^{n+1}$ by the matrix

$$\alpha^{n+1}\beta^{n+1} = \begin{bmatrix} \text{id}_{B^{n+1}} & 0 & \text{id}_{B^{n+1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\alpha^{n+1}\beta^{n+1} - \text{id}_{\text{cyl}(B)^{n+1}} = \begin{bmatrix} 0 & 0 & \text{id}_{B^{n+1}} \\ 0 & -\text{id}_{B^{n+2}} & 0 \\ 0 & 0 & -\text{id}_{B^{n+1}} \end{bmatrix}. \tag{21}$$

Also, we have

$$\begin{aligned}
&d^n_{\mathrm{cyl}(B)} s^n + s^{n+1} d^{n+1}_{\mathrm{cyl}(B)} \\
&= \begin{bmatrix} d^n_B & \mathrm{id}_{B^{n+1}} & 0 \\ 0 & -d^{n+1}_B & 0 \\ 0 & -\mathrm{id}_{B^{n+1}} & d^n_B \end{bmatrix} s^n + s^{n+1} \begin{bmatrix} d^{n+1}_B & \mathrm{id}_{B^{n+2}} & 0 \\ 0 & -d^{n+2}_B & 0 \\ 0 & -\mathrm{id}_{B^{n+2}} & d^{n+1}_B \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \mathrm{id}_{B^{n+1}} \\ 0 & 0 & -d^{n+1}_B \\ 0 & 0 & -\mathrm{id}_{B^{n+1}} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{id}_{B^{n+2}} & d^{n+1}_B \\ 0 & 0 & 0 \end{bmatrix} \\
&= \alpha^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}},
\end{aligned}$$

by (21), as needed.

(d) As before, we have $\beta\alpha' = \mathrm{id}_B$, thus to conclude that α' is a chain homotopy equivalence, it suffices to show that $\alpha'\beta$ is chain homotpic to $\mathrm{id}_{\mathrm{cyl}(B)}$. Since $\mathbf{K}(A)$ is a category, we know that composition of morphisms is compatible with chain homotopy. Thus:

$$\alpha'\beta \sim \mathrm{id}_{\mathrm{cyl}(B)}\alpha'\beta \sim (\alpha\beta)\alpha'\beta \sim \alpha\beta \sim \mathrm{id}_{\mathrm{cyl}(B)},$$

as needed.

Now we wish to find maps $\{t^n : \mathrm{cyl}(B)^{n+1} \to \mathrm{cyl}(B)^n\}_{n\in\mathbb{Z}}$ such that

$$d^n_{\mathrm{cyl}(B)} t^n + t^{n+1} d^{n+1}_{\mathrm{cyl}(B)} = (\alpha')^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}}. \tag{22}$$

Define a chain map $\varphi : \mathrm{cyl}(B) \to \mathrm{cyl}(B)$ by $\varphi^n(x, y, z) = (-z, y, -x)$ for $x, z \in B^n$ and $y \in B^{n+1}$. This is in fact a chain map because

$$\begin{aligned}
d^n_{\mathrm{cyl}(B)}\varphi^n &= \begin{bmatrix} d^n_B & \mathrm{id}_{B^{n+1}} & 0 \\ 0 & -d^{n+1}_B & 0 \\ 0 & -\mathrm{id}_{B^{n+1}} & d^n_B \end{bmatrix} \begin{bmatrix} 0 & 0 & -\mathrm{id}_{B^n} \\ 0 & \mathrm{id}_{B^{n+1}} & 0 \\ -\mathrm{id}_{B^n} & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \mathrm{id}_{B^{n+1}} & -d^n_B \\ 0 & -d^{n+1}_B & 0 \\ -d^n_B & -\mathrm{id}_{B^{n+1}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -\mathrm{id}_{B^{n+1}} \\ 0 & \mathrm{id}_{B^{n+2}} & 0 \\ -\mathrm{id}_{B^{n+1}} & 0 & 0 \end{bmatrix} \begin{bmatrix} d^n_B & \mathrm{id}_{B^{n+1}} & 0 \\ 0 & -d^{n+1}_B & 0 \\ 0 & -\mathrm{id}_{B^{n+1}} & d^n_B \end{bmatrix} \\
&= \varphi^{n+1} d^n_{\mathrm{cyl}(B)}.
\end{aligned}$$

We also note that (i) $\varphi\varphi = \mathrm{id}_{\mathrm{cyl}(B)}$, (ii) $\varphi\alpha = -\alpha'$, and (iii) $\beta\varphi = -\beta$. Recall that the maps $\{s^n : \mathrm{cyl}(B)^{n+1} \to \mathrm{cyl}(B)^n\}_{n\in\mathbb{Z}}$ from the previous subexercise of this problem satisfy for all $n \in \mathbb{Z}$:

$$\begin{aligned}
d^n_{\mathrm{cyl}(B)} s^n + s^{n+1} d^{n+1}_{\mathrm{cyl}(B)} &= \alpha^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}} \\
d^n_{\mathrm{cyl}(B)}(\varphi^n s^n \varphi^{n+1}) + (\varphi^{n+1} s^{n+1} \varphi^{n+2}) d^{n+1}_{\mathrm{cyl}(B)} &= (\alpha')^{n+1}\beta^{n+1} - \mathrm{id}_{\mathrm{cyl}(B)^{n+1}}.
\end{aligned}$$

Thus if for each integer n we define $t^n : \mathrm{cyl}(B)^{n+1} \to \mathrm{cyl}(B)^n$ by $t^n = \varphi^n s^n \varphi^{n+1}$, then the desired equation (22) is satisfied. Specifically, for $x, z \in B^{n+1}$ and $y \in B^{n+2}$, we have

$$t^n(x, y, z) = \varphi^n s^n(-z, y, -x) = \varphi^n(0, -x, 0) = (0, -x, 0).$$

(ii) (a) We first determine the mapping cone $\mathrm{cone}(0_A)$ of $0_A : A \to A$. The object in degee n of the complex $\mathrm{cone}(0_A)$ is $A^{n+1} \oplus A^n$, and the differential $d^n : \mathrm{cone}(0_A)^n \to \mathrm{cone}(0_A)^{n+1}$ is defined to be

$$d^n = (-d^{n+1}_A, d^n_A). \tag{23}$$

Recall that the translate $A[-1]$ has nth object $A[-1]^n = A^{n+1}$ and nth differential $-d_A^{n+1}$. Thus $\mathrm{cone}(0_A)$ is precisely the cochain complex $A[-1] \oplus A$. It follows that we have a strict (and therefore exact) triangle

$$\begin{tikzcd}
& A[-1] \oplus A \arrow[dl] & \\
A \arrow[rr, "0_A"] & & A. \arrow[ul]
\end{tikzcd}$$

Next, we find an exact triangle for the identity morphism $1_A : A \to A$. The mapping cone $\mathrm{cone}(1_A)$ of the identity on A is by definition the cone $\mathrm{cone}(A)$ of the complex A. In Week 4 (cf. Proposition 3.9 of Week 4's lecture notes), we saw that $\mathrm{cone}(A)$ is split exact and therefore contractible. Hence there is a homotopy equivalence $h : \mathrm{cone}(1_A) \to 0^\bullet$, where $0^\bullet$ is the zero cochain complex. Thus we have a diagram

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "1_A"] & A \arrow[r] \arrow[d, "1_A"] & 0^\bullet \arrow[r] \arrow[d, "h"] & A[-1] \arrow[d] \\
A \arrow[r, "1_A"] & A \arrow[r] & \mathrm{cone}(1_A) \arrow[r] & A[-1],
\end{tikzcd}$$

in which the vertical maps are homotopy equivalences and each square commutes up to homotopy. We conclude that the triangle

$$\begin{tikzcd}
& 0^\bullet \arrow[dl] & \\
A \arrow[rr, "1_A"] & & A \arrow[ul]
\end{tikzcd}$$

is exact.

(b, i) We first assume that the triangle (u, v, w) on A, B, C is the strict triangle on $u : A \to B$, i.e., $C = \mathrm{cone}(u)$ and $v : B \to \mathrm{cone}(u)$ and $w : \mathrm{cone}(u) \to A[-1]$ are the usual maps. We want to find a homotopy equivalence $\beta : A[-1] \to \mathrm{cone}(v)$ such that we have a diagram

$$\begin{tikzcd}
B \arrow[r, "v"] \arrow[d, "1"] & C \arrow[r, "w"] \arrow[d, "1"] & A[-1] \arrow[r, "-u[-1]"] \arrow[d, "\beta"] & B[-1] \arrow[d, "1"] \\
B \arrow[r, "v"] & C \arrow[r, "w'"] & \mathrm{cone}(v) \arrow[r, "\delta"] & B[-1]
\end{tikzcd} \tag{24}$$

that commutes up to homotopy, where the bottom row is the strict triangle on $v : B \to C$. We first clarify what the complex $\mathrm{cone}(v)$ is. Its nth object is

$$\mathrm{cone}(v)^n = B^{n+1} \oplus C^n = B^{n+1} \oplus \mathrm{cone}(u)^n = B^{n+1} \oplus A^{n+1} \oplus B^n,$$

and its nth differential is

$$d^n = \begin{bmatrix} -d_B^{n+1} & 0 \\ -v^{n+1} & d^n_{\mathrm{cone}(u)} \end{bmatrix} = \begin{bmatrix} -d_B^{n+1} & 0 & 0 \\ 0 & -d_A^{n+1} & 0 \\ -\mathrm{id}_{B^{n+1}} & -u^{n+1} & d_B^n \end{bmatrix}.$$

Define $\beta : A[-1] \to \mathrm{cone}(v)$ by letting $\beta^n : A^{n+1} \to \mathrm{cone}(v)^n$ be the morphism $\beta^n = (-u^{n+1}, \mathrm{id}_{A^{n+1}}, 0)$

for each $n \in \mathbb{Z}$. This is a chain map because for any n:

$$\begin{aligned}
\beta^{n+1} d^n_{A[-1]} &= -\beta^{n+1} d^{n+1}_A \\
&= \begin{bmatrix} u^{n+2} d^{n+1}_A \\ -d^{n+1}_A \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} d^{n+1}_B u^{n+1} \\ -d^{n+1}_A \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -d^{n+1}_B & 0 & 0 \\ 0 & -d^{n+1}_A & 0 \\ -\mathrm{id}_{B^{n+1}} & -u^{n+1} & d^n_B \end{bmatrix} \begin{bmatrix} -u^{n+1} \\ \mathrm{id}_{A^{n+1}} \\ 0 \end{bmatrix} \\
&= d^n_{\mathrm{cone}(v)} \beta^n.
\end{aligned}$$

Next, define $\gamma : \mathrm{cone}(v) \to A[-1]$ by letting $\gamma^n : \mathrm{cone}(v)^n \to A^{n+1}$ be the morphism $\gamma^n = (0, \mathrm{id}_{A^{n+1}}, 0)$. This is a morphism of chain complexes because

$$\begin{aligned}
d^n_{A[-1]} \gamma^n &= (0, -d^{n+1}_A, 0) \\
&= (0, \mathrm{id}_{A^{n+2}}, 0) \begin{bmatrix} -d^{n+1}_B & 0 & 0 \\ 0 & -d^{n+1}_A & 0 \\ -\mathrm{id}_{B^{n+1}} & -u^{n+1} & d^n_B \end{bmatrix} \\
&= \gamma^{n+1} d^n_{\mathrm{cone}(v)}.
\end{aligned}$$

Moreover, we have $\gamma\beta = \mathrm{id}_{A[-1]}$, thus to show that $\beta : A[-1] \to \mathrm{cone}(v)$ is a homotopy equivalence, it suffices to show that $\beta\gamma$ is homotopic to the identity on $\mathrm{cone}(v)$. For each integer n, define $s^n : \mathrm{cone}(v)^{n+1} \to \mathrm{cone}(v)^n$ by $(x, y, z) \mapsto (z, 0, 0)$. Then:

$$\begin{aligned}
&d^n_{\mathrm{cone}(v)} s^n + s^{n+1} d^{n+1}_{\mathrm{cone}(v)} = \\
&\begin{bmatrix} -d^{n+1}_B & 0 & 0 \\ 0 & -d^{n+1}_A & 0 \\ -\mathrm{id}_{B^{n+1}} & -u^{n+1} & d^n_B \end{bmatrix} \begin{bmatrix} 0 & 0 & \mathrm{id}_{B^{n+1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & \mathrm{id}_{B^{n+2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -d^{n+2}_B & 0 & 0 \\ 0 & -d^{n+2}_A & 0 \\ -\mathrm{id}_{B^{n+2}} & -u^{n+2} & d^{n+1}_B \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -d^{n+1}_B \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id}_{B^{n+1}} \end{bmatrix} + \begin{bmatrix} -\mathrm{id}_{B^{n+2}} & -u^{n+2} & d^{n+1}_B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{id}_{B^{n+2}} & -u^{n+2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id}_{B^{n+1}} \end{bmatrix}.
\end{aligned} \tag{25}$$

On the other hand, we have $\beta^n \gamma^n (x, y, z) = \beta^n(y) = (-u^{n+1}(y), y, 0)$, thus:

$$\begin{aligned}
&\beta^{n+1}\gamma^{n+1} - \mathrm{id}_{\mathrm{cone}(v)^{n+1}} \\
&= \begin{bmatrix} 0 & -u^{n+2} & 0 \\ 0 & +\mathrm{id}_{A^{n+2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathrm{id}_{B^{n+2}} & 0 & 0 \\ 0 & \mathrm{id}_{A^{n+2}} & 0 \\ 0 & 0 & \mathrm{id}_{B^{n+1}} \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{id}_{B^{n+2}} & -u^{n+2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id}_{B^{n+1}} \end{bmatrix} \\
&= d^n_{\mathrm{cone}(v)} s^n + s^{n+1} d^{n+1}_{\mathrm{cone}(v)},
\end{aligned}$$

by (25), and we conclude that $\beta\gamma$ is homotopic to $\mathrm{id}_{\mathrm{cone}(v)}$. Thus $\beta : A[-1] \to \mathrm{cone}(v)$ is a homotopy equivalence. It remains to show that the squares in (24) commute up to homotopy. We first note that for $x \in A[-1]^n = A^{n+1}$, we have

$$\delta^n \beta^n(x) = \delta^n(-u^{n+1}(x), x, 0) = -u^{n+1}(x) = -u[-1]^n(x),$$

thus $\delta\beta = -u[-1]$, and the right square in (24) commutes. Since the left square commutes trivially, it remains to check that the central square commutes up to homotopy, i.e., $\beta w \sim w'$. Recalling that $w : \mathrm{cone}(u) \to A[-1]$ is the usual map from the mapping cone of $u : A \to B$ to $A[-1]$, we find $\gamma w' = w$, thus $\gamma w' \sim w$. But γ is a homotopy equivalence and its inverse is β, thus $w' \sim \beta w$, as needed. This completes the proof that

$$\begin{tikzcd}
& A[-1] \arrow[dl, "{-u[-1]}"'] & \\
B \arrow[rr, "v"] & & C \arrow[ul, "w"']
\end{tikzcd} \tag{26}$$

is exact when (u, v, w) is a strict triangle on A, B, C. Since an exact triangle is by definition isomorphic to a strict triangle, the rotate in (26) remains exact when (u, v, w) is an exact triangle on A, B, C.

(b, ii) Once again, we assume that the triangle (u, v, w) on A, B, C is the strict triangle on $u : A \to B$, i.e., C is the mapping cone of u. We let δ be the map $-w[1] : C[1] \to A$, and want to find a homotopy equivalence $\sigma : B \to \mathrm{cone}(\delta)$ such that the diagram

$$\begin{tikzcd}
C[1] \arrow[r, "\delta"] \arrow[d, "1"] & A \arrow[r, "u"] \arrow[d, "1"] & B \arrow[r, "v"] \arrow[d, "\sigma"] & C \arrow[d, "1"] \\
C[1] \arrow[r, "\delta"] & A \arrow[r, "u'"] & \mathrm{cone}(\delta) \arrow[r, "v'"] & C
\end{tikzcd}$$

commutes up to homotopy. We first demonstrate that $\mathrm{cone}(\delta) = \mathrm{cyl}(u)$. For any integer n, we have

$$\mathrm{cone}(\delta)^n = C[1]^{n+1} \oplus A^n = C^n \oplus A^n = A^{n+1} \oplus B^n \oplus A^n \cong \mathrm{cyl}(u)^n.$$

Moreover, the differential $d^n_{\mathrm{cone}(\delta)} : \mathrm{cone}(\delta)^n \to \mathrm{cone}(\delta)^{n+1}$ is

$$d^n_{\mathrm{cone}(\delta)} = \begin{bmatrix} -d^{n+1}_{C[1]} & 0 \\ -\delta^{n+1} & d^n_A \end{bmatrix} = \begin{bmatrix} d^n_C & 0 \\ w^n & d^n_A \end{bmatrix} = \begin{bmatrix} -d^{n+1}_A & 0 & 0 \\ -u^{n+1} & d^n_B & 0 \\ \mathrm{id}_{A^{n+1}} & 0 & d^n_A \end{bmatrix}.$$

It is straightforward to verify that the maps $\vartheta^k : \mathrm{cone}(\delta)^k \to \mathrm{cyl}(u)^k$ given by $\vartheta^k(x, y, z) = (z, x, y)$ define an isomorphism $\vartheta : \mathrm{cone}(\delta) \to \mathrm{cyl}(u)$ of chain complexes. Our task thus becomes to find a homotopy equivalence $\tilde{\sigma} : B \to \mathrm{cyl}(u)$ such that the diagram

$$\begin{tikzcd}
C[1] \arrow[r, "\delta"] \arrow[d, "1"] & A \arrow[r, "u"] \arrow[d, "1"] & B \arrow[r, "v"] \arrow[d, "\tilde{\sigma}"] & C \arrow[d, "1"] \\
C[1] \arrow[r, "\delta"] & A \arrow[r, "\tilde{u}"] & \mathrm{cyl}(u) \arrow[r, "\tilde{v}"] & C
\end{tikzcd} \tag{27}$$

is commutative up to homotopy, where $\tilde{u} : A \to \mathrm{cyl}(U)$ is the inclusion of A in $\mathrm{cyl}(u)$ and $\tilde{v} : \mathrm{cyl}(u) \to C$ is the projection.

Before continuing, we note that subexercise (d) of Problem 1 holds in a more general context (cf. Exercise 1.5.4 in Weibel): if $u : A \to B$ is a morphism of chain complexes, and we define a map $\alpha' : B \to \mathrm{cyl}(u)$ by $\alpha'(b) = (0, 0, b)$ and a map $\beta : \mathrm{cyl}(u) \to B$ by $\beta(a', a, b) = f(a') + b$, then α' is a homotopy equivalence with inverse β. In particular, we may define $\tilde{\sigma} : B \to \mathrm{cyl}(u)$ to be the map α' to obtain a homotopy equivalence such that the diagram in (27) commutes up to homotopy. Indeed, we have $v = \tilde{v}\tilde{\sigma}$, so the right square in (27) commutes. As for the central square, we have $\beta\tilde{u} = u$, thus

$$\beta\tilde{u} \sim u \implies \tilde{u} \sim \tilde{\sigma}u,$$

since $\tilde{\sigma}$ is the inverse of the homotopy equivalence β. This completes the proof that given a strict triangle (u, v, w) on A, B, C, the rotate

$$\begin{tikzcd}
& B \arrow[dl, "v"'] & \\
C[+1] \arrow[rr, "-w[1]"] & & A \arrow[ul, "u"']
\end{tikzcd}$$

is an exact triangle, and by the same reasoning as before, we conclude that this remains true when (u, v, w) is only an exact triangle on A, B, C.

(iii) We let $\mathbb{Z}/2[0]$ and $\mathbb{Z}/4[0]$ be the cochain complexes concentrated in degree 0. The claim is that there is no morphism $w : \mathbb{Z}/2[0] \to \mathbb{Z}/2[-1]$ such that

$$\begin{tikzcd}
& \mathbb{Z}/2[0] \arrow[dl, "w"'] & \\
\mathbb{Z}/2[0] \arrow[rr, "2"] & & \mathbb{Z}/4[0] \arrow[ul, "1"']
\end{tikzcd}$$

is an exact triangle. We suppose toward a contradiction that there is such a w.

Let $\varphi : \mathbb{Z}/2[0] \to \mathbb{Z}/4[0]$ be the extension of the map $\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4$. Then the exactness of $(\varphi, 1, w)$ on $\mathbb{Z}/2[0], \mathbb{Z}/4[0], \mathbb{Z}/2[0]$, the exactness of the strict triangle on $\varphi : \mathbb{Z}/2[0] \to \mathbb{Z}/4[0]$, and the the TR3 axiom of the triangulated category $\mathbf{K}(A)$ yield a morphism $\psi : \mathbb{Z}/2[0] \to \operatorname{cone}(\psi)$ such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{Z}/2[0] \arrow[r, "\varphi"] \arrow[d, "1"'] & \mathbb{Z}/4[0] \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{Z}/2[0] \arrow[r, "w"] \arrow[d, dashed, "\exists\psi"] & \mathbb{Z}/2[-1] \arrow[d, "1"] \\
\mathbb{Z}/2[0] \arrow[r, "\varphi"] & \mathbb{Z}/4[0] \arrow[r] & \operatorname{cone}(\varphi) \arrow[r] & \mathbb{Z}/2[-1].
\end{tikzcd}$$

The 5-lemma for exact triangles tells us that ψ is in fact a homotopy equivalence. Thus we have a quasi-isomorphism between $\mathbb{Z}/2[0]$ and $\operatorname{cone}(\varphi)$. But $H^{-1}(\mathbb{Z}/2[0]) = \{0\}$, whereas

$$H^{-1}(\operatorname{cone}(\varphi)) = \frac{\ker(\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4)}{\operatorname{im}(0 \to \mathbb{Z}/2)} = \ker(\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4) \neq \{0\}.$$

This contradiction tells us that there is no morphism w such that $(2, 1, w)$ is an exact triangle on $(\mathbb{Z}/2, \mathbb{Z}/4, \mathbb{Z}/2)$.

(iv) Let $\mathcal{D}$ be a triangulated category, and suppose we have a diagram

$$\begin{tikzcd}
A \arrow[r, "u"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] & TA \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'
\end{tikzcd},$$

where the rows are exact triangles. We assume that $v'gu = 0$, we want to show that there are maps $f : A \to A'$ and $h : C \to C'$ which assemble with g to get a map of exact triangles, i.e. fit into the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'
\end{tikzcd}.$$

To do this, recall that for all $X \in \mathcal{D}$, we have that $\operatorname{Hom}(X, -) : \mathcal{D} \to \mathrm{Ab}$ is a cohomological functor. In particular the following is an exact sequence of abelian groups

$$\operatorname{Hom}(A, A') \xrightarrow{(u')^*} \operatorname{Hom}(A, B') \xrightarrow{(v')^*} \operatorname{Hom}(A, C') \xrightarrow{(w')^*} \operatorname{Hom}(A, TA') \ .$$

The assumptions imply that $gu \in \operatorname{Hom}(A, B')$ is mapped to 0 in $\operatorname{Hom}(A, C')$. By exactness this gives a map $f : A \to A'$ such that post-composing with u' yields the composition gu. Now axiom (TR3) gives us a map $h : C \to C'$ such that

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'
\end{tikzcd}$$

commutes, which is what we wanted to show. This concludes the proof of this exercise.

(v) Let $\mathcal{A}$ be an abelian category, and consider the category of graded $\mathcal{A}$-objects viewed as the functor category $\mathcal{A}^{\mathbb{Z}}$, where $\mathbb{Z}$ is the set $\mathbb{Z}$ viewed as a discrete category. There is an obvious automorphism $\mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given by precomposing a functor $\mathbb{Z} \to \mathcal{A}$ by the the "+1" map, so that $T(A_\bullet)_n = A_{n-1}$ for some $\mathbb{Z}$-graded object in $\mathcal{A}$. We call a triangle $(A_\bullet, B_\bullet, C_\bullet, u, v, w)$ exact if for all n the sequence

$$A_n \xrightarrow{u} B_n \xrightarrow{v} C_n \xrightarrow{w} A_{n-1}$$

is exact in A.
Consider the case $\mathcal{A} = \mathrm{Ab}$, we claim that in this case the category of graded objects with the degree shift automorphisms satisfies (TR1) and (TR2), but not (TR3). The fact that satisfies (TR2) is immediate, and as such we omit the details. Now suppose we have a morphism $A_\bullet \xrightarrow{u_\bullet} B_\bullet$, we want to fit it in an exact triangle. For this, the graded object $C_n = \operatorname{coker}(u_n) \oplus A_{n-1}$ will do the job. Indeed, for the map $B_\bullet \to C_\bullet$ in degree n take the composite $B_n \to \operatorname{coker}(u_n) \to \operatorname{coker}(u_n) \oplus A_{n-1}$, and for the map $C_\bullet \to A_{\bullet-1}$ in degree n take the obvious projection map. It is now clear from the fact that exactness in a functor category can be checked objectwise that the image of $B_\bullet \to C_\bullet$ is the kernel of $C_\bullet \to A_{\bullet-1}$.
We now show that this category doesn't satisfy (TR3). Notice that exact triangles $(A_\bullet, B_\bullet, C_\bullet)$ with $A_\bullet$ concentrated in degree -1 and both $B_\bullet$ and $C_\bullet$ concentrated in degree 0 correspond to short exact sequences of abelian groups in an obvious way, and under this correspondence, if our category $\mathcal{A}^{\mathbb{Z}}$ were to satisfy (TR3) then any partial map of short exact sequences of abelian group

$$\begin{tikzcd}
0 \arrow[r] & B_0 \arrow[r] \arrow[d] & C_0 \arrow[r] & A_{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B'_0 \arrow[r] & C'_0 \arrow[r] & A'_{-1} \arrow[r] & 0
\end{tikzcd},$$

could be completed into a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & B_0 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & A_{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B'_0 \arrow[r] & C'_0 \arrow[r] & A'_{-1} \arrow[r] & 0
\end{tikzcd}.$$

This would in particular enable us to obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathbb{Z}/4\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd},$$

which would by the 5 lemma imply that $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which is obviously false, thus yielding the desired contradiction.
We will now show that if instead $\mathcal{A}$ is the category of vector spaces over a field, then the category of graded objects with the shift automorphism is in fact a triangulated category. The fact that axiom

(TR1) and (TR2) are satisfied follow by a reasoning perfectly analogous to the abelian group case. Now let us show that $(k-\mathrm{Vect})^{\mathbb{Z}}$ satisfies (TR3). Assume we have a partial map of exact triangles

$$\begin{tikzcd}
U_\bullet \arrow[r] \arrow[d] & V_\bullet \arrow[r] \arrow[d] & W_\bullet \arrow[r] & U_{\bullet-1} \arrow[d] \\
U'_\bullet \arrow[r] & V'_\bullet \arrow[r] & W'_\bullet \arrow[r] & U'_{\bullet-1}
\end{tikzcd}.$$

We need to find a map $W_\bullet \to W'_\bullet$ making the above diagram commute. This is equivalent to defining maps w_n for all $n \in \mathbb{Z}$ fitting in the following commutative diagrams with exact rows

$$\begin{tikzcd}
U_n \arrow[r] \arrow[d] & V_n \arrow[r] \arrow[d] & W_n \arrow[r] & U_{n-1} \arrow[d] \\
U'_n \arrow[r] & V'_n \arrow[r] & W'_n \arrow[r] & U'_{n-1}
\end{tikzcd}.$$

The fact that there exists a map $w_n : W_n \to W'_n$ making the above diagram commutes follows from basic linear algebra.

Finally, we show that this category satisfies the (TR4) axiom. To do this we introduce a different perspective on this axiom than the one seen in class, coming from the stacks project [**stacks-project**]. We want to insist on the fact that this is nothing more than a shift in perspective.

Suppose $\mathcal{D}$ is a category with an automorphism $T : \mathcal{D} \to FD$ and collection of distinguished triangles. We say that it satisfies (TR4) if given two composable morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ and distinguished triangles (A, B, Q_1, f, p_1, d_1), $(A, C, Q_2, g \circ f, p_2, d_2)$ and (B, C, Q_3, g, p_3, d_3), then there exists a fourth distinguished triangle $(Q_1, Q_2, Q_3, a, b, T(p_1) \circ d_3)$. And furthermore we require that the triple

$$(\mathrm{Id}_X, g, a) : (X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$$

is a morphism of triangles and

$$(f, \mathrm{Id}_Z, b) : (X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$$

as well. When using (TR4), we will allow ourselves to only refer to the morphism f, g as input, leaving the rest of the input as implicit.

We now prove that the category of graded vector spaces with the degree shift automorphism satisfies (TR4). First notice that by the (standard) 5-lemma and (TR2), up to isomorphism any exact triangle $(V_\bullet, V'_\bullet, V''_\bullet, a, b, c)$ of graded vector spaces can be identified with $(V_\bullet, V'_\bullet, \mathrm{coker}(a_\bullet) \oplus V_{\bullet-1}, a, \iota \circ q, \pi)$, where $\iota : \mathrm{coker}(a_\bullet) \to \mathrm{coker}(a_\bullet) \oplus V_{\bullet-1}$ is the obvious inclusion, $q : B \to \mathrm{coker}(a_\bullet)$ is the canonical map and $\pi : \mathrm{coker}(a_\bullet) \oplus V_{\bullet-1} \to V_{\bullet-1}$ is the obvious projection. With this in hand, we see that we may start with a diagram of the form

$$\begin{tikzcd}
U_\bullet \arrow[r, "f"] \arrow[drr, "g\circ f"'] & V_\bullet \arrow[rr] \arrow[dr, "g"] & & V_\bullet/U_\bullet \oplus U_{\bullet-1} \arrow[r] & U_{\bullet-1} \\
 & & W_\bullet \arrow[r] \arrow[dr] & W_\bullet/U_\bullet \oplus U_{\bullet-1} \arrow[r] & U_{\bullet-1} \\
 & & & W_\bullet/V_\bullet \oplus V_{\bullet-1} \arrow[r] & V_{\bullet-1}
\end{tikzcd}$$

where we denote the cokernel of a map by the quotient of the codomain by the domain and where the sequences of composable moprhisms which look like exact triangles are exact triangles. If we construct an exact triangle on the second to last column, along with some moprhisms of triangles, we will be done. This will be given by the following triangle, where we will detail the definitions of the morphisms but will leave the verification of exactness to the reader

$$V_\bullet/U_\bullet \oplus U_{\bullet-1} \xrightarrow{\tilde{g}\oplus \mathrm{Id}} W_\bullet/U_\bullet \oplus U_{\bullet-1} \xrightarrow{q\oplus f} W_\bullet/V_\bullet \oplus V_{\bullet-1} \xrightarrow{q\circ\pi} V_{\bullet-1}/U_{\bullet-1} \oplus U_{\bullet-2}.$$

So as to not overclutter the notation, we have omitted placeholder subscripts in our notation. We now detail all the maps: the map $\tilde{g}$ is the map induced by g on the quotients; the map $q : W_\bullet/U_\bullet \to W_\bullet/V_\bullet$ is the obvious quotient map; the map $\pi : W_\bullet/V_\bullet \oplus V_{\bullet-1} \to V_{\bullet-1}$ is the obvious projection map; and $q : V_{\bullet-1} \to V_{\bullet-1}/U_{\bullet-1}$ is the obvious quotient map. As we stated above, we leave the verification that all of these maps make $(V_\bullet/U_\bullet \oplus U_{\bullet-1}, W_\bullet/U_\bullet \oplus U_{\bullet-1}, W_\bullet/V_\bullet \oplus V_{\bullet-1})$ an exact triangle.
It only remains to verify that

$$(\mathrm{Id}, g, \tilde{g} \oplus \mathrm{Id}) : (U_\bullet, V_\bullet, V_\bullet/U_\bullet \oplus U_{\bullet-1} \to (U_\bullet, W_\bullet, W_\bullet/U_\bullet \oplus U_{\bullet-1})$$

and

$$(f, \mathrm{Id}, q \oplus f) : (U_\bullet, W_\bullet, W_\bullet/U_\bullet \oplus U_{\bullet-1}) \to (V_\bullet, W_\bullet, W_\bullet/V_\bullet \oplus V_{\bullet-1})$$

are moprhisms of exact triangles (we have allowed ourselves the abuse of notation of only referring to the objects of the exact triangles). But of these are completely obvious from our choice of maps.

(vi) Let $\mathcal{D}$ be a triangulated category and consider a commutative square in this category

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "u"'] & B \arrow[d] \\
A' \arrow[r] & B'
\end{tikzcd} .$$

We want to show that we can always extend such a square to a diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "u"'] & B \arrow[r, "j"] \arrow[d] & C \arrow[r, "k"] \arrow[d] & TA \arrow[d, "Tu"] \\
A' \arrow[r] \arrow[d, "v"'] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & TA' \arrow[d, "Tv"] \\
A'' \arrow[r] \arrow[d, "w"'] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & TA'' \arrow[d, "Tw"] \\
TA \arrow[r, "Ti"'] & TB \arrow[r, "Tj"'] & TC \arrow[r, "Tk"'] & T^2A
\end{tikzcd}$$

where all the rows and columns are exact triangles and all the squares commute except the bottom right one which anti-commutes. We will allow ourselves the abuse notation of suppressing the morphisms when specifying a triangle. First, we use four applications of (TR1) to extend our square to the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "u"'] & B \arrow[r, "j"] \arrow[d] & C \arrow[r, "k"] & TA \arrow[d, "Tu"] \\
A' \arrow[r] \arrow[d, "v"'] & B' \arrow[r] \arrow[d] & C' \arrow[r] & TA' \arrow[d, "Tv"] \\
A'' \arrow[d, "w"'] & B'' \arrow[d] & & TA'' \arrow[d, "Tw"] \\
TA \arrow[r, "Ti"'] & TB \arrow[r, "Tj"'] & TC \arrow[r, "Tk"'] & T^2A
\end{tikzcd} .$$

The commutativity of this diagram follows from the fact that the composite of two morphism of an exact triangle are 0. To conclude, we want to use axiom (TR4), and to do this, we add a temporary object, which will be useful in constructing the remaining morphisms, but won't appear in our final diagram. This object is obtained by applying axiom (TR1) to the map $A \to B'$, yielding the following diagram (where we have also added names to all of our maps for convenience):

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "u"'] \arrow[dr] & B \arrow[r, "j"] \arrow[d, "f"] & C \arrow[rr, "k"] & & TA \arrow[d, "Tu"] \\
A' \arrow[r, "\alpha"] \arrow[d, "v"'] & B' \arrow[rr, "\beta"] \arrow[dd, "g"] \arrow[dr, "m"] & & C' \arrow[r, "\gamma"] & TA' \arrow[d, "Tv"] \\
A'' \arrow[dd, "w"'] & & D \arrow[dr, "n"] & & TA'' \arrow[dd, "Tw"] \; . \\
& B'' \arrow[d, "h"] & & TA & \\
TA \arrow[r, "Ti"'] & TB \arrow[r, "Tj"'] & TC \arrow[rr, "Tk"'] & & T^2A
\end{tikzcd}$$

We will use the variant on axiom (TR4) mentionned in the previous exercise, which we do not recall here. First we have two applications of (TR4). One on the composition $A \xrightarrow{i} B \xrightarrow{f} B'$ and one on the composition $A \xrightarrow{u} A' \xrightarrow{\alpha} B'$. The first of these gives us an exact triangle $(C, D, B'', t_1, s_1, T(j) \circ h)$ along with the maps of triangles

$$(\mathrm{Id}, f, t_1) : (A, B, C) \to (A, B', D)$$

and

$$(i, \mathrm{Id}, s_1) : (A, B', D) \to (B, B', B'').$$

The second application of (TR4) gives us an exact trainge $(A'', D, C', t_2, s_2, Tv \circ \gamma)$ along with maps of triangles

$$(\mathrm{Id}, \alpha, t_2) : (A, A', A'') \to (A, B', D)$$

and

$$(u, \mathrm{Id}, s_2) : (A, B', D) \to (A', B', C').$$

Now considering the composite $s_1 \circ t_2 : A'' \to B''$, and applying (TR1) to get an exact triangle $(A'', B'', C'', s_1 \circ t_2, \sigma, \tau)$ and (TR2) on $(C, D, B'', t_1, s_1, T(j) \circ h)$ to get an exact triangle $(D, B'', TC, s_1, T(j) \circ h, -Tt_1)$, we have the necessary set up to apply (TR4) to the composition $s_1 \circ t_2$. This gives us an exact triangle $(C', C'', TC, p, q, Ts_2 \circ (-Tt_1))$, and two morphisms of triangles

$$(\mathrm{Id}, s_1, p) : (A'', D, C') \to (A'', B'', C'')$$

and

$$(t_2, \mathrm{Id}, q) : (A'', B'', C'') \to (D, B'', TC).$$

We can rotate the triangle $(C', C'', TC, p, q, Ts_2 \circ (-Tt_1))$, to obtain the triangle $(C, C', C'', s_2 \circ t_1, p, q)$. We can use all of this to construct the following diagram, whose commutativity will occupy us for the rest of this exercise

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "u"'] & B \arrow[r, "j"] \arrow[d, "f"] & C \arrow[r, "k"] \arrow[d, "s_2 \circ t_1"] & TA \arrow[d, "Tu"] \\
A' \arrow[r, "\alpha"] \arrow[d, "v"'] & B' \arrow[r, "\beta"] \arrow[d, "g"] & C' \arrow[r, "\gamma"] \arrow[d, "p"] & TA' \arrow[d, "Tv"] \; . \\
A'' \arrow[r, "s_1 \circ t_2"] \arrow[d, "w"] & B'' \arrow[r, "\sigma"] \arrow[d, "h"] & C'' \arrow[r, "\tau"] \arrow[d, "q"] & TA'' \arrow[d, "Tw"] \\
TA \arrow[r, "Ti"] & TB \arrow[r, "Tj"] & TC \arrow[r, "Tk"] & T^2A
\end{tikzcd}$$

In what follows we will refer to triangles only by their objects. Commutativity of the three squares of the first column follows from composing the morphisms of triangles $(A, A', A'') \to (A, B', D)$ and

$(A, B', D) \to (B, B', B'')$. Similarly, the three sqaures of the top row commute by composing the morphisms of triangles $(A, B, C) \to (A, B', D)$ and $(A, B', D) \to (A', B', C')$. The middle square commutes, by adding D in the middle, and the fact that in the following diagram all 4 triangles commute, as can be seen by inspecting the various morphisms of triangles (TR4) has granted us

$$\begin{tikzcd}
B' \arrow[rr, "\beta"] \arrow[dr, "m"] \arrow[dd, "g"'] & & C' \arrow[dd, "p"] \\
& D \arrow[ur, "s_2"'] \arrow[dl, "s_1"] \arrow[dr, "p \circ s_2"] & \\
B'' \arrow[rr, "\sigma"] & & C''
\end{tikzcd} .$$

The bottom middle square commute thanks to the morphism of triangles $(A'', B'', C'') \to (D, B'', TC)$ and because the map $B'' \to TC$ is $T(j) \circ h$. Similarly, the middle right square commutes thanks to the map of triangles $(A'', D, C') \to (A'', B'', C'')$ and the fact that the map $C' \to TA''$ factors as $C' \to TA \to TA''$. For anticommutativity of the bottom right diagram, because $(A'', B'', C'') \to (D, B'', TC)$ is a morphism of triangles, we have that the following square commutes

$$\begin{tikzcd}
C'' \arrow[r, "\tau"] \arrow[d, "q"] & TA'' \arrow[d, "Tt_2"] \\
TC \arrow[r, "-Tt_1"] & TD
\end{tikzcd} .$$

We can now post compose by $Tn : TD \to T^2A$, which because $(A, A', A'') \to (A, B', D)$ and $(A, B, C) \to (A, B', D)$ are morphisms of triangles, gives us a diagram

$$\begin{tikzcd}
C'' \arrow[r, "\tau"] \arrow[d, "q"] & TA'' \arrow[d, "Tt_2"] \arrow[ddr, bend left, "Tw"] & \\
TC \arrow[r, "-Tt_1"] \arrow[drr, bend right, "Tk"'] & TD \arrow[dr, "Tn"] & \\
& & T^2A
\end{tikzcd} ,$$

which proves the deisred anticomutativity. This concludes the exercise.

6.12 Week 12 (by Haotian Lyu and Runchi Tan)

Week 12 Ex 1

Proof. First, the boundary map of $(C \oplus A[-1]) \oplus (B[-1] \oplus A[-2])$ is given by the following matrix

$$\mathcal{A} = \begin{pmatrix} \partial & -vu & -v & 0 \\ 0 & -\partial & 0 & -\mathrm{id} \\ 0 & 0 & -\partial & u \\ 0 & 0 & 0 & \partial \end{pmatrix}.$$

Let d denote the boundary map of $C \oplus A[-1]$.

(i) By definition, one can check

$$(\gamma \circ (\mathrm{id} \oplus u) - \iota)(c, a) = (0, -a, u(a), 0).$$

Define the chain contraction map

$$s_n : C_{n-1} \oplus A_{n-2} \to C_n \oplus A_{n-1} \oplus B_{n-1} \oplus A_{n-2}$$

by $(c, a) \mapsto (0, 0, 0, a)$. By direct calculation, it is clear that

$$\gamma \circ (\mathrm{id} \oplus u) - \iota = \mathcal{A}s + sd.$$

Then we finish the proof of (a).

(ii) By direct calculation it is easy to check that $g \circ \gamma = \mathrm{id}_{C \oplus B[-1]}$. To show that γ defines a homotopy equivalence between the two chain complexes, only need to check $\gamma \circ g$ is homotopic to $\mathrm{id}_{C_n \oplus A_{n-1} \oplus B_{n-1} \oplus A_{n-2}}$. Since

$$\gamma \circ g : C_n \oplus A_{n-1} \oplus B_{n-1} \oplus A_{n-2} \to C_n \oplus A_{n-1} \oplus B_{n-1} \oplus A_{n-2}$$

$$(c, a, b, a') \mapsto (c, 0, u(a) + b, 0).$$

Define the chain contraction map

$$t_n : C_{n-1} \oplus A_{n-2} \oplus B_{n-2} \oplus A_{n-3} \to C_n \oplus A_{n-1} \oplus B_{n-1} \oplus A_{n-2}$$

$$(c, a, b, a') \mapsto (0, 0, 0, a).$$

Then one can check that $\gamma \circ g - \mathrm{id} = \mathcal{A}t + t\mathcal{A}$. Therefore we conclude that γ with g defines a homotopy equivalence between the two chain complexes.

□

Week 12 Ex 2

(a)

Proof. Given $Z \simeq 0$ in $\mathcal{C}$, then for any $A, B \in \mathcal{S}^{-1}\mathcal{C}$ we can construct $f : A \to Z$ and $g : Z \to B$ for morphisms f, g in $\mathcal{C}$. Therefore we can construct morphisms $A = A \to Z$ and $Z = Z \to B$ who live inside $\mathcal{S}^{-1}\mathcal{C}$. This proves that $q(Z)$ is both inital and terminal in $\mathcal{S}^{-1}\mathcal{C}$. Therefore $q(0) \simeq 0 \in \mathcal{S}^{-1}\mathcal{C}$, and

$$q(X) \simeq 0 \simeq q(0).$$

Assume $\mathcal{S}$ contains zero map $0 : X \to X$. Equivalently, we can apply Corollary 10.3.9., $0\,\mathrm{id} = 00 = 0$ implies we can identify $0, \mathrm{id} : X \to X$ in $\mathcal{S}^{-1}\mathcal{C}$. And this proves $q(X) \simeq 0 \in \mathcal{S}^{-1}\mathcal{C}$. □

(b)

Proof. Since $X \times Y$ exists in $\mathcal{C}$, then there exists a unique morphism $X \times Y \to A$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & X \times Y \arrow[d, dashed, "\exists ! f"] & Y \arrow[l] \arrow[dl] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
q(X) \arrow[r] \arrow[dr] & q(X \times Y) \arrow[r] \arrow[d, dashed] & q(Y) \arrow[dl] \\
& q(A) = A &
\end{tikzcd}$$

Here we define the morphism between $q(X \times Y) = q(X \times Y) \to q(A)$ by $q(f) \circ \mathrm{id}$. Uniqueness and existence implies $q(X \times Y) \simeq q(X) \times q(Y)$. □

(c)

Proof. Firstly, we'll check $\mathcal{S}^{-1}\mathcal{C}$ is **Ab**-category. We'll define addition of two fractions from $X \to Y$ as follows:

$$\begin{tikzcd}
& {} \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
X & X_2 \arrow[l, dashed, "s"] \arrow[u, "g_1'"] \arrow[d, "g_2'"'] \arrow[r, dashed, bend left, "f_1'"] \arrow[r, dashed, bend right, "f_2'"'] & Y \\
& {} \arrow[ul, "s_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& {} \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
X & X_2 \arrow[l, "s"] \arrow[u, "g_1'"] \arrow[r, "f_2'"] \arrow[d, "\mathrm{id}"] & Y \\
& X_2 \arrow[ul, "s"] \arrow[ur, "f_2'"'] &
\end{tikzcd}$$

Given $f_1 s_1^{-1}, f_2 s_2^{-1}$, applying Ore condition we'll find an $g_1', g_2' \in \mathcal{S}$ such that the square on the left commutes. Thus we obtain $s := s_1 \circ g_1' = s_2 \circ g_2'$, $f_1' := f_1 \circ g_1'$, and $f_2' := f_2 \circ g_2'$. Now we define addition on $\mathrm{Hom}_{\mathcal{S}^{-1}\mathcal{C}}(X, Y)$ to be

$$f_1 s_1^{-1} + f_2 s_2^{-1} := (f_1' + f_2') s^{-1}.$$

The diagram on the right shows that by filling identities appropriately it follows $f_1 s^{-1} \sim f_1' s_1^{-1}$ and similarly $f_2 s_2^{-1} \sim f_2' s^{-1}$. We need to verify the sum above is well-defined:

$$\begin{tikzcd}
 & {} \arrow[dl, "s_1"'] \arrow[drrr, "f_1"] & & {} \arrow[dlll, "s_{11}"'] \arrow[dr, "f_{11}"] & \\
X & & {} \arrow[ll, bend right, "s'"'] \arrow[ll, bend left, "s"] & & Y \\
 & {} \arrow[ul, "s_2"] \arrow[urrr, "f_2"'] & & {} \arrow[ulll, "s_{22}"] \arrow[ur, "f_{22}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {} \arrow[dl, "s"'] \arrow[dr, "f_1'+f_2'"] & \\
X & {} \arrow[l, dashed] \arrow[r, dashed] \arrow[u, "h_1"] \arrow[d, "h_2"'] & Y \\
 & {} \arrow[ul, "s'"] \arrow[ur, "f_{11}'+f_{21}'"'] &
\end{tikzcd}$$

Suppose we choose an entirely different but equivalent fractions as above, with the original set of fractions produce s and new set of morphisms produce s'. Then apply Ore condition gives rise to the diagram on the right, with $f_{11}' + f_{21}'$ defined by applying exactly same argument of the above definition. Notice that by construction both $s, s' \in \mathcal{S}$, then Ore condition gives rise to a middle term as in the right diagram with h_1, h_2, then we can define the remaining by composing h_1 with $s, f_1' + f_2'$ separately. Hence it remains to check $(f_1' + f_2')h_1 = (f_{11}' + f_{21}')h_2$. Recall the definition of $f_1' = f_1 \circ g_1'$. Then we replace f_1' by $f_1 \circ g_1'$, while $f_1' s^{-1} \sim f_{11}' s'^{-1}$, then we must have $f_1' h_1 = f_{11}' h_2$. Hence the addition is indeed well-defined.
And we can easily verify the composition is distributive over addition. By previous two parts we know $\mathcal{S}^{-1}\mathcal{C}$ admits 0 and product of two objects, therefore it's an additive category.
It remains to check q is an additive functor. We need to check the following is a group homomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{S}^{-1}\mathcal{C}}(qX, qY)$$

Consider two parallel morphisms $f, g : X \to Y$ in $\mathcal{C}$, then $qf : X = X \to Y$, $qg : X = X \to Y$. And we have

$$qf + qg = f\,\mathrm{id}^{-1} + g\,\mathrm{id}^{-1} = (f + g)\,\mathrm{id}^{-1} = q(f + g) : X = X \to Y.$$

This confirms that q is indeed an additive functor. □

Week 12 Ex 3

Proof.

- We first check that $\textbf{mod-}S^{-1}R \cong \sum^{-1}\textbf{mod-}R$. Denote $\textbf{mod-}R$ by $\mathcal{C}$ and $\textbf{mod-}S^{-1}R$ by $\mathcal{D}$. Let $q : \mathcal{C} \to \mathcal{D}$ be the natural functor that sends an $R-$mod to its localization by S.
    (i) By the definition of $\sum$, it is clear that $q(s)$ is always an isomorphism.
    (ii) For each functor $F : \mathcal{C} \to \mathcal{B}$ such that $F(s)$ is an isomorphism for all $s \in S$. For each $M \in \mathcal{D}$ (or $M \to N \in \mathcal{D}$), it is naturally to be regarded as an R-mod M (or an $R-$ morphism $M \to N$). Therefore we can define a functor $\tilde{F} : \mathcal{D} \to \mathcal{B}$. According to (a) $\tilde{F}$ is well defined. It is unique because every $S^{-1}R-$mod has a lift as an $R-$mod.

    Therefore we conclude that $\mathcal{D} \cong \sum^{-1} \mathcal{C}$.
- Next we prove that $\sum$ is a multiplicative system.
    (i) The composition of two isomorphisms is still an isomorphism, then $\sum$ is closed under composition.
    (ii) (Ore condition) Let $t : Z \to Y$ is in $\sum$ and $g : X \to Y$ is in $\mathcal{C}$. Set

    $$W := \{(x, z) \in X \times Z | g(x) = t(z)\}.$$

I.e. the following diagram is a pullback diagram, where s and f are natural projections.

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "s"'] & Z \arrow[d, "t"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

Now we prove $s \in \sum$. In fact, for each $x/l \in S^{-1}X$. Consider its image $g(x/l)$, which corresponds to z/l' in $S^{-1}Z$. Now we claim $(x/l, z/l') \in S^{-1}W$. This is because $t(z/l') = g(x/l)$ implies there exists $a \in S$ such that $al \cdot t(z) = al' \cdot g(x)$. Therefore $(al'x, alz) \in W$ and $(x/l, z/l') = \frac{(al'x,alz)}{all'} \in S^{-1}W$. Thus $s : S^{-1}W \to S^{-1}X$ is surjective. As for injectivity, notice that if $(x, z)/l$ is sent to 0, i.e. $x/l = 0 \in S^{-1}X$, then $t(z/l) = g(x/l) = 0$. But t induces an isomorphism, which means $z/l = 0$. Then we have proved the injectivity. Therefore we conclude $s \in \sum$.

Dually, we consider the case with all arrows reversed. We choose W as follows:

$$X \oplus Z/\{(g(y), -t(y))|y \in Y\}$$

I.e. the following diagram is a pushout diagram, where s and f are composition of natural injections into $X \oplus Z$ and the quotient map onto $X \oplus Z/\{(g(y), -t(y))|y \in Y\}$.

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "t"'] & X \arrow[d, "s"] \\
Z \arrow[r, "f"'] & W
\end{tikzcd}$$

Now we prove $s \in \sum$.

- Surjectivity: for each $(x, z)/l \in S^{-1}W$, consider $y/l' \in S^{-1}Y$ such that $t(y/l') = z/l$. Then $(x, z)/l = (x/l + g(y)/l', 0)$ in $S^{-1}W$. Thus $(x, z)/l$ has preimage $x/l + g(y)/l'$ in $S^{-1}X$, which proves the surjectivity.
- Injectivity: If for some $x/l \in S^{-1}X$, such that $(x/l, 0) = 0$ in $S_{-1}W$. This means $(x/l, 0) = (g(y), -t(y)) \in S^{-1}(X \oplus Z)$ for some $y \in S^{-1}Y$. But $t(y) = 0$ implies $y = 0$,which means $x/l = g(y) = 0$. This proves the injectivity.

(iii) (Cancellation) In this case, both (a) and (b) in the textbook, mean that f and g induce the same morphism $S^{-1}X \to S^{-1}Y$. To finish the proof, we only need to check that if f and g induce the same morphism $S^{-1}X \to S^{-1}Y$, then there exist $s \in S$ with source Y and $t \in S$ with target X such that $sf = sg$ and $ft = gt$.

As for s, choose $s : Y \in S^{-1}Y$, then two compositions $X \to S^{-1}Y$ can be extended to $S^{-1}X \to S^{-1}Y$ naturally, which are just induced by f and g. Hence $sf = sg$.

As for t, choose t to be the natural injection

$$Z := \{x \in X|f(x) = g(x)\} \to X.$$

Now we prove $t \in \sum$. Since t is injective, the map $S^{-1}Z \to S^{-1}X$ induced by t is also injective. For each $x/s \in S^{-1}X$, we have $f(x/s) = g(x/s)$ since f and g induce the same map. Thus we have $f(x)/s = g(x)/s$, i.e. there exists $s' \in S$ such that $ss'f(x) = ss'g(x)$. Thus $(ss')^{-1}x \in S^{-1}W$ is the preimage, which proves the surjectivity.

Therefore $\sum$ is a multiplicative system.

- In the end we prove $\sum^{-1} \mathcal{D} \to \sum^{-1} \mathcal{C}$ is fully faithful. It is trivial because $\sum^{-1} \mathcal{D} = \mathcal{D}$.

Hence we finish the proof. □

Week 12 Ex 4

(a)

(i) We need to show Σ is closed under compositions and contains all identities. Clearly, all identities belong to Σ for $\operatorname{Ker}\mathrm{id} = \operatorname{Coker}\mathrm{id} = 0 \in \mathcal{B}$. Fix two morphisms $f, g \in \Sigma$. Consider the following diagram

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "gf"] & B \arrow[r] \arrow[d, "g"] & \operatorname{Coker} f \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & C \arrow[r, "\mathrm{id}"] & C \arrow[r] & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & A \arrow[r, "\mathrm{id}"] \arrow[d, "f"] & A \arrow[r] \arrow[d, "gf"] & 0 \\
0 \arrow[r] & \operatorname{Ker} g \arrow[r] & B \arrow[r, "g"] & C &
\end{tikzcd}$$

Applying Snake's Lemma to the diagrams yields the following exact sequence

$$0 \to \mathrm{Ker} f \to \mathrm{Ker} gf \to \mathrm{Ker} g \to \mathrm{Coker} f \to \mathrm{Coker} gf \to \mathrm{Coker} g \to 0,$$

which will implies that $\mathrm{Ker} gf, \mathrm{Coker} gf \in \mathcal{B}$. Hence Σ is closed under composition.

(ii) Ore condition. In Abelian category, pullback exists. Therefore we obtain the following diagram with all objects in $\mathcal{A}$.

$$\begin{tikzcd}
D \arrow[r] \arrow[d, "s"] & C \arrow[d, "f \in \Sigma"] \\
A \arrow[r, "\forall\ g \in \mathcal{A}"] & B
\end{tikzcd}$$

Here we start with a morphism $f \in \Sigma$ and fix a morphism $g \in \mathcal{A}$. According to Tag 08N3, we know $\mathrm{Ker} s = \mathrm{Ker} f$. By construction $D = \mathrm{Ker}(A \oplus C \to B)$, which gives us $\mathrm{Coker} s \to \mathrm{Coker} f$ is surjective(or we can fill the square diagram into exact sequences where Snake's Lemma implies it's sujective). This proves that $s \in \Sigma$ as desired.

(iii) For Cancellation requirement(Tag 02MN), consider parallel morphisms $f, g : B \to C$. Suppose there exists $s \in \Sigma$ such that $fs = gs$. Then $(f - g)s = 0$, which implies a surjection

$$\mathrm{Coker}(s) \to \mathrm{Image}(f - g) \to 0.$$

Then we can consider the morphism

$$B \overset{f-g}{\to} C \to C/\mathrm{Image}(f - g) \overset{\pi}{\to} 0.$$

The composition equals to 0 implies we've found a morphism π such that $\pi f = \pi g$ as expected. For converse direction it's similar. Therefore we've checked the cancellation condition.

Therefore Σ is a multiplicative system in $\mathcal{A}$.

(b)

Proof. Let q denotes the map $q : \mathcal{A} \to \mathcal{A}/\mathcal{B} = \Sigma^{-1}\mathcal{A}$. Then $q(X) \simeq 0$ iff $0 : X \to X$ contained in Σ by Exercise 2. While $X = \mathrm{Ker} 0$, therefore $X \in \mathcal{B}$. Conversely, we assume $X \in \mathcal{B}$. Then $X \to 0 \to X = 0$ implies $0 : X \to X$ lies in $\mathcal{B}$, hence $q(X) = 0$. $\square$

(c)

Proof. We will show that multiplicative system Σ is locally small (on the left) in the category $\mathcal{A}$. Fix an object $X \in \mathcal{A}$. Since $\mathcal{B}$ is small, then we can consider the set as follows

$$\Sigma_X = \{f: \ A \to X \mid f \in \Sigma, \ A \in \operatorname{Obj} \mathcal{B}\}.$$

For any given morphism in $\mathcal{A}$ such that $(g : Y \to X) \in \Sigma$. Then the composition

$$\mathrm{Ker} g \to Y \overset{g}{\to} X \in \Sigma_X$$

since $\mathrm{Ker} g \in \mathcal{B}$. Therefore we know that Σ is locally small. $\square$

(d)

Proof. By Exercise 2, since $\mathcal{A}$ is additive, then $\mathcal{A}/\mathcal{B}$ is additive with functor q being additive. The proof of previous parts show that Σ is a *right multiplicative system* in the sense of Tag 04VB. We can run the argument of part (a) again to prove Σ is in fact a left multiplicative system. Therefore a multiplicative system, in which we're ready to apply Tag 05QG to conclude that $\mathcal{A}/\mathcal{B}$ is abelian and q is an exact functor. $\square$

(e)

Proof. Notice that B-$\mathrm{Mod}_S R$ being *Serre subcategory* translates to the fact that if we fix a short exact sequence with objects in B-$\mathrm{Mod}R$,

$$0 \to A \to B \to C \to 0.$$

Then $A, C \in B$-$\mathrm{Mod}_S R$ iff $B \in B$-$\mathrm{Mod}_S R$. While localization(ring theoretically) is an exact functor, then this proves the equivalence. So B-$\mathrm{Mod}_S R$ is a Serre subcategory of B-Mod_R.
Notice that the following two sets are exactly the same because localization is exact:

- all morphisms in B-$\mathrm{Mod} - R$ with kernel and cokernel in B-$\mathrm{Mod}_S R$
- the morphism described in Exercise 3: the collection of morphisms in B-$\mathrm{Mod} - R$ such that the induced morphism on $S^{-1}A \to S^{-1}B$ is an isomorphism.

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r] & A \arrow[r] & B \arrow[r] & \operatorname{Coker} f \\
0 \arrow[r] & S^{-1}A \arrow[r] & S^{-1}B \arrow[r] & 0
\end{tikzcd}$$

Therefore we can apply Exercise 3, which gives us

$$B\text{-Mod} - S^{-1}R \simeq \Sigma^{-1}B\text{-Mod} - R \simeq B\text{-Mod} - R/B\text{-Mod}_S R.$$

$\square$

(i) First, we define a **special triangle** is a triangle

$$X \to Y \to Z \to TA$$

satisfies that for each object W of $\mathcal{K}$ the long sequence of abelian groups

$$\cdots \to \mathrm{Hom}_{\mathcal{K}}(W, X) \to \mathrm{Hom}_{\mathcal{K}}(W, Y) \to \mathrm{Hom}_{\mathcal{K}}(W, Z) \to \mathrm{Hom}_{\mathcal{K}}(W, TX) \to \dots$$

is exact. It is obvious that the direct sum of two special triangles is still special, and exact triangles are special. We need the following lemma.

Lemma 6.8. *Let $\mathcal{K}$ be a triangulated category. Let*

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

be a morphism of special triangles. If two among a, b, c are isomorphisms so is the third.

Proof. Assume that a and c are isomorphisms. For any object W of $\mathcal{D}$ write $H_W(-) = Hom_{\mathcal{D}}(W, -)$. Then we get a commutative diagram of abelian groups

$$\begin{tikzcd}
H_W(T^{-1}Z) \arrow[r] \arrow[d] & H_W(X) \arrow[r] \arrow[d] & H_W(Y) \arrow[r] \arrow[d] & H_W(Z) \arrow[r] \arrow[d] & H_W(TX) \arrow[d] \\
H_W(T^{-1}Z') \arrow[r] & H_W(X') \arrow[r] & H_W(Y') \arrow[r] & H_W(Z') \arrow[r] & H_W(TX')
\end{tikzcd}$$

By assumption the right two and left two vertical arrows are bijective. By five lemma it follows that the middle vertical arrow is an isomorphism. Hence by Yoneda's lemma, we see b is an isomorphism.This implies the other cases by rotating. ☐

Back to the problem. Since (A, B, C) and (A', B', C') are exact triangle, then $(A \oplus A', B \oplus B', C \oplus C')$ is special. By (TR1) we can extend $A \oplus A' \to B \oplus B'$ to an exact triangle $(A \oplus A', B \oplus B', D)$. Thus by (TR 3) we have a natural morphism $(A, B, C) \to (A \oplus A', B \oplus B', D)$ induced by $A \to A \oplus A'$ and $B \to B^{\oplus}B'$. Similarly we have $(A', B', C') \to (A \oplus A', B \oplus B', D)$. These two morphism induce a morphism

$$(\mathrm{id}, \mathrm{id}, \varphi) : (A \oplus A', B \oplus B', C \oplus C') \to (A \oplus A', B \oplus B', D).$$

According to the above lemma φ is an isomorphism. Therefore by (TR1) $(A \oplus A', B \oplus B', C \oplus C')$ is also an exact triangle.

(ii) According to (TR1), the triangles $(A, A, 0)$ and $(C, C, 0)$ are exact. By rotating we have $(0, C, C)$ is exact. By the above point, it follows

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, equal, "\mathrm{id}"'] & B \arrow[r, "v"] \arrow[d] & C \arrow[r, "w"] \arrow[d, equal, "\mathrm{id}"'] & TA \arrow[d, equal, "\mathrm{id}"] \\
A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r, "0"] & 0
\end{tikzcd}$$

is a morphism of exact triangles. By the lemma above, we have

$$B \cong A \oplus C.$$

(iii) Extend $f : A \to B$ to an exact triangle $A \to B \to D \to TA$. By rotating we have an exact triangle $T^{-1}D \to A \to B \to D$. Since the composition $A \to D$ is zero and f is monic, we have $B \to D$ is zero. Therefore $B \cong A \oplus T^{-1}D$.

(iv) Denote $\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ by ψ. If $K(\mathbb{A}b)$ is abelian, we have an exact triangle

$$\ker(\psi) \to \mathbb{Z}/p^2\mathbb{Z} \to D \to \ker(\psi)[-1]$$

where the first arrow is monic. But $\mathbb{Z}/p^2\mathbb{Z}$ can not be decomposed, which is a contradiction.

6.13 Week 13 (by Jorge Martín and Alexandre Pons).

1 (i) Consider the cochain complex in $\mathbf{Ch}(\mathbf{Ab})$

$$A : 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \to 0,$$

continued by zeros at both sides. Set $f := \mathrm{id}_A$ to be the identity on A. Then f has the required properties, i.e. $f = 0$ in $\mathbb{D}(\mathbf{Ab})$ but is not null-homotopic.

We have seen that id_A is null-homotopic if and only if the complex A is split. Here, since $\mathsf{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$, then A is not split and the second claim about f follows.

To see that $f = 0$ in $\mathbb{D}(\mathbf{Ab})$, it suffices to show that $sf = s0 = 0$ for some quasi-isomorphism s, using the characterisation of identified maps after localizing. We may take $s = 0 : A \to 0$. Observe that A is exact, and therefore its cohomology vanishes. Hence, the map induced by s in cohomology is the isomorphism $0 \to 0$ at all levels, and therefore s is indeed a quasi-isomorphism.

(ii) Here, consider the cochain complexes X, Y and the map $g : X \to Y$ as in the following diagram:

$$\begin{tikzcd}
X : \arrow[d, "g"] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "1"] & \mathbb{Z} \arrow[r] \arrow[d, "2"] & 0 \\
Y : & 0 \arrow[r] & \mathbb{Z} \arrow[r, "1"] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Then g induces the zero map in cohomology, but does not vanish in $\mathbb{D}(\mathbf{Ab})$. For the first claim, we directly compute the induced diagram in cohomology and check that the induced map vanishes:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "0"] & 0 \arrow[r] & 0
\end{tikzcd}$$

However, let us see that $g \neq 0$ in $\mathbb{D}(\mathbf{Ab})$. Otherwise, suppose there exists a complex W and a quasi-isomorphism $s : Y \to W$ such that $sg = 0$. Then, given that s induces isomorphisms in cohomology, the cohomology of W is isomorphic to that of Y. In particular, at level 2 in the diagram above we have $H^2(W) \cong H^2(Y) \cong \mathbb{Z}$. Hence s induces a non-zero isomorphism and thus $s^2 : Y^2 \to W^2$ is not the zero map. Since g^2 is an isomorphism, then s^2g^2 is non-zero and therefore also $sg \neq 0$. We conclude that the composition of g with any quasi-isomorphism is non-zero, i.e. $g \neq 0$ in $\mathbb{D}(\mathbf{Ab})$.

2 We show that $\mathcal{S}^{-1}\mathcal{K}$ satisfies the axioms TR1, TR2 and TR3 of a triangulated category.

Notice that the definition of triangles in $\mathcal{S}^{-1}\mathcal{K}$ in Weibel's book was replaced in the erratum by the following:

Definition 6.9. A tuple (u, v, w) is an exact triangle in $\mathcal{S}^{-1}\mathcal{K}$ if it is isomorphic, in the sense of (TR1), to the image under $\mathcal{K} \to \mathcal{S}^{-1}\mathcal{K}$ of an exact triangle in $\mathcal{K}$.

We will show the statement using this definition, as the one in the original version is somewhat incomplete.

(TR1) Recall that the first axiom consists of three statements. For the first one, given a morphism $A \overset{s}{\leftarrow} A' \overset{f}{\to} B$ in $\mathcal{S}^{-1}\mathcal{K}$, we need to show that fs^{-1} fits into an exact triangle in $\mathcal{S}^{-1}\mathcal{K}$.

We know that f fits into an exact triangle

$$\Delta' : A' \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} T(A')$$

in the triangulated category $\mathcal{K}$. Its image under $\mathcal{K} \to \mathcal{S}^{-1}\mathcal{K}$ is represented by

$$A' \overset{\mathrm{id}_{A'}}{=} A' \overset{f}{\to} B \overset{\mathrm{id}_B}{=} B \overset{g}{\to} C \overset{\mathrm{id}_C}{=} C \overset{h}{\to} T(A')$$

In the following, we will abuse notation and consider indistinctly a triangle in $\mathcal{K}$ and its image in $\mathcal{S}^{-1}\mathcal{K}$, in this case Δ'. We will also omit the middle object in fractions when it is clear. Next, we may consider in $\mathcal{S}^{-1}\mathcal{K}$ the triangle

$$\Delta : A \overset{fs^{-1}}{\to} B \overset{g}{\to} C \overset{T(s)h}{\to} T(A).$$

Then, the following commutative diagram represents a morphism of triangles $\Delta' \to \Delta$ in $\mathcal{S}^{-1}\mathcal{K}$

$$\begin{tikzcd}
A' \arrow[r, "f"] \arrow[d, "s"] & B \arrow[r, "g"] \arrow[d, "\mathrm{id}_B"] & C \arrow[r, "h"] \arrow[d, "\mathrm{id}_C"] & T(A') \arrow[d, "T(s)"] \\
A \arrow[r, "fs^{-1}"] & B \arrow[r, "g"] & C \arrow[r, "T(s)h"] & T(A)
\end{tikzcd}$$

This is actually an isomorphism, as the maps s, id_B and id_C are invertible in $\mathcal{S}^{-1}\mathcal{K}$. Thus Δ is isomorphic to (the image of) the exact triangle Δ', and hence is an exact triangle.

The second and third statements follow trivially from the definition. Indeed, the triangle $(\mathrm{id}_A, 0, 0)$ in $\mathcal{S}^{-1}\mathcal{K}$ is an exact triangle in $\mathcal{K}$, whereas a triangle isomorphic to an exact triangle is exact by transitivity.

(TR2) Given an exact triangle (u, v, w) in $\mathcal{S}^{-1}\mathcal{K}$, we need to see that its rotates are also exact triangles. For that, consider an exact triangle (u', v', w') in $\mathcal{K}$ isomorphic to (u, v, w) in $\mathcal{S}^{-1}\mathcal{K}$, via a morphism (s, t, r). Then, inspecting the corresponding diagrams allows to verify that $(v, w, -T(u))$ is isomorphic to $(v', w', -T(u'))$ via $(t, r, T(s))$ and $(-T^{-1}(w), u, v)$ is isomorphic to $(-T^{-1}(w'), u', v')$ via $(T^{-1}(r), s, t)$. Therefore, both rotates are exact triangles.

(TR3) Consider triangles $A \overset{u}{\to} B \overset{v}{\to} C \overset{w}{\to} T(A)$ and $A' \overset{u'}{\to} B' \overset{v'}{\to} C' \overset{w'}{\to} T(A')$ in $\mathcal{S}^{-1}\mathcal{K}$ and morphisms $f : A \to A'$, $g : B \to B'$ such that $u'f = gu$. Then, we need to find h such that (f, g, h) is a morphism of triangles.

Let us define objects X, Y and morphisms $s, t, \tilde{f}, \tilde{g}$ with $f = \tilde{f}s^{-1}$ and $g = \tilde{g}t^{-1}$, as in the following diagram.

$$\begin{tikzcd}
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & T(A) \\
X \arrow[u, "s"'] \arrow[d, "\tilde{f}"'] & Y \arrow[u, "t"'] \arrow[d, "\tilde{g}"'] & Z \arrow[u, dashed, "r"'] \arrow[d, dashed, "\tilde{h}"'] & T(X) \arrow[u, "T(s)"'] \arrow[d, "T(\tilde{f})"'] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & T(A')
\end{tikzcd}$$

Thus, our goal is to find Z and a morphism $h : C \overset{r}{\leftarrow} Z \overset{\tilde{h}}{\to} C'$ that makes the diagram commutative. For that, we want to complete the middle row and use (TR3) of the category $\mathcal{K}$. First, the Ore condition gives X' and maps $\tilde{u}$ in $\mathcal{K}$ and $\tilde{t}$ in $\mathcal{S}$ such that the diagram

$$\begin{tikzcd}
X' \arrow[r, dashed, "\tilde{u}"] \arrow[d, dashed, "\tilde{t}"'] & Y \arrow[d, "t"] \\
X \arrow[r, "us"] & B
\end{tikzcd}$$

commutes. We can now replace X by X' in the diagram above, as the fraction through X' also represents f. Indeed, this holds because the diagram

$$\begin{tikzcd}
 & X' \arrow[dl, "s\tilde{t}"'] \arrow[dr, "\tilde{f}\tilde{t}"] \arrow[dd, "\tilde{t}"] & \\
A & & A' \\
 & X \arrow[ul, "s"] \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

is commutative. Now, we observe that the top left square in the main diagram commutes in $\mathcal{K}$, since $t\tilde{u} = (us)\tilde{t} = u(s\tilde{t})$. In turn, we can only assert that the bottom left square commutes in $\mathcal{S}^{-1}\mathcal{K}$, since $u'\tilde{f}s^{-1} = \tilde{g}t^{-1}u = \tilde{g}\tilde{u}(s\tilde{t})^{-1}$, and thus $u'f\tilde{t} = \tilde{g}\tilde{u}$, using that s^{-1} is an isomorphism. However, this means that there exists a morphism $q : X'' \to X'$ such that $u'f\tilde{t}q = \tilde{g}\tilde{u}q$ in $\mathcal{K}$. We replace X' by X'' and by the same reasoning as above, the fraction $A \leftarrow X'' \to A'$ is equivalent to f. Setting $s'' := s\tilde{t}q, f'' := \tilde{f}\tilde{t}q$, the main diagram reads now

$$\begin{tikzcd}
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & T(A) \\
X'' \arrow[u, "s''"'] \arrow[d, "f''"'] \arrow[r, "\tilde{u}"] & Y \arrow[u, "t"'] \arrow[d, "\tilde{g}"'] \arrow[r, dashed, "\tilde{v}"] & Z \arrow[u, dashed, "r"'] \arrow[d, dashed, "\tilde{h}"'] \arrow[r, dashed, "\tilde{w}"] & T(X'') \arrow[u, "T(s'')"'] \arrow[d, "T(f'')"'] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & T(A')
\end{tikzcd}$$

where the two squares in the left are commutative. Furthermore, we have completed $X'' \overset{\tilde{u}}{\to} Y$ to an exact triangle $(\tilde{u}, \tilde{v}, \tilde{w})$ in $\mathcal{K}$.

Now, the three rows represent exact triangles in $\mathcal{K}$. Using (TR3), we find $\tilde{h} : Z \to C'$ and $r : Z \to C$ giving respective morphisms of triangles $(f'', \tilde{g}, \tilde{h})$ and (s'', t, r). Furthermore, since $\mathcal{S}$ arises from a cohomological functor and $s'', t \in \mathcal{S}$, then r is again in $\mathcal{S}$. Thus, the morphism $h := \tilde{h}r^{-1} : C \to C'$ is a well-defined morphism in $\mathcal{S}^{-1}\mathcal{K}$ that makes the diagram commutative, and therefore (f, g, h) is a morphism of triangles in $\mathcal{S}^{-1}\mathcal{K}$.

Remark: for the original definition of Weibel to be equivalent to the one used here, one needs to require the existence of a map $w' : C' \to T(A')$ such that $w = T(t_1)w'$. This arrow completes the triangle in $\mathcal{K}$ with vertices (A', B', C'). With this, the following diagrams provide the triangles required in the proofs of (TR1) and (TR2). We omit (TR3) as it involves messy diagrams, showing the convenience of the alternative definition.

(TR1)

$$\begin{tikzcd}
A' \arrow[r, "f"] \arrow[dd, "1"'] & B \arrow[r, "g"] \arrow[d, "1"'] & C \arrow[r, "h"] \arrow[d, "1"'] & T(A') \arrow[r, "T(s)"] & T(A) \\
 & B \arrow[r, "g"'] \arrow[d, "1"'] & C & & \\
A' \arrow[r, "f"'] \arrow[d, "s"'] & B & & & \\
A & & & &
\end{tikzcd}$$

(TR2)

$$\begin{tikzcd}
B' \arrow[r, "v'"] \arrow[dd] & C' \arrow[r, "w'"] \arrow[d, "1"'] & T(A') \arrow[r, "-T(u')"] \arrow[d, "T(t_1)"] & T(B') \\
 & C' \arrow[r, "w"'] \arrow[d, "s_3"'] & T(A) & \\
B'' \arrow[r, "v"'] \arrow[d, "s_2"'] & C & & \\
B & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^{-1}(C') \arrow[r, "-T^{-1}(w')"] \arrow[dd, "1"'] & A' \arrow[r, "u'"] \arrow[d] \arrow[dd, bend right, "t_1"'] & B' \arrow[r, "v'"] \arrow[d, "s_2"] & C' \\
 & A'' \arrow[r, "u"'] \arrow[d, "s_1"] & B & \\
T^{-1}(C') \arrow[r, "-T^{-1}(w)"'] \arrow[d, "T^{-1}(s_3)"'] & A & & \\
T^{-1}(C) & & &
\end{tikzcd}$$

3 (i) Let (u, v, w) be an exact triangle on (A, B, C). We want to show that the following long sequence is exact in FA/FB:

$$\cdots \overset{w^*}{\to} H(A[-i]) \overset{u^*}{\to} H(B[-i]) \overset{v^*}{\to} H(C[-i]) \overset{w^*}{\to} H(A[-(i+1)]) \overset{u^*}{\to} \cdots$$

By exercise 12.4(d), π is exact so it suffices to show that the following is exact in $\mathcal{A}$:

$$\cdots \overset{w^*}{\to} H^0(A[-i]) \overset{u^*}{\to} H^0(B[-i]) \overset{v^*}{\to} H^0(C[-i]) \overset{w^*}{\to} H^0(A[-(i+1)]) \overset{u^*}{\to} \cdots$$

But this is clear by Corollary 10.1.4 of Weibel's book.

(ii) By definition, $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ is the full subcategory of $\mathcal{K}(\mathcal{A})$ consisting of those objects A with $H^i(A) = 0$ for any i (see Exercise 10.2.5 of Weibel's book). That means an object X of $\mathcal{K}(\mathcal{A})$ is in $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ if and only if $\pi H^i(X) = 0$ for all i. By exercise 12.4(b) this is equivalent to asking that $H^i(X)$ is in $\mathcal{B}$ for any i, which is what we wanted to show.

(iii) By Lemma 10.3.13.2 of Weibel's book, to show that $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ is a localizing subcategory of $\mathcal{K}(\mathcal{A})$, it suffices to prove that for any quasi-isomorphism $Y \to X$ with X an object of $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$, then there exists another object X' of $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ and a morphism $X' \to Y$ such that the composite $X' \to Y \to X$ is a quasi-isomorphism. We claim that actually, Y already is in $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$, so that taking $X' := Y$ gives us what we want. Indeed, $Y \to X$ being a quasi-isomorphism, by definition it induces an isomorphism $H^i(Y)\xrightarrow{\sim}H^i(X)$ for any i. But X being an object of $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$ is, by point (b), equivalent to saying that $H^i(X)$ is an object of $\mathcal{B}$ for any i. By the fact that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ and that $0 \to H^i(Y)\xrightarrow{\sim}H^i(X) \to 0 \to 0$ is short exact for all i, one has that $H^i(Y)$ is in $\mathcal{B}$ for all i, which equivalent to saying that Y is in $\mathcal{K}_{\mathcal{B}}(\mathcal{A})$, and we have proven the point. In particular $\mathbb{D}_{\mathcal{B}}(\mathcal{A})$ is a triangulated subcategory of $\mathbb{D}(\mathcal{A})$ by point (a) .

(iv) Denote by $\mathcal{K}^+(\mathcal{I}_{\mathcal{B}})$ (*resp.* by $\mathcal{K}^+(\mathcal{I}_{\mathcal{A}})$) the full subcategories of $\mathcal{K}^+(\mathcal{A})$ consisting of bounded below cochain complexes of injectives of $\mathcal{B}$ (*resp.* of $\mathcal{A}$). Denote moreover by $\mathcal{K}^+_{\mathcal{B}}(\mathcal{I}_{\mathcal{A}})$ the full subcategory of $\mathcal{K}^+_{\mathcal{B}}(\mathcal{A})$ consisting of bounded below cochain complexes of injectives $I^\bullet$ such that $H^i(I) \in \mathcal{B}$ for all i. By Theorem 10.4.8 of Weibel's book, one has equivalences of categories

$$\mathbb{D}^+(\mathcal{B}) \cong \mathcal{K}^+(\mathcal{I}_{\mathcal{B}}), \mathbb{D}^+(\mathcal{A}) \cong \mathcal{K}^+(\mathcal{I}_{\mathcal{A}}), \text{ and } \mathbb{D}^+_{\mathcal{B}}(\mathcal{A}) \cong \mathcal{K}^+_{\mathcal{B}}(\mathcal{I}_{\mathcal{A}}).$$

Note that through these equivalences, the functor $F : \mathbb{D}^+(\mathcal{B}) \to \mathbb{D}^+(\mathcal{A})$ is fully faithful, as $\mathcal{K}^+(\mathcal{I}_{\mathcal{B}})$ and $\mathcal{K}^+(\mathcal{I}_{\mathcal{A}})$ both are full subcategories of $\mathcal{K}(\mathcal{A})$.

Now we will show that $\mathbb{D}^+_{\mathcal{B}}(\mathcal{A})$ is in the essential image of F in two steps: first we will show the case of bounded complexes by induction on the number of non-zero terms of the complex, and then we will show the bounded below case from the bounded case. Let then $C^\bullet$ be a bounded complex in $\mathcal{A}$ with n non-zero terms (where $n \in \mathbb{Z}_{>0}$), and with cohomology in $\mathcal{B}$. If $n = 1$, then the only non-zero term is isomorphic to the only non-zero cohomology object, whence is in $\mathcal{B}$. Assume then that $n > 1$ and that if $D^\bullet$ is another complex with k non-zero terms, where $k < n$, and with cohomology in $\mathcal{B}$, then $D^\bullet$ is quasi-isomorphic to a complex in $\mathcal{B}$. Write $C^{<m}$ for the canonical truncation of $C^\bullet$, i.e. for the complex

$$\cdots \to C^{m-2} \to \ker(d^{m-1}) \to 0 \to \cdots ,$$

where d^{m-1} is the differential $C^{m-1} \to C^m$. Samely, write $C^{\geq m}$ for the complex

$$\cdots \to 0 \to \operatorname{coker}(d^{m-1}) \to C^{m+1} \to \cdots .$$

Choose m such that both $C^{<m}$ and $C^{\geq m}$ have strictly less than n non-zero terms, so that they actually are in the essential image of F by induction hypothesis, as the cohomology objects $H^i(C^{<m})$ and $H^i(C^{\geq m})$ are either 0 or isomorphic to cohomology objects $H^i(C)$. Notice that then $C^\bullet$ is quasi-isomorphic to the cone of the 0 map $C^{\geq m} \to C^{<m}[1]$. It follows that we have a triangle

$$C^{<m} \to C^\bullet \to C^{\geq m} \to C^{<m}[1]$$

in $\mathbb{D}^+_{\mathcal{B}}(\mathcal{A})$. On the other hand we can also form an exact triangle in $\mathbb{D}^+(\mathcal{B})$:

$$D^\bullet \to E^\bullet \to B^\bullet \to D^\bullet[1],$$

where $F(D^\bullet) \cong C^{\geq m}$ and $F(E^\bullet) \cong C^{<m}[1]$. In particular, we have an isomorphism $F(B^\bullet[-1]) \cong C^\bullet$ by (TR3), coming from the map of triangles:

$$\begin{tikzcd}
F(D^\bullet[-1]) \arrow[r] \arrow[d, "\cong"'] & F(E^\bullet[-1]) \arrow[r] \arrow[d, "\cong"'] & F(B^\bullet[-1]) \arrow[r] \arrow[d, dashed, "\exists"'] & F(D^\bullet) \arrow[d, "\cong"'] \\
C^{\geq m}[-1] \arrow[r] & C^{<m} \arrow[r] & C^\bullet \arrow[r] & C^{\geq m}
\end{tikzcd}$$

which concludes the bounded case.

As for the general case, note that the truncations $C^{\leq m}$ of a bounded below complex $C^\bullet$ with cohomology in $\mathcal{B}$ themselves have cohomology in $\mathcal{B}$. Hence every truncation $C^{\leq m}$ is in the essential image of F by the bounded case. Through the equivalence $\mathbb{D}^+_{\mathcal{B}}(\mathcal{A}) \cong \mathcal{K}^+_{\mathcal{B}}(\mathcal{I}_{\mathcal{A}})$, the complexes $C^{\leq m}$ then correspond to injective resolutions $I^\bullet_m$ in $\mathcal{K}^+(\mathcal{I}_{\mathcal{B}})$, given by the total complexes of the Cartan–Eilenberg resolutions of $C^{\leq m}$. In particular by construction (see Lemma 5.7.2 of Weibel's book), as the first $m-1$ columns of the Cartan–Eilenberg resolution of $C^{\leq m}$ are the same as the ones of the Cartan–Eilenberg resolution of $C^{\leq k}$ for any $k \geq m$, one has then that $I^j_m = I^j_k$ for any $k \geq m$ and any $j \leq m$. Moreover, again by construction of the Cartan–Eilenberg resolution, the m-th column of the resolution of $C^{\leq m}$ is a subobject of the m-th columns of the resolutions of $C^{\leq k}$ for any $k \geq m$; therefore I^j_m is a subobject of I^j_k for any $k, j \geq m$. This gives us that the colimit in $\mathcal{K}^+(\mathcal{I}_{\mathcal{A}})$ of the complexes $I^\bullet_m$ actually exists and is the complex

$$I^\bullet = \cdots \to I^m \to I^{m+1} \to \cdots ,$$

which is in $\mathcal{K}^+(\mathcal{I}_{\mathcal{B}})$ as every I^m is in $\mathcal{B}$. But again through the equivalence $\mathbb{D}^+_{\mathcal{B}}(\mathcal{A}) \cong \mathcal{K}^+_{\mathcal{B}}(\mathcal{I}_{\mathcal{A}})$, the colimit of the complexes $I^\bullet_m$ must correspond to the colimit of the truncated complexes $C^{\leq m}$ – as indeed equivalence of categories preserves colimits – which is $C^\bullet$. Therefore $C^\bullet$ actually is in the essential image of F, concluding the bounded below case.

Finally, we show that the essential image of F is in $\mathbb{D}^+_{\mathcal{B}}(\mathcal{A})$ by observing that, for any injective complex in $\mathcal{K}^+(\mathcal{I}_{\mathcal{B}})$ and for any $m \in \mathbb{Z}$ we have exact sequences in $\mathcal{A}$ as follows:

$$0 \to \text{im } d^{m-1} \to \ker d^m \to H^m(I) \to 0 \ \text{ and}$$

$$0 \to \ker d^m \to I^m \to \text{im } d^m \to 0.$$

In particular, as $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ and I^m is in $\mathcal{B}$, then $H^m(I)$ itself is in $\mathcal{B}$, i.e. $I^\bullet$ actually is in $\mathcal{K}^+_{\mathcal{B}}(\mathcal{I}_{\mathcal{A}})$. Consequently $\mathcal{K}^+(\mathcal{I}_{\mathcal{B}})$ is a subcategory of $\mathcal{K}^+_{\mathcal{B}}(\mathcal{I}_{\mathcal{A}})$, and through the equivalences $\mathbb{D}^+(\mathcal{B}) \cong \mathcal{K}^+(\mathcal{I}_{\mathcal{B}})$ and $\mathbb{D}^+_{\mathcal{B}}(\mathcal{A}) \cong \mathcal{K}^+_{\mathcal{B}}(\mathcal{I}_{\mathcal{A}})$, this exactly means that the essential image of F is in $\mathbb{D}^+_{\mathcal{B}}(\mathcal{A})$, proving the point.

4 Let R be a Noetherian ring and $\mathbb{M}(R)$ the category of all finitely generated R-modules. Consider the category $\mathbb{D}_{\text{fg}}(R) := \mathbb{D}_{\mathbb{M}(R)}(\mathbf{mod} - R)$ with the notation of the exercise 3 above.

First, we show that $\mathbb{D}_{\text{fg}}(R)$ is triangulated. By exercise 12.4, $\mathbb{M}(R)$ is a Serre subcategory of $\mathbf{mod} - R$. Then, we are under the conditions of exercise 3. In particular, we deduce that $\mathbb{D}_{\text{fg}}(R)$ is a triangulated subcategory of $\mathbb{D}(\mathbf{mod} - R)$. By definition of triangulated subcategory, $\mathbb{D}_{\text{fg}}(R)$ is a triangulated category itself.

Secondly, we show that there is an equivalence $\mathbb{D}^-(\mathbb{M}(R)) \cong \mathbb{D}^-_{\text{fg}}(R)$ between the respective derived categories of bounded above complexes. In exercise 3, again identifying $\mathcal{A} = \mathbf{mod} - R$ and $\mathcal{B} = \mathbb{M}(R)$, the relation is shown for the derived categories of bounded below complexes under certain assumptions on the injective objects of $\mathcal{B}$. Here we will use the dual result, thus we need to verify the corresponding properties on the projective objects of $\mathbb{M}(R)$. In particular, we need to show that $\mathbb{M}(R)$ has enough projectives and that every projective of $\mathbb{M}(R)$ is also a projective of $\mathbf{mod} - R$. Then the stated equivalence will follow.

Given a finitely generated module M, we may consider a set of generators of M and let F be the free module on those generators. Then F is projective in $\mathbf{mod} - R$ because it is free, thus also in $\mathbb{M}(R)$, and there exists a surjection $F \twoheadrightarrow M$, namely the quotient map modulo the relations defining M. Hence, $\mathbb{M}(R)$ has enough projectives.

Regarding the second property, consider a projective object P in $\mathbb{M}(R)$. We claim that P is a direct summand of a free, finitely generated module. Indeed, similarly as in the proof in $\mathbf{mod} - R$, we have a surjection $F \twoheadrightarrow P$ with F free and finitely generated. Since P is projective, there exists a map $P \to F$ completing the diagram

$$\begin{tikzcd} & F \arrow[d, two heads] \\ P \arrow[ur, dashed] \arrow[r, "1"] & P \end{tikzcd}$$

Thus, there is a split short exact sequence $0 \to P \to F \to F/P \to 0$, and we conclude that $F \cong P \oplus F/P$. In particular, P is a direct summand of a free R-module and therefore is projective in $\mathbf{mod} - R$.

5 (i) Recall that any morphism $f : A \to B$ factors uniquely through the image of f as

$$\begin{tikzcd}
A \arrow[r, two heads, "\varphi"] \arrow[dr, "f"'] & \operatorname{im} f \arrow[d, hook, "\psi"] \\
& B
\end{tikzcd}$$

where ψ is a monomorphism and φ is an epimorphism. Then the diagram written above fits in the following diagram:

$$\begin{tikzcd}
& & & 0 \arrow[d] & \\
0 \arrow[r] & \ker \varphi \arrow[r, hook] & A \arrow[r, two heads, "\varphi"] \arrow[dr, "f"'] & \operatorname{im} f \arrow[r] \arrow[d, hook, "\psi"] & 0 \\
& & & B \arrow[d, two heads] & \\
& & & \operatorname{coker} \psi \arrow[d] & \\
& & & 0 &
\end{tikzcd}$$

where the triangle commutes, and the row and column are exact. But as $\mathcal{A}$ is semisimple, the row and column split, giving us the existence of $i : \operatorname{im} f \to A$ and of $j : B \to \operatorname{im} f$ such that $i\varphi = \psi j = \mathrm{id}_{\operatorname{im} f}$. We claim that $ij : B \to A$ will be our pseudo-inverse of f. But as $f = \psi\varphi$, we directly compute

$$ijfij = ij\psi\varphi ij = i(j\psi)(\varphi i)j = ij, \text{ and}$$

$$fijf = \psi\varphi ij\psi\varphi = \psi(\varphi i)(j\psi)\varphi = \psi\varphi = f.$$

(ii) "⇒" Let $\mathcal{K}$ be an abelian triangulated category and let $f : A \to B$ be any morphism. As $\mathcal{K}$ is abelian, f factors uniquely through its image as $f = \psi\varphi$, where ψ is monic and φ is epic. Write furthermore $A' := \ker f, B' := \operatorname{coker} f$ and $I := \operatorname{im} f$. In this setup, it suffices to show that ψ and φ both split, as indeed in this case one has $A \cong A' \oplus I$ and $B \cong I \oplus B'$, so that $f : A \to B$ actually is isomorphic to $f' := \begin{pmatrix} 0 & 1_I \\ 0 & 0 \end{pmatrix} : A' \oplus I \to I \oplus B'$.

First, by TR1, one can embed ψ in an exact triangle (ψ, v, w) on (I, B, C). Since any composition in an exact triangle is 0 (this is Exercise 10.2.1 of Weibel's book, that we have proven in the course), in particular $\psi(T^{-1}w) = 0$. By monicness of ψ, this is equivalent to saying that $T^{-1}w = 0$, which is equivalent to $w = 0$. With this in mind, since $\operatorname{Hom}_{\mathcal{K}}(C, -)$ is a cohomological functor (see Example 10.2.8 of Weibel's book), the sequence

$$\operatorname{Hom}_{\mathcal{K}}(C, B) \xrightarrow{v \circ -} \operatorname{Hom}_{\mathcal{K}}(C, C) \xrightarrow{0} \operatorname{Hom}_{\mathcal{K}}(C, TI)$$

is exact (where $v \circ -$ is the postcomposition by v), i.e. $v \circ -$ is surjective. In particular the identity on C has a preimage through this morphism, i.e. v admits a right-inverse. Samely, since $\operatorname{Hom}_{\mathcal{K}}(-, I)$ is also a (contravariant) cohomological functor, the sequence

$$\operatorname{Hom}_{\mathcal{K}}(B, I) \xrightarrow{- \circ \psi} \operatorname{Hom}_{\mathcal{K}}(I, I) \xrightarrow{0} \operatorname{Hom}_{\mathcal{K}}(T^{-1}C, I)$$

is exact (where $- \circ \psi$ is the precomposition by ψ), giving us a left-inverse for ψ. Therefore the exact triangle (ψ, v, w) is isomorphic to the "direct sum" exact triangle

$$I \to I \oplus C \to C \xrightarrow{0} TI,$$

i.e. ψ splits as desired.

Now, by TR1 again, we embed φ in an exact triangle (φ, s, t) on (A, I, D). Since φ is an epimorphism and $s\varphi = 0$, it follows that $s = 0$. As for ψ, applying $\mathrm{Hom}_{\mathcal{K}}(-, I)$ yields that φ has a right-inverse, whereas applying $\mathrm{Hom}_{\mathcal{K}}(D, -)$ gives a left-inverse for t. In particular, the exact triangle (φ, s, t) is isomorphic to the following exact triangle:

$$T^{-1}D \oplus I \to I \xrightarrow{0} D \to D \oplus TI.$$

This exactly means that φ splits as well, and we are done.

"$\Leftarrow$" Suppose now that $\mathcal{K}$ is a triangulated category such that every $f : A \to B$ is isomorphic to f'. Note that f' has kernel A' and cokernel B'. Moreover, f' is monic if and only $A' \cong 0$, and is epic if and only if $B' \cong 0$. In particular, f is monic if and only if f' is monic, and in this case we have a short exact sequence

$$0 \to I \xrightarrow{f'} I \oplus B' \to B' \to 0,$$

i.e. f' is the kernel of its cokernel $I \oplus B' \to B'$. Conversly, f is epic if and only if f' is epic, and we conclude similarly that f' must be the cokernel of its kernel $A' \to A' \oplus I$.

(iii) Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Denote by $A[0]^\bullet$ the chain complex associated to A *concentrated in degree* 0, namely $0 \to A \to 0$, and by Q the localization functor $\mathcal{K}(\mathcal{A}) \to \mathbb{D}(\mathcal{A})$. By Verdier's lemma, one has an isomorphism of short exact sequences in $\mathbb{D}(\mathcal{A})$:

$$\begin{tikzcd}
0 \to Q(A[0]^\bullet) \arrow[r] \arrow[d] & Q(B[0]^\bullet) \arrow[r] \arrow[d] & Q(C[0]^\bullet) \to 0 \arrow[d] \\
0 \to Q(A[0]^\bullet) \arrow[r] & Q(A[0]^\bullet) \oplus Q(C[0]^\bullet) \arrow[r] & Q(C[0]^\bullet) \to 0
\end{tikzcd}$$

It follows that there is a quasi-isomorphism between short exact sequences in $\mathcal{K}(\mathcal{A})$:

$$\begin{tikzcd}
0 \arrow[r] & A[0]^\bullet \arrow[r] \arrow[d] & B[0]^\bullet \arrow[r] \arrow[d] & C[0]^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A[0]^\bullet \arrow[r] & (A \oplus C)[0]^\bullet \arrow[r] & C[0]^\bullet \arrow[r] & 0
\end{tikzcd}$$

Therefore, computing the zero-th homology group of the objects, this quasi-isomorphism in particular induces an isomorphism in $\mathcal{A}$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

which exactly says that the exact sequence splits.

(iv) Consider the abelian category $\mathcal{A}^{\mathbb{Z}}$ where the objects are collections of objects of $\mathcal{A}$ indexed by $\mathbb{Z}$. We claim that $\mathbb{D}(\mathcal{A})$ is equivalent to $\mathcal{A}^{\mathbb{Z}}$. To show this claim, we consider the functor $\mathcal{F} : \mathcal{K}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$ which, on objects, sends a cochain complex $C^\bullet$ on the collection $(H^n(C))_{n\in\mathbb{Z}}$. Note that any two quasi-isomorphic cochain complexes have isomorphic cohomology objects, so that $\mathcal{F}$

passes through $\mathbb{D}(\mathcal{A})$ via Q, i.e. there is a functor $\mathcal{G} : \mathbb{D}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{K}(\mathcal{A}) \arrow[r, "\mathcal{F}"] \arrow[d, "Q"'] & \mathcal{A}^{\mathbb{Z}} \\
\mathbb{D}(\mathcal{A}) \arrow[ur, dashed, "\exists \mathcal{G}"'] &
\end{tikzcd}$$

Moreover define the functor $\iota : \mathcal{A}^{\mathbb{Z}} \to \mathcal{K}(\mathcal{A})$ on objects by $\iota((C^n)_n) = C^\bullet$ where the differentials are 0. We want now to show that $\mathcal{G}$ and $Q\iota$ are the wanted equivalence of categories. Note that $\mathcal{G}Q\iota$ clearly is the identity functor as

$$\mathcal{G}Q\iota((C^n)_n) = \mathcal{F}\iota((C^n)_n) = \mathcal{F}(C^\bullet) = (H^n(C))_n = (C^n)_n$$

where the last equality stands as the differentials of $H^\bullet(C)$ are 0. Conversly, we want to define mutually inverse natural transformations $\varepsilon : \mathrm{id}_{\mathbb{D}(\mathcal{A})} \to Q\iota\mathcal{G}$ and $\eta : Q\iota\mathcal{G} \to \mathrm{id}_{\mathbb{D}(\mathcal{A})}$.

Before doing this, let us recall that for any complex $C^\bullet$ with differentials d^n, one has short exact sequences

$$0 \to B^n(C) \to Z^n(C) \to H^n(C) \to 0, \text{ and}$$
$$0 \to Z^n(C) \to C^n \to B^{n+1}(C) \to 0,$$

where $B^n(C) = \mathrm{im}\ d^{n-1}$ and $Z^n(C) = \ker d^n$. As $\mathcal{A}$ is semisimple, these short exact sequences split so that actually one can decompose C^n as $C^n \cong B^n(C) \oplus H^n(C) \oplus B^{n+1}(C)$. Under this isomorphism, the differentials become

$$d^n = \begin{pmatrix} 0 & 0 & 1_{B^{n+1}(C)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note now that, on objects, $Q\iota\mathcal{G}(C^\bullet) = H^\bullet(C)$, where the complex $H^\bullet(C)$ is taken with the 0 differentials. Define then ε and η on objects as

$$\varepsilon^n_{C^\bullet} : \begin{array}{ccc} B^n(C) \oplus H^n(C) \oplus B^{n+1}(C) & \longrightarrow & H^n(C) \\ (b^n, h^n, b^{n+1}) & \longmapsto & h^n \end{array}, \text{ and}$$

$$\eta^n_{C^\bullet} : \begin{array}{ccc} H^n(C) & \longrightarrow & B^n(C) \oplus H^n(C) \oplus B^{n+1}(C) \\ h^n & \longmapsto & (0, h^n, 0) \end{array}.$$

In particular,

$$\varepsilon^n_{C^\bullet} \circ \eta^n_{C^\bullet}(h^n) = h^n \text{ for any } n, \text{ and}$$
$$\eta^n_{C^\bullet} \circ \varepsilon^n_{C^\bullet}(b^n, h^n, b^{n+1}) = (0, h^n, 0).$$

The first one is an isomorphism in $\mathcal{K}(\mathcal{A})$, whereas the second one only is a quasi-isomorphism in $\mathcal{K}(\mathcal{A})$, because it induces an isomorphism

$$H^n(\eta_{C^\bullet} \circ \varepsilon_{C^\bullet})(h^n) = h^n \in H^n(H^\bullet(C)) = H^n(C)$$

on cohomology. But a quasi-isomorphism in $\mathcal{K}(\mathcal{A})$ is exactly an isomorphism in $\mathbb{D}(\mathcal{A})$, so we have shown that $\mathbb{D}(\mathcal{A})$ and $\mathcal{A}^{\mathbb{Z}}$ are equivalent.

6.14 Week 14 (by Virgile Constantin and Claudio Pfammatter).

(i) Since $\mathcal{A}$ has enough injectives, we know that $\mathbf{D}^+(\mathcal{A}) \cong \mathbf{K}^+(\mathcal{I}) \subseteq \mathbf{K}^+(\mathcal{A})$. In exercise 3 of last week, we showed that $\mathbf{K}^+_{\mathcal{B}}(\mathcal{A}) \subseteq \mathbf{K}^+(\mathcal{A})$ is a localizing subcategory. This means that we can compute its derived category as the subcategory of $\mathbf{D}^+(\mathcal{A}) \cong \mathbf{K}^+(\mathcal{I})$ consisting of those cochain complexes of injectives with cohomology in $\mathcal{B}$. In formula this is

$$\mathbf{D}^+_{\mathcal{B}}(\mathcal{A}) \cong \mathbf{K}^+_{\mathcal{B}}(\mathcal{I}) \subseteq \mathbf{K}^+(I) \cong \mathbf{D}^+(\mathcal{A}).$$

Now the same proof as in Theorem 10.5.6 of Weibel's book shows that the derived functor of $\mathbf{K}^+_{\mathcal{B}}(\mathcal{A}) \to \mathbf{K}^+(\mathcal{A}) \xrightarrow{F} \mathbf{K}^+(\mathcal{C})$ is given by the composition

$$\mathbf{D}^+_{\mathcal{B}}(\mathcal{A}) \cong \mathbf{K}^+_{\mathcal{B}}(\mathcal{I}) \to \mathbf{K}^+_{\mathcal{B}}(\mathcal{A}) \to \mathbf{K}^+(\mathcal{A}) \xrightarrow{F} \mathbf{K}^+(\mathcal{C}) \xrightarrow{q} \mathbf{D}^+(\mathcal{C}).$$

This is precisely the restriction of $\mathbf{R}^+F$ to $\mathbf{D}^+_{\mathcal{B}}(\mathcal{A})$ since the following square commutes:

$$\begin{tikzcd}
\mathbf{K}^+_{\mathcal{B}}(\mathcal{A}) \arrow[r, hook] & \mathbf{K}^+(\mathcal{A}) \\
\mathbf{D}^+_{\mathcal{B}}(\mathcal{A}) \arrow[r, hook] \arrow[u, hook] & \mathbf{D}^+(\mathcal{A}) \arrow[u, hook]
\end{tikzcd}$$

(ii) For this exercise let R denote the ring $\mathbb{Z}/4\mathbb{Z}$ and consider the cochain complex

$$\ldots \to R \xrightarrow{\cdot 2} R \to \ldots$$

which we denote by $P^\bullet$.

(a) It is obvious that $P^\bullet$ consists of projective modules since R is a free module over itself and free modules are projective. Moreover we observe that

$$\ker(R \xrightarrow{\cdot 2} R) = 2 \cdot \mathbb{Z}/4\mathbb{Z}$$

as well as

$$\operatorname{im}(R \xrightarrow{\cdot 2} R) = 2 \cdot \mathbb{Z}/4\mathbb{Z}$$

and thus $H^i(P^\bullet) = 0$ for all $i \in \mathbb{Z}$. This shows that $P^\bullet$ is quasi-isomorphic to the zero complex as wanted.

(b) Now we consider the complex $P^\bullet \otimes_R \mathbb{Z}/2\mathbb{Z}$ which is given by

$$\ldots \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z} \to \ldots$$

since

$$R \otimes_R \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$$

and therefore $H^i(P^\bullet \otimes_R \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for all $i \in \mathbb{Z}$. We conclude that $P^\bullet \otimes_R \mathbb{Z}/2\mathbb{Z}$ can't be quasi-isomorphic to the zero complex.

(iii) (a)

(b) We assume that F has finite cohomological dimension and we want to show that $\mathbf{R}F$ exists on $\mathbf{D}(\mathcal{A})$. As suggested by the hint we consider the full subcategory $\mathbf{K}'$ of $\mathbf{K}(\mathcal{A})$ consisting of complexes of F-acyclic objects in $\mathcal{A}$. We have to show that every complex $X^\bullet$ admits a quasi-isomorphism $X^\bullet \to X'^\bullet$ where $X'^\bullet$ is a complex of F-acyclic objects to be able to conclude by the Generalized Existence Theorem and part a).

To do this take a Cartan-Eilenberg resolution $X^\bullet \to I^{\bullet,\bullet}$ and let $\tau I^{\bullet,\bullet}$ be the double subcomplex of $I^{\bullet,\bullet}$ obtained by taking the good truncation $\tau_{\leq n}(I^{p,\bullet})$ of each column (see Truncations 1.2.7. in Weibel's book for more details on this construction) where n is the cohomological dimension of the functor F. As each $X^p \to I^{p,\bullet}$ is an injective resolution, each $\tau_{\leq n}(I^{p,\bullet})$ is a finite resolution of X^p by F-acyclic objects and therefore $X'^\bullet = \mathrm{Tot}(\tau I^{\bullet,\bullet})^\bullet$ is a cochain complex of F-acyclic objects. Then the bounded spectral sequence

$$H^p H^q(\tau I^{\bullet,\bullet}) \Rightarrow H^{p+q}(X'^\bullet)$$

degenerates to yield $H^*(X^\bullet) \cong H^*(X'^\bullet)$ which proves that $X^\bullet \to X'^\bullet$ is a quasi-isomorphism.

As already mentioned we conclude the exercise by invoking the Generalized Existence Theorem and part a).

(iv) Let R be a commutative ring. Recall that during the talk we took a cochain complex $A^\bullet$ of right R-modules and considered the functor

$$F(B^\bullet) = \mathrm{Tot}^\oplus(A^\bullet \otimes_R B^\bullet)$$

from $\mathbf{K}(R-\mathbf{mod})$ to $\mathbf{K}(\mathbf{AB})$. As the category $R-\mathbf{mod}$ has enough projectives, its total left derived functor

$$\mathbf{L}^- F : \mathbf{D}^-(R-\mathbf{mod}) \to \mathbf{D}(\mathbf{Ab})$$

exists by the Existence Theorem. We defined the total tensor product of a bounded above cochain complex $A^\bullet$ of right R-modules and a cochain complex $B^\bullet$ of left R-modules by

$$A^\bullet \otimes_R^{\mathbf{L}} B^\bullet = \mathbf{L}^-\mathrm{Tot}^\oplus(A^\bullet \otimes_R B^\bullet).$$

However, for a fixed cochain complex $B^\bullet$ of left R-modules we could also look at the functor $G(A^\bullet) = \mathrm{Tot}^\oplus(A^\bullet \otimes_R B^\bullet)$ where $A^\bullet$ is a cochain complex of right R-modules. Again the total left derived functor

$$\mathbf{L}^- G : \mathbf{D}^-(\mathbf{mod}-R) \to \mathbf{D}(\mathbf{Ab})$$

exists by the Existence Theorem.

Let us now take bounded above cochain complexes $A^\bullet$ and $B^\bullet$ of right respectively left R-modules. We want to show that $A^\bullet \otimes_R^{\mathbf{L}} B^\bullet$ is naturally isomorphic to $\mathbf{L}^-\mathrm{Tot}^\oplus(- \otimes_R B^\bullet)(A^\bullet)$.

As in the proof of lemma 10.6.2. of Weibel's book we are free to change $A^\bullet$ and $B^\bullet$ up to quasi-isomorphism and assume that both are bounded above complexes of flat modules (this can be done using Cartan-Eilenberg resolution and invoking exercise 1 of sheet 10). In this case we obtain

$$A^\bullet \otimes_R^{\mathbf{L}} B^\bullet = \mathrm{Tot}^\oplus(A^\bullet \otimes_R B^\bullet) = \mathbf{L}^-\mathrm{Tot}^\oplus(- \otimes_R B^\bullet)(A^\bullet)$$

by Example 10.5.5. of Weibel's book which provides the desired natural isomorphism.

(v) Let R, R_1 and R_2 be a rings. Our goal is to refine the total tensor product functor

$$- \otimes_R^{\mathbf{L}} - : \mathbf{D}^-(\mathbf{mod}-R) \times \mathbf{D}^-(R-\mathbf{mod}) \to \mathbf{D}^-(\mathbf{Ab})$$

to obtain a functor

$$- \otimes_R^{\mathbf{L}} - : \mathbf{D}^-(R_1-\mathbf{mod}-R) \times \mathbf{D}^-(R-\mathbf{mod}-R_2) \to \mathbf{D}^-(R_1-\mathbf{mod}-R_2)$$

such that the diagram of the exercise description commutes (see below).

To do this, notice first that if $A^\bullet$ and $B^\bullet$ are bounded above cochain complexes of $R_1 - R$ bimodules respectively $R - R_2$ bimodules then the tensor product double complex $A^\bullet \otimes_R B^\bullet$ admits naturally the structure of an $R_1 - R_2$ bimodule double complex. Now to show that we can indeed refine the total tensor product functor in the desired way, we may assume that $A^\bullet$ and $B^\bullet$ are bounded above cochain complexes consisting of flat modules (see the solution of exercise 4 of this sheet for more details on why one can do this). In this situation we have

$$A^\bullet \otimes_R^{\mathbf{L}} B^\bullet = \mathrm{Tot}^\oplus(A^\bullet \otimes_R B^\bullet)$$

by Example 10.5.5. of Weibel's book and $\mathrm{Tot}^\oplus(A^\bullet \otimes_R B^\bullet)$ admits naturally the structure of an $R_1 - R_2$ bimodule since the tensor product double complex $A^\bullet \otimes_R B^\bullet$ admits the structure of an $R_1 - R_2$ bimodule double complex and the direct sum of $R_1 - R_2$ bimodules is an $R_1 - R_2$ bimodule. Now the diagram

$$\begin{tikzcd}
\mathbf{D}^-(R_1-\mathbf{mod}-R) \times \mathbf{D}^-(R-\mathbf{mod}-R_2) \arrow[r, "-\otimes_R^{\mathbf{L}}-"] \arrow[d] & \mathbf{D}^-(R_1-\mathbf{mod}-R_2) \arrow[d] \\
\mathbf{D}^-(\mathbf{mod}-R) \times \mathbf{D}^-(R-\mathbf{mod}) \arrow[r, "-\otimes_R^{\mathbf{L}}-"'] & \mathbf{D}^-(\mathbf{Ab})
\end{tikzcd}$$

where the vertical functors are forgetful functors commutes trivially.

Assume now that R is commutative. By what we did so far we have the refined total tensor product functor

$$- \otimes_R^{\mathbf{L}} - : \mathbf{D}^-(R - \mathbf{mod}) \times \mathbf{D}^-(R - \mathbf{mod}) \to \mathbf{D}^-(R - \mathbf{mod})$$

by commutativity of the ring R. The natural isomorphism follows easily in the case where $A^\bullet$ and $B^\bullet$ are bounded above cochain complexes of flat modules because then

$$A^\bullet \otimes_R^{\mathbf{L}} B^\bullet = \mathrm{Tot}^{\oplus}(A^\bullet \otimes_R B^\bullet) = \mathrm{Tot}^{\oplus}(B^\bullet \otimes_R A^\bullet) = B^\bullet \otimes_R^{\mathbf{L}} A^\bullet.$$

By the same argument as above (replacing any bounded above cochain complex in $\mathbf{D}^-(R - \mathbf{mod})$ by a quasi-isomorphic bounded above complex of flat modules) the desired result follows.

(vi) In this exercise we want to establish some sort of associativity property of the composition of total derived functors up to natural transformations. Let us first recall the universal property of the total right derived functor $\mathbf{R}H : \mathbf{D} \to \mathbf{D}'$ where we denote by $\mathbf{D}$ the derived category associated to $\mathbf{K}$. There is a natural transformation $\xi^H : qH \Rightarrow \mathbf{R}Hq$ with the property that for any functor $L : \mathbf{D} \to \mathbf{D}'$ equipped with a natural transformation $\tau : qH \Rightarrow Lq$ there exists a unique natural transformation $\eta : \mathbf{R}H \Rightarrow L$ such that $\tau = \eta_q \circ \xi^H$. One can keep the following diagram in mind

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "H"] \arrow[rd, "q"'] & \mathbf{K}' \arrow[r, "q"] \arrow[d, Rightarrow, "\xi^H"'] & \mathbf{D}' \\
& \mathbf{D} \arrow[ru, bend left, "\mathbf{R}H"{name=U}] \arrow[ru, bend right=60, "L"'{name=V}] &
\arrow[from=U, to=V, Rightarrow, "\exists!\eta"]
\end{tikzcd}$$

Moreover, the total right derived functors $\mathbf{R}G$ and $\mathbf{R}F$ come equipped with natural transformations $\xi^G : qG \Rightarrow \mathbf{R}Gq$ and $\xi^F : qF \Rightarrow \mathbf{R}Fq$ and the same thing is true for the total right derived functors $\mathbf{R}(FG)$, $\mathbf{R}(GH)$ and $\mathbf{R}(FGH)$. Recall that by the Composition Theorem one has commutative diagrams

$$\begin{tikzcd}
qGH \arrow[r, Rightarrow, "\xi^G_H"] \arrow[d, Rightarrow, "\xi^{GH}"'] & (\mathbf{R}G)qH \arrow[d, Rightarrow, "\mathbf{R}G\xi^H"] \\
\mathbf{R}(GH)q \arrow[r, Rightarrow, "\zeta_q^{G,H}"'] & \mathbf{R}G \circ \mathbf{R}Hq
\end{tikzcd} \tag{28}$$

and

$$\begin{tikzcd}
qFG \arrow[r, Rightarrow, "\xi^F_G"] \arrow[d, Rightarrow, "\xi^{FG}"'] & (\mathbf{R}F)qG \arrow[d, Rightarrow, "\mathbf{R}F\xi^G"] \\
\mathbf{R}(FG)q \arrow[r, Rightarrow, "\zeta_q^{F,G}"'] & \mathbf{R}F \circ \mathbf{R}Gq
\end{tikzcd} \tag{29}$$

In fact the statement of the exercise contains a slight abuse of notation, we have to show that $\zeta^{G,H} \circ \zeta^{F,GH} = \zeta^{F,G} \circ \zeta^{FG,H}$ which boils down to prove that the natural transformations

$$\mathbf{R}(FGH) \overset{\zeta^{FG,H}}{\Longrightarrow} \mathbf{R}(FG) \circ \mathbf{R}H \overset{\zeta^{F,G}_{\mathbf{R}H}}{\Longrightarrow} \mathbf{R}F \circ \mathbf{R}G \circ \mathbf{R}H$$

and

$$\mathbf{R}(FGH) \overset{\zeta^{F,GH}}{\Longrightarrow} \mathbf{R}F \circ \mathbf{R}(GH) \overset{\mathbf{R}F\zeta^{G,H}}{\Longrightarrow} \mathbf{R}F \circ \mathbf{R}G \circ \mathbf{R}H$$

are equal.

The current situation can be summarized in the following diagram

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "H"] \arrow[ddrr, "q"'] & \mathbf{K}' \arrow[r, "G"] & \mathbf{K}'' \arrow[r, "F"] \arrow[dd, Rightarrow, "\xi^{FGH}"'] & \mathbf{K}(\mathcal{D}) \arrow[r, "q"] & \mathbf{D}(\mathcal{D}) & \\
 & & & & \arrow[r, Rightarrow, "\zeta^{F,G}"] & \mathbf{D}'' \arrow[ul, bend right, "\mathbf{R}F"'] \\
 & & \mathbf{D} \arrow[uurr, bend left, "\mathbf{R}(FGH)"] \arrow[urrr, "\mathbf{R}(GH)"] \arrow[rr, "\mathbf{R}H"'] & \arrow[dr, Rightarrow, "\zeta^{G,H}"'] & \mathbf{D}' \arrow[uu, "\mathbf{R}(FG)"] \arrow[ur, bend right, "\mathbf{R}G"'] &
\end{tikzcd}$$

and thus if we show that

$$\zeta^{F,G}_{\mathbf{R}Hq} \circ \zeta^{FG,H}_{q} \circ \xi^{FGH} = \mathbf{R}F\zeta^{G,H}_{q} \circ \zeta^{F,GH}_{q} \circ \xi^{FGH}$$

then we get the desired equality by the universal property of the total right derived functor $\mathbf{R}(FGH)$.

We compute

$$\begin{aligned}
\zeta^{F,G}_{\mathbf{R}Hq} \circ \zeta^{FG,H}_{q} \circ \xi^{FGH} &= \mathbf{R}F\mathbf{R}G\xi^{H} \circ \zeta^{F,G}_{qH} \circ \xi^{FG}_{H} \\
&= \mathbf{R}F\mathbf{R}G\xi^{H} \circ \mathbf{R}F\xi^{G}_{H} \circ \xi^{F}_{GH} \\
&= \mathbf{R}F(\mathbf{R}G\xi^{H} \circ \xi^{G}_{H}) \circ \xi^{F}_{GH} \\
&= \mathbf{R}F(\zeta^{G,H}_{q} \circ \xi^{GH}) \circ \xi^{F}_{GH} \\
&= \mathbf{R}F\zeta^{G,H}_{q} \circ \zeta^{F,GH}_{q} \circ \xi^{FGH}
\end{aligned}$$

where we used the commutativity of diagram (29) in the second equality and the commutativity of diagram (28) in the fourth equality, the first and the last equality follow from the construction of the natural transformations in question.

As already mentioned this concludes the proof by applying the universal property of the total right derived functor $\mathbf{R}(FGH)$.