Student Seminar in Pure Mathematics

Toric Varieties

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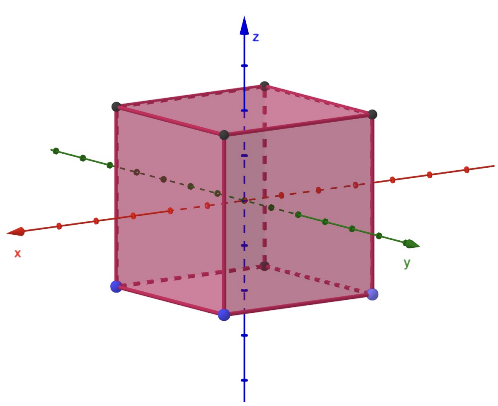

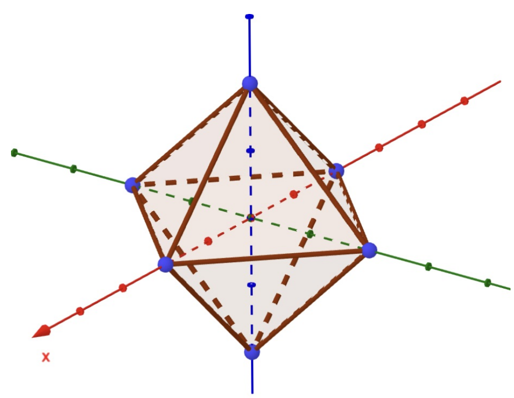

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Chapter 0. Preface

The present manuscript grew out of the student seminar in pure mathematics at EPFL (organized by Dimitri Wyss as main lecturer, and Sergej Monavari as teaching assistant), which took place in the Fall semester of 2023. The goal was to get familiar with the basics of toric varieties following the book of Cox-Little-Shenck [CLS][1].

The seminar consisted of 2 hours of lectures and 2 hours of exercises per week for 14 weeks. Lectures, notes and solutions were given and written by the students participating the seminar course.

In the first chapters, we recall the basics of affine varieties, and all the background material concerning normality, smoothness and other basic properties of affine varieties. Then we introduce algebraic tori and affine toric varieties, and the combinatorial language needed to describe them, comprising cones, polyhedra, lattices and operations among them. In the second part of the course, projective varieties and abstract varieties are recalled, in order to study projective toric varieties – and their associated fan – and abstract toric varieties, arising from their combinatorial counterpart, the normal fans. At the end, the language of toric varieties is applied to present a (part of the) solution of McMullen's conjecture, following Stanley[2]. Again, the necessary background for the singular cohomology and spectral sequences is introduced.

Every chapter ends with a few exercises, whose solutions can be found at the end of the notes.

We finish by taking all responsabilities for possible typos and mistakes that you could find in these notes.

[1] D. Cox, J. Little, H. Schenck, *Toric varieties*, Graduate Studies in Mathematics 124. Providence, RI: American Mathematical Society (AMS) 841 p. (2011).

[2] R. Stanley, *The number of faces of a simplicial convex polytope*, Adv. Math. 35, 236-238 (1980).

Chapter 1. Affine varieties

Chapter written by Prof. Dimitri Wyss

1.1 Reminder on affine varieties

For the duration of the seminar an *affine (algebraic) variety* V is the zero-set of an ideal $I \subset S = \mathbb{C}[x_1, \dots, x_n]$ for some positive integer n i.e.

$$V = \mathbf{V}(I) = \{p \in \mathbb{C}^n \mid f(p) = 0 \text{ for all } f \in I\}.$$

To V we may associate its *coordinate ring*

$$\mathbb{C}[V] = S/\mathbf{I}(V),$$

where $\mathbf{I}(V) = \{f \in S \mid f(p) = 0 \text{ for all } p \in V\}$. One should think of $\mathbb{C}[V]$ as the ring of functions on V in the category of algebraic varieties. Next we recall a few facts about these objects:

- (Hilbert's basis theorem) Any $I \subset S$ is finitely generated, thus V is the zero-set of finitely many polynomial equations.
- (Hilbert's Nullstellensatz) Since $\mathbb{C}$ is algebraically closed

  $$\mathbf{I}(\mathbf{V}(I)) = \sqrt{(}I) = \{f \in S \mid f^s \in I \text{ for some } s \geq 1\}.$$

  In particular, $\mathbb{C}[V]$ is reduced i.e. does not contain any non-zero nilpotent elements.
- $\mathbb{C}[V]$ is an integral domain $\iff$ $\mathbf{I}(V)$ is prime $\iff$ V is irreducible.[3]
- The category of affine algebraic varieties is equivalent to the (opposite) category of finitely generated reduced $\mathbb{C}$-algebras. In particular $V \cong W$ if and only if $\mathbb{C}[W] \cong \mathbb{C}[V]$.
- Affine subvarieties of V correspond to ideals in $\mathbb{C}[V]$. In particular a point $p \in V$ corresponds to the maximal ideal

  $$m_{V,p} = m_p = \{f \in \mathbb{C}[V] \mid f(p) = 0\}.$$

Given a finitely generated reduced $\mathbb{C}$-algebras R we use the notation $\mathrm{Spec}(R)$ for the corresponding variety. Usually the spectrum of a ring consists as a set of all prime ideals in R, but in light of the last bullet point, one should rather think of $\mathrm{Spec}(R)$ as the set of all maximal ideals in R in this seminar.

1.1.1 Topologies and open affines

An affine variety V admits two natural topologies. The *classical* or *Euclidean* topology by considering V as a subspace of $\mathbb{C}^n$ and the *Zarisiki* topology, where the closed sets are subvarieties of V. For now we will work with the Zariski topology.

An important class of open subset of an affine variety V are the affine opens: For any $f \in \mathbb{C}[V] \setminus \{0\}$ consider

$$V_f = \{p \in V \mid f(p) \neq 0\} = V \setminus \{f = 0\}.$$

If we let $g \in \mathbb{C}[x_1, \dots, x_n]$ be a lift of f, then we can identify V_f with the affine variety

$$\mathbf{V}(I, gx_{n+1} - 1) \subset \mathbb{C}^{n+1}.$$

If V is irreducible, $\mathbb{C}[V]$ is an integral domain and can describe the coordinate ring of V_f as a subring of the field of fractions $\mathbb{C}(V)$:

$$\mathbb{C}[V_f] = \{h/f^l \in \mathbb{C}(V) \mid h \in \mathbb{C}[V],\ l \geq 0\}.$$

In other words $\mathbb{C}[V_f] = \mathbb{C}[V]_f$, where $\mathbb{C}[V]_f$ denotes the localization of $\mathbb{C}[V]$ at the multiplicative subset $\{1, f, f^2, \dots\}$. The following example will be crucial in this seminar:

[3] Other authors often require an (affine) algebraic variety to be irreducible

Example 1.1. If $V = \mathbb{C}^n$ and $f = x_1x_2\dots x_n$. Then $V_f = (\mathbb{C}^*)^n$ and

$$\mathbb{C}[V_f] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

the ring of *Laurent polynomials*.

1.1.2 Normality and smoothness

Normality is a somewhat strange property of a variety, but will have a very nice interpretation for toric varieties. Recall that an integral domain R with field of fractions K is *normal*, or *integrally closed*, if every element of K which is integral over R (i.e. the root of a monic polynomial with coefficients in R) already lies in R. For example any UFD is normal.

Definition 1.2. An irreducible affine variety V is *normal* if $\mathbb{C}[V]$ is.

Example 1.3.

- $\mathbb{C}^n$ is normal since $\mathbb{C}[x_1, \dots, x_n]$ is a UFD.
- $V = \mathbf{V}(x^3 - y^2) \subset \mathbb{C}^2$ is not normal. Let $\overline{x}, \overline{y}$ the the images of x, y in $\mathbb{C}[V]$. Then a small computation shows that $\overline{y}/\overline{x} \in \mathbb{C}(V) \setminus \mathbb{C}[V]$ but $(\overline{y}/\overline{x})^2 = \overline{x}$.

Given an irreducible affine variety V one can always pass to a normal variety V' by considering the *integral closure*

$$\mathbb{C}[V]' = \{\alpha \in \mathbb{C}(V) \mid \alpha \text{ is integral over } \mathbb{C}[V]\}.$$

Then $\mathbb{C}[V]'$ is a reduced (contained in $\mathbb{C}(V)$), integrally closed and finitely generated (this is non-trivial!) $\mathbb{C}$-algebra. We define

$$V' = \operatorname{Spec}(\mathbb{C}[V]')$$

and call it the *normalization* of V. The inclusion $\mathbb{C}[V] \subset \mathbb{C}[V]'$ corresponds to a morphism $V' \to V$ called the *normalization map*.

Example 1.4. For $V = \mathbf{V}(x^3 - y^2)$ the normalization is given by $\mathbb{C}[\overline{y}/\overline{x}]$ and the map $\mathbb{C} \to V$, $t \mapsto (t^2, t^3)$ is the normalization map.

1.2 Exercises

Exercise 1.1. Let $V \subset \mathbb{C}^n$ be an affine algebraic variety and $f \in \mathbb{C}[V] \setminus \{0\}$. Show that there is a natural bijection between the open $V_f \subset V$ and $\mathbf{V}(\mathbf{I}(V), x_{n+1}g - 1) \subset \mathbb{C}^{n+1}$, where $g \in \mathbb{C}[x_1, \dots, x_n]$ is any lift of f. If V is irreducible, deduce from this the identification $\mathbb{C}[V_f] \cong \mathbb{C}[V]_f$, where $\mathbb{C}[V]_f$ denotes the localization of $\mathbb{C}[V]$ with respect to $\{1, f, \dots\}$.

Exercise 1.2. Prove that any UFD is normal.

Exercise 1.3. Let $V = \mathbf{V}(x^3 - y^2)$ be a cusp. Show that the normalization of V is given by $\mathbb{C}[\overline{y}/\overline{x}] \subset \mathbb{C}(V)$, where $\overline{y}, \overline{x}$ denote the images of x, y in $\mathbb{C}[V]$. Deduce that the morphism $\mathbb{C} \to V$, $t \mapsto (t^2, t^3)$ is the normalization map.

Exercise 1.4.

(a) Let R be a normal domain with field of fractions K and let $S \subset R$ be a multiplicative subset. Prove that the localization $S^{-1}R$ is normal.

(b) Let R_i, $i \in I$ be normal domains with the same field of fractions K. Prove that the intersection $\bigcap_{i \in I} R_i$ is normal.

Chapter 2. Smoothness and algebraic tori

Chapter written by Julie Bannwart after the talk of Dimitri Wyss

2.1 Smooth affine varieties

Intuitively, smoothness for a variety carries the same idea as for manifolds: the absence of "corners", "wrinkles", "cusps" etc. In the algebraic setting, there are different characterizations of this notion.

Definition 2.1. Let V be an affine variety and $p \in V$. We denote by m_p the maximal associated with p in $\mathbb{C}[V]$. The *local ring* $\mathcal{O}_{V,p}$, or simply $\mathcal{O}_p$, of V at p is the localization of $\mathbb{C}[V]$ at m_p:

$$\mathcal{O}_p = (\mathbb{C}[V] \setminus m_p)^{-1}\mathbb{C}[V],$$

and we denote its unique maximal ideal by $\mathfrak{m}_p$.

Note that if V is irreducible, then $\mathbb{C}[V]$ is a domain and $\mathcal{O}_p = \left\{ \frac{f}{g} \in \mathbb{C}(V) \middle| g(p) \neq 0 \right\}$ can be seen as a subring of the fraction field $\mathbb{C}(V)$ of $\mathbb{C}[V]$.

Smooth varieties should come, like smooth manifolds, with a notion of a tangent space at any point:

Definition 2.2. The *Zariski tangent space* of V at p is the $\mathbb{C}$-vector space:

$$T_p(V) = \operatorname{Hom}_{\mathbb{C}}\left(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{C}\right) = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*,$$

where $(-)^*$ denotes the dual of a $\mathbb{C}$-vector space.

Example 2.3. Let $f \in \mathbb{C}[x,y]$ non constant, and $V := \mathbf{V}(f)$ the associated affine plane curve. Assume $p = (0,0) \in V$ and let $\overline{x}$, $\overline{y}$ be the images of x and y in $\mathcal{O}_{p,V}$, so that $\mathfrak{m}_p = (\overline{x}, \overline{y})$. Then:

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = (\overline{x},\overline{y}) \Big/ (\overline{x}^2, \overline{xy}, \overline{y}^2).$$

We have $\dim_{\mathbb{C}} T_p(V) = \dim_{\mathbb{C}}\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) \leq 2$ because this $\mathbb{C}$-vector space is generated by the classes of $\overline{x}$ and $\overline{y}$ in the quotient.
Actually this dimension is even equal to 2 unless there is some relation between $\overline{x}$ and $\overline{y}$, due to f, i.e. unless f contains a linear summand, or equivalently if its derivative does not vanish at the origin.

Lemma 2.4. *Let $V \subseteq \mathbb{C}^n$ be an affine variety and $p \in V$. Let $f_1, \ldots, f_s$ be generators of $\mathbb{I}(V)$ (which exist by Noetherianity of $\mathbb{C}[x_1, \ldots, x_n]$), and define for $1 \leq i \leq s$ linear polynomials:*

$$d_p(f_i) = \frac{\partial f_i}{\partial x_1}(p) \cdot x_1 + \cdots + \frac{\partial f_i}{\partial x_n}(p) \cdot x_n \in \mathbb{C}[x_1, \ldots, x_n].$$

Then $T_p(V)$ is isomorphic to the vector subspace of $\mathbb{C}^n$ defined by the linear equations $d_p(f_1) = \cdots = d_p(f_s) = 0$. In particular, $\dim_{\mathbb{C}} T_p(V) \leq n$.

Proof. Omitted. □

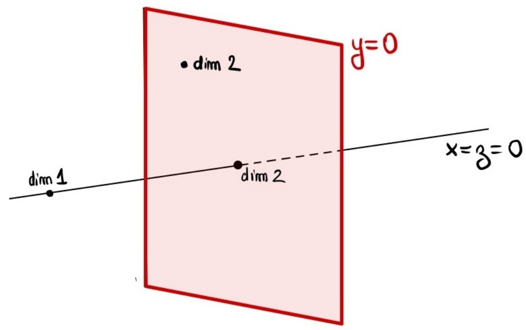
Continuing our analogy with manifolds, smoothness means that there are as many tangent directions at any point as the dimension of our object, and we have a local approximation of it by a vector space. Therefore to define smoothness, we first need a local notion of dimension for varieties:

Definition 2.5. The *dimension of V at p* for V an affine variety and $p \in V$ is the Krull dimension of $\mathcal{O}_{p,V}$:

$$\dim_p(V) = \dim_{\text{Krull}} \mathcal{O}_{p,V}$$

This definition may hide some subtleties when V is not irreducible, the dimension might not be the same in every point. But if V is irreducible, it is basically the Krull dimension of the coordinate ring of the variety. Geometrically, $\dim_p(V)$ is the maximum of the dimensions of the irreducible components of V containing p.

Example 2.6. Let $V := \mathbf{V}(xy, yz) \subseteq \mathbb{C}^3$. It decomposes in irreducible components as $\mathbf{V}(x, z) \cup \mathbf{V}(y)$. We may represent this variety, and several points together with the dimension of the variety at these points, if we imagine working over $\mathbb{R}$, as follows:



According to the above discussion about smoothness being characterized by the number of tangent directions, we now want to define:

Definition 2.7.

- A point p in an affine variety V is called *smooth* or *non singular* if $\dim_p(V) = \dim_{\mathbb{C}} T_p(V)$.
- The point p is *singular* if it is not smooth.
- The variety V is *smooth* if all its points are smooth.

Remark 2.8. Any point lying on two or more irreducible components of V is singular, in particular a smooth and connected variety is irreducible. We leave the proof of this fact as an exercise.

Lemma 2.9. *(Jacobi criterion) For V irreducible, and $p \in V = \mathbf{V}(f_1, \ldots, f_s)$ with $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$, the point p is smooth if and only if the* Jacobian matrix

$$J_p(f_1, \ldots, f_s) = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i \leq s,\ j \leq n}$$

has rank $n - \dim_p(V)$ (the dimension doesn't actually depend on p here).

Example 2.10. Let $V = \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$. Then for $f = xy - zw$, we have $J_p(f) = (y_p, x_p, -w_p, -z_p)$ for any $p = (x_p, y_p, z_p, w_p) \in \mathbb{C}^4$. Therefore by the Jacobi criterion (since f is irreducible), the singular points of V are exactly the points p such that $\operatorname{rank}(J_p(f)) \neq 4 - \dim(V) = 1$, i.e. the points such that $J_p(f) = 0$. Therefore V has a unique singular point $p = (0, 0, 0, 0)$.

Combining the definitions above, we get that a point $p \in V$ is smooth if and only if $\mathcal{O}_p$ is such that $\dim_{\mathbb{C}} \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) = \dim_{\mathrm{Krull}} \mathcal{O}_p$. We give a special name to this property:

Definition 2.11. A local ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$ is called *regular* if $\dim_k (\mathfrak{m}/\mathfrak{m}^2) = \dim_{\mathrm{Krull}} R$.

This property has a basic consequence that we will not prove:

Theorem 2.12. *Any regular local ring is a UFD.*

Remark 2.13. For local, finitely generated $\mathbb{C}$-algebras of Krull dimension 1, being regular is equivalent to being a PID. Indeed, by Theorem 2.12, regularity implies being a UFD, but by standard facts of commutative algebra, a Noetherian UFD with Krull dimension 1 is a DVR, and for a DVR being a UFD is the same as being a PID (and finitely generated $\mathbb{C}$-algebras are Noetherian). Conversely any local PID that has Krull dimension 1 is not a field, and is regular: let p be a generator of the maximal ideal, then $p \neq 0$ and $(p)/(p)^2$ admits as a basis over $\mathbb{C}$ the image of p in the quotient.

Proposition 2.14. *Any smooth irreducible affine variety is normal.*

Proof. If V is an irreducible variety, then $\mathbb{C}[V]$ is a domain, so by standard commutative algebra, we have:

$$\mathbb{C}[V] = \bigcap_{\substack{\mathfrak{m} \leq \mathbb{C}[V] \\ \text{maximal ideal}}} \mathbb{C}[V]_{\mathfrak{m}} = \bigcap_{p \in V} \mathcal{O}_p,$$

viewing localizations of $\mathbb{C}[V]$ as subrings of its fraction field $\mathbb{C}(V)$.
If further V is smooth, by Theorem 2.12 above, $\mathcal{O}_p$ is a UFD for all $p \in V$, so by Exercise 1.1, $\mathcal{O}_p$ is normal for all $p \in V$. By Exercise 1.4, $\mathbb{C}[V] = \bigcap_{p \in V} \mathcal{O}_p$ is then itself normal. By definition, this means that the variety V is normal. □

Normality is one type of "regularity condition" that one can impose on a variety. Smoothness is another one, and it is stronger, by the proposition we just showed. Actually, smoothness is the strongest regularity condition for varieties that one usually asks for.

2.2 Product varieties

Given two affine varieties $V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^m$, their product as a set is naturally again an affine variety. Indeed, letting $I = \mathbb{I}(V)$ and $J = \mathbb{I}(W)$, we have that $V \times W = \mathbf{V}((I, J)) \subseteq \mathbb{C}^{n+m}$ where I is embedded in $\mathbb{C}[x_1, \ldots, x_{n+m}]$ in variables $x_1, \ldots, x_n$ and J is embedded in the same polynomial ring by considering the polynomials in J as having variables $x_{n+1}, \ldots x_{n+m}$. As for the ring of regular functions:

Proposition 2.15. *In the setting above,* $\mathbb{C}[V \times W] = \mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[W]$.

Proof. Exercise 2.2. □

We could also have defined $V \times W$ as the variety associated with the ring $\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[W]$, if we proved that the latter was a finitely generated reduced $\mathbb{C}$-algebra, and then check that $V \times W$ defined in this way enjoys the universal property of the product for V and W as varieties.

Remark 2.16. The Zariski topology on $V \times W$ as an affine variety is in general not the product topology. For instance, $\mathbb{C} \times \mathbb{C}$ with the product topology has irreducible closed sets:

$$\{\emptyset, \mathbb{C} \times \mathbb{C}\} \cup \{\{c\} \times \mathbb{C}, \{c\} \times \{d\}, \mathbb{C} \times \{d\} \mid c, d \in \mathbb{C}\},$$

whereas $\mathbb{C}^2$ has more irreducible closed subsets in the Zariski topology, for instance all irreducible plane curves like the parabola.

2.3 Algebraic tori

Definition 2.17. A *torus* is an affine algebraic variety T isomorphic to $(\mathbb{C}^*)^n$ for some $n \in \mathbb{N}$ (the latter is a variety, see example 1.1 in notes for week 1). In particular, T is an algebraic group since $(\mathbb{C}^*)^n$ is. Let then $\dim T = n$.

We do not define a torus as being the variety $(\mathbb{C}^*)^n$ itself for some $n \in \mathbb{N}$ in the same way as we do not define a finite dimensional vector space over $\mathbb{C}$ to be $\mathbb{C}^n$ itself for some $n \in \mathbb{N}$, the latter corresponding to fixing a basis in our vector space. We do not want to make such non-canonical choices, which is one of the reasons why we only ask for something *isomorphic* to one of the varieties $(\mathbb{C}^*)^n$, for $n \in \mathbb{N}$.

2.3.1 Characters of tori

Definition 2.18. A *character* of a torus T is a homomorphism of (algebraic) groups $\chi : T \to \mathbb{C}^*$.

Example 2.19. Consider the torus $(\mathbb{C}^*)^n$ itself for some $n \in \mathbb{N}$. Every n-uple of integers $m := (a_1, \dots, a_n) \in \mathbb{Z}^n$ defines a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ mapping $(t_1, \dots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}$.

Proposition 2.20. *All characters of the torus $(\mathbb{C}^*)^n$ are of the form χ^m for some $m \in \mathbb{Z}^n$, as defined in Example 2.19.*

Proof. (Sketch) Let χ be a character on this torus. Then it corresponds to a map between the rings of regular functions on the varieties $\mathbb{C}^*$ and $(\mathbb{C}^*)^n$, i.e. Example 1.1, to a ring homomorphism

$$\varphi : \mathbb{C}[x, x^{-1}] \longrightarrow \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}].$$

Since x is a unit in the first ring, it gets mapped to a unit, and one can show that this implies that $\varphi(x) = \lambda \cdot x_1^{a_1} \cdots x_n^{a_n}$ for some $m = (a_1, \dots, a_n) \in \mathbb{Z}^n$ and $\lambda \in \mathbb{C}^*$. Since χ is a group homomorphism, $\chi(1, \dots, 1) = 1$, so really identifying the polynomials in the domain and codomain of φ as regular functions on our varieties, we have: $\varphi(x)(1, \dots, 1) = \lambda = x(\chi(1, \dots, 1)) = 1$. Going back from rings to varieties, we get $\chi(t_1, \dots, t_n) = t_1^{a_1} \cdots t_n^{a_n}$ $\forall (t_1, \dots, t_n) \in (\mathbb{C}^*)^n$. $\square$

Proposition 2.21. *Characters on a given torus T form an abelian group $(M, +)$, with $(\chi + \chi')(t) := \chi(t)\chi'(t)$ for all $t \in T$. This group of characters is isomorphic to $\mathbb{Z}^n$, i.e. the characters of T form a* lattice of rank $\dim T$ *(namely a free abelian group of this rank).*

Indeed, note that in Example 2.19 above, the addition of characters we defined, i.e. pointwise multiplication, correspond to addition of the associated vectors in $\mathbb{Z}^n$.

Notation. We write $\chi^m : T \to \mathbb{C}^*$ for the character represented by $m \in M$.

Proposition 2.22.

- *Let $\varphi : T_1 \to T_2$ be an (algebraic) group homomorphism of tori. Then the image of φ is a closed subtorus of T_2.*
- *Let T be a torus and $H \leq T$ be an irreducible subgroup. Then H is a torus.*

Note that this result does not hold for non-irreducible subgroups in general: consider the subgroup $\mathbf{V}(x^n - 1) \subseteq \mathbb{C}^*$ in a one-dimensional torus (the group of n-th roots of unity in $\mathbb{C}^*$) for $n \geq 2$. It is not a torus, but consists instead of n distinct points (so it is not irreducible).

Proposition 2.23. *Let T be a torus acting by linear maps on a finite dimensional $\mathbb{C}$-vector space W, i.e. there is a group homomorphism $S : T \to GL_{\mathbb{C}}(W)$. Then there exists a basis of W, such that the induced map $T \to GL_n(\mathbb{C})$ factors through the diagonal torus $(\mathbb{C}^*)^n \subseteq GL_n(\mathbb{C})$ (corresponding to diagonal matrices).*

Namely we can simultaneously diagonalize all matrices in the image of S. This is a generalization of the fact that if two diagonalizable matrices commute, then there exists a basis in which they are both diagonal. Here all matrices in the image of S commute, because T is abelian as a group.

In representation theory, characters can be used to decompose representations. Characters on tori can be used with the same purpose, and provide an analog of Proposition 2.23 without having to choose a basis:

Proposition 2.24. *In the setting of Proposition 2.23, define for any character $m \in M$ the subspace:*

$$W_m = \{w \in W \mid S(t)(w) = \chi^m(t)w \ \forall t \in T\}.$$

Then W decomposes as a direct sum: $W = \bigoplus_{m \in M} W_m$.

The spaces W_m correspond to some kind of common eigenspaces for all transformations in the image of S. In particular, if $W_m \neq \{0\}$, it means that the character χ^m "detects eigenvalues": for any $t \in T$, $\chi^m(t)$ is an eigenvalue of $S(t)$.

2.3.2 One-parameter subgroups of tori

Definition 2.25. A *one-parameter subgroup* of a torus T is a homomorphism of algebraic groups $\lambda : \mathbb{C}^* \to T$.

Example 2.26. Similarly to Example 2.19, any n-tuple of integers corresponds to a one-parameter subgroup of $(\mathbb{C}^*)^n$: indeed to $u := (b_1, \dots, b_n) \in \mathbb{Z}^n$, we can associate the one-parameter subgroup λ^u such that $\lambda^u(t) = (t^{b_1}, \dots, t^{b_n})$ for all $t \in \mathbb{C}^*$.

Proposition 2.27. *All one parameters subgroups of $(\mathbb{C}^*)^n$ are of the form λ^u for some $u \in \mathbb{Z}^n$, as defined in Example 2.26. In general one-parameter subgroups of a torus T form a lattice N of rank* $\dim T$.

Notation. We write $\lambda^u : \mathbb{C}^* \to T$ for the one-parameter subgroup represented by $u \in N$.

2.3.3 Duality between characters and one-parameter subgroups

Proposition 2.28. *One-parameter subgroups on a torus T are dual to characters on T, in the sense that there exists a bilinear, perfect (i.e. non-degenerate) pairing:*

$$\langle \bullet, \bullet \rangle : M \times N \longrightarrow \mathbb{Z},$$

associating to $(m, u) \in M \times N$ the unique integer $\ell \in \mathbb{Z}$ such that the composition $\chi^m \circ \lambda^u : \mathbb{C}^* \to \mathbb{C}^*$ sends any $t \in \mathbb{C}^*$ to t^ℓ.

The integer ℓ exists because $\chi^m \circ \lambda^u$ becomes a character on the torus $\mathbb{C}^*$, and we have seen that all characters on such a torus are of the form $t \mapsto t^\ell$ for some $\ell \in \mathbb{Z}$.
Let us make this pairing more explicit if we choose an isomorphism of our torus T with $(\mathbb{C}^*)^n$. Such an isomorphism induces identifications $M \cong \mathbb{Z}^n$ and $N \cong \mathbb{Z}^n$ where n-tuples of integers correspond to characters and one-parameters subgroups as in Examples 2.19 and 2.26. Let $m = (a_1, \dots, a_n) \in \mathbb{Z}^n \cong M$ and $u = (b_1, \dots, b_n) \in \mathbb{Z}^n \cong N$. Then $\langle m, u \rangle = \sum_{i=1}^n a_i b_i$, which corresponds to the standard dot product, and this proves in particular that the pairing is perfect. This formula comes from the fact that for $t \in T \cong (\mathbb{C}^*)^n$, $\chi^m(\lambda^u(t)) = \chi^m((t^{b_1}, \dots, t^{b^n})) = (t^{b_1})^{a_1} \cdots (t^{b_n})^{a_n} = t^{a_1 b_1 + \cdots + a_n b_n}$.

By non-degeneracy of the pairing, we get isomorphisms of abelian groups:

$$\begin{aligned} N &\cong \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \\ u &\mapsto \langle \bullet, u \rangle \end{aligned} \qquad\qquad \begin{aligned} M &\cong \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \\ m &\mapsto \langle m, \bullet \rangle. \end{aligned}$$

Proposition 2.29. *Let T be a torus. There is an isomorphism of groups:*

$$\begin{aligned} N \otimes_{\mathbb{Z}} \mathbb{C}^* &\xrightarrow{\sim} T \\ u \otimes t &\longmapsto \lambda^u(t). \end{aligned}$$

Hence for a lattice N we can consider the torus associated with N, *defined as $T_N = N \otimes_Z \mathbb{C}^*$.*

2.4 Exercises

Exercise 2.1. Show that a point lying in the intersection of at least two irreducible components of an affine variety cannot be smooth. In particular, a connected, smooth variety is irreducible. (*Hint: You may use that any regular local ring is a domain*).

Exercise 2.2. Let V and W be affine varieties and let $S \subset V$ be a subset.

(a) Show that $\mathbb{C}[V \times W] = \mathbb{C}[V] \otimes \mathbb{C}[W]$.

(b) Prove that $\overline{S} \times W = \overline{S \times W}$, where $\overline{(\cdot)}$ denotes the Zariski closure.

(c) Assume that V and W are irreducible. Prove that $V \times W$ is irreducible.

Exercise 2.3. Let $I \subset \mathbb{C}[x_0, \ldots, x_d]$ be the ideal generated by $x_i x_{j+1} - x_{i+1} x_j$ for $0 \leq i < j \leq d-1$ and $\widehat{C}_d$ the surface parametrized by

$$\Phi(s,t) = (s^d, s^{d-1}t, \ldots, st^{d-1}, t^d) \in \mathbb{C}^{d+1}.$$

(a) Prove that $\widehat{C}_d = \mathbf{V}(I)$.

(b) Prove that $\widehat{C}_d$ is irreducible.
Hint: Write $\widehat{C}_d$ as the Zariski-closure of a torus.

In the next chapter we will see that $\widehat{C}_d$ is an example of an affine toric variety.

Chapter 3. Affine toric varieties

Chapter written by Matthias Schuller after the talk of Joel Hakavuori and Isak Sundelius

3.1 The definition of affine toric varieties

Definition 3.1. An *affine toric variety* is an irreducible affine variety V that contains a torus T as a Zariski open subset, and such that the action of T on itself extends to V, meaning an action $T \times V \to V$ given by a morphism.

Example 3.2. Obvious examples are $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$. For the latter the action is component-wise multiplication and clearly extends.
Another example is the curve $C = \mathbf{V}(x^3 - y^2)$. Its torus is $C \backslash \{0\}$ which is isomorphic to $\mathbb{C}^*$ via

$$\begin{aligned} \mathbb{C}^* &\to C\backslash\{0\} \\ t &\mapsto (t^2, t^3). \end{aligned}$$

The action extends as follows :

$$\begin{aligned} \mathbb{C}^* \times C &\to C \\ (t, (s^2, s^3)) &\mapsto ((ts)^2, (ts)^3) \end{aligned}$$

where we use $C = \{(s^2, s^3) | s \in \mathbb{C}\}$.

3.2 Toric varieties from lattices

Given a torus T_N with character lattice M and a set $\mathcal{A} = \{m_1, ..., m_s\} \subset M$, consider the map

$$\Phi_{\mathcal{A}} : T_N \to \mathbb{C}^s$$

defined by

$$\Phi_{\mathcal{A}}(t) = (\chi^{m_1}(t), ..., \chi^{m_s}(t)) \,.$$

We then define $Y_{\mathcal{A}} \subset \mathbb{C}^s$ to be the Zariski closure of the image of $\Phi_{\mathcal{A}}$.

Proposition 3.3. *The above constructed $Y_{\mathcal{A}}$ is an affine toric variety with $\mathbb{Z}\mathcal{A}$ as character lattice, where $\mathbb{Z}\mathcal{A} \subset M$ is the sublattice generated by $\mathcal{A}$.*

Proof. The map $\Phi_{\mathcal{A}}$ can be regarded as a map of tori $T_N \to (\mathbb{C}^*)^s$. Because it is defined from characters, it is indeed a group homomorphism. Then by Proposition 2.22, $T := \Phi_{\mathcal{A}}(T_N)$ is a torus that is closed in $(\mathbb{C}^*)^s$. Since $Y_{\mathcal{A}}$ is the Zariski closure of T, it follows that $Y_{\mathcal{A}} \cap (\mathbb{C}^*)^s = T$, and then that T is open in $Y_{\mathcal{A}}$, because $(\mathbb{C}^*)^s$ is open in $\mathbb{C}^s$. As a torus, T is irreducible, hence so is its Zariski closure $Y_{\mathcal{A}}$.
To finish proving that $Y_{\mathcal{A}}$ is an affine toric variety with torus T, let's consider the action of T. For $t \in T \subset (\mathbb{C}^*)^s$ we have an action on $\mathbb{C}^s$ inherited by $(\mathbb{C}^*)^s$ which takes varieties to varieties. Then we have

$$T = t \cdot T \subset t \cdot Y_{\mathcal{A}}.$$

Any variety in $\mathbb{C}^s$ containing T contains its closure $Y_{\mathcal{A}}$, so $Y_{\mathcal{A}} \subset t \cdot Y_{\mathcal{A}}$. Repeating that for t^{-1} gives $Y_{\mathcal{A}} = t \cdot Y_{\mathcal{A}}$. The action of T extends on $Y_{\mathcal{A}}$, therefore $Y_{\mathcal{A}}$ is an affine toric variety.
It remains to compute the character lattice of $Y_{\mathcal{A}}$. Let's denote it by M'. We have the following commutative diagram :

$$\begin{array}{ccc} T_N & \xrightarrow{\Phi_{\mathcal{A}}} & (\mathbb{C}^*)^s \\ & \searrow & \uparrow \\ & & T. \end{array}$$

A map of tori induces a map on character lattice via pre-composition. Hence the above diagram induces a commutative diagram of character lattices

$$\begin{array}{ccc} M & \xleftarrow{\widehat{\Phi}_{\mathcal{A}}} & \mathbb{Z}^s \\ & \nwarrow & \downarrow \\ & & M' \end{array}$$

where $\widehat{\Phi}_{\mathcal{A}} : \mathbb{Z}^s \to M$ is the map induced by $\Phi_{\mathcal{A}}$. This map takes the standard basis $e_1, ..., e_s$ to $m_1, ..., m_s$, thus its image is the sublattice $\mathbb{Z}\mathcal{A}$ generated by $\mathcal{A} = \{m_1, ..., m_s\}$. From the diagram we then obtain $M' \cong \widehat{\Phi}_{\mathcal{A}}(\mathbb{Z}^s) = \mathbb{Z}\mathcal{A}$, which concludes the proof. □

3.3 Toric ideals

Let $Y_{\mathcal{A}}$ be defined as before, and define

$$L := \ker(\widehat{\Phi}_{\mathcal{A}}) = \left\{ (l_1, ..., l_s) \in \mathbb{Z}^s \;\middle|\; \sum_{i=1}^{s} l_i m_i = 0 \right\}.$$

Given $l = (l_1, ..., l_s) \in \mathbb{Z}^s$, set $l_+ = \sum_{l_i > 0} l_i e_i$ and $l_- = -\sum_{l_i < 0} l_i e_i$, then $l = l_+ - l_i$.
We define

$$I_L = \langle x^{l_+} - x^{l_-} \mid l \in L \rangle \subset \mathbb{C}[x_1, ..., x_s]$$

where $x^l = \prod x_i^{l_i}$.

Proposition 3.4. *The ideal of the affine toric variety $Y_{\mathcal{A}}$ is*

$$\mathbf{I}(Y_{\mathcal{A}}) = I_L = \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s, \alpha - \beta \in L \rangle.$$

Proof. The second equality of the statement corresponds to Exercise 3.1 and we will use it here as a description of I_L.
Let's start by showing the inclusion $I_L \subset \mathbf{I}(Y_{\mathcal{A}})$. Take $\alpha, \beta \in \mathbb{N}^s$ such that $\alpha - \beta \in L$. Then we have

$$\sum_{i=1}^{s} (\alpha_i - \beta_i) m_i = 0 \implies \sum_{i=1}^{s} \alpha_i m_i = \sum_{i=1}^{s} \beta_i m_i.$$

Let $f = x^\alpha - x^\beta$. Then for $p = (\chi^{m_1}(t), ..., \chi^{m_s}(t)) \in \operatorname{im} \Phi_{\mathcal{A}}$, we see that

$$f(p) = \chi^{\sum \alpha_i m_i}(t) - \chi^{\sum \beta_i m_i}(t) = 0,$$

so $x^\alpha - x^\beta \in \mathbf{I}(\operatorname{im} \Phi_{\mathcal{A}})$. Now, since $Y_{\mathcal{A}}$ is the Zariski closure of $\operatorname{im} \Phi_{\mathcal{A}}$, we must have $\mathbf{I}(Y_{\mathcal{A}}) = \mathbf{I}(\operatorname{im} \Phi_{\mathcal{A}})$, otherwise $\mathbf{V}(\mathbf{I}(\operatorname{im} \Phi_{\mathcal{A}}))$ would be a closed subset containing $\operatorname{im} \Phi_{\mathcal{A}}$ and strictly smaller than $Y_{\mathcal{A}}$. Hence $x^\alpha - x^\beta$ is in $\mathbf{I}(Y_{\mathcal{A}})$.
Next we show the inclusion $\mathbf{I}(Y_{\mathcal{A}}) \subset I_L$. Fix a monomial order on $\mathbb{C}[x_1, ..., x_n]$. Also, for simplicity, fix an isomorphism $T_N \cong (\mathbb{C}^*)^n$ so that we may assume $M = \mathbb{Z}^n$ and $\chi^{m_i}(t) = t^{m_i}$.
Suppose $\mathbf{I}(Y_{\mathcal{A}}) \not\subset I_L$. Pick $f \in \mathbf{I}(Y_{\mathcal{A}}) \backslash I_L$ with minimal leading term, denote it by x^α. Since $f(t^{m_1}, ..., t^{m_s})$ is zero as a polynomial in $t_1, ..., t_s$, there must be some cancellation happening involving the terms coming from x^α. That is, there is some monomial x^β in f such that

$$\prod (t^{m_i})^{\alpha_i} = \prod (t^{m_i})^{\beta_i}$$

which implies

$$\sum \alpha_i m_i = \sum \beta_i m_i.$$

This gives $\alpha - \beta \in L$. But then $x^\alpha - x^\beta \in I_L \subset \mathbf{I}(Y_{\mathcal{A}})$ so $f - x^\alpha + x^\beta \in \mathbf{I}(Y_{\mathcal{A}}) \backslash I_L$ and the latter has strictly smaller leading term that f, which is a contradiction. Therefore $\mathbf{I}(Y_{\mathcal{A}}) = I_L$. □

Definition 3.5. Let $L \subset \mathbb{Z}^s$ be a sublattice.

- The ideal $I_L = \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s, \alpha - \beta \in L \rangle$ is called a *lattice ideal*.
- A *toric ideal* is a prime lattice ideal.

Proposition 3.6. *The ideal $I \subset \mathbb{C}[x_1, ..., x_s]$ is toric if and only if it is prime and generated by binomials.*

Proof. If I is toric then by definition it is prime and generated by binomials.
Suppose I is a prime ideal generated by binomials of the form $x^\alpha - x^\beta$. Then $T := \mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is nonempty since it contains $(1, ..., 1)$. Furthermore, if t, t' are in T it is easy to see that $t \cdot t'$ and t^{-1} also are in T, hence T is a subgroup of $(\mathbb{C}^*)^s$. Since I is prime, $\mathbf{V}(I) \subset \mathbb{C}^s$ is irreducible, so $T = \mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is an irreducible subvariety of $(\mathbb{C}^*)^s$ that is also a subgroup. Proposition 2.22 then tells us that T is a torus.
Projecting of the i-th coordinate gives a character of T, which we write as $\chi^{m_i} : T \to \mathbb{C}^*$ for m_i in M, the character lattice of T. Let $\mathcal{A} = \{m_1, ..., m_s\}$. For $t \in T$ we have

$$\Phi_{\mathcal{A}}(t) = (\chi^{m_1}(t), ..., \chi^{m_s}(t)) = t,$$

so $\operatorname{Im} \Phi_{\mathcal{A}} = T$. Since $T = \mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is dense in $\mathbf{V}(I)$, we get $Y_{\mathcal{A}} = V(\mathbf{I})$. Because I is prime, we have $I = \mathbf{I}(Y_{\mathcal{A}})$ by the Nullstellensatz. Therefore, by Proposition 3.4, I is toric. ☐

3.4 Affine semigroups

Definition 3.7. A *semigroup* is a set S with an associative binary operation and identity. An *affine semigroup* is a semigroup S satisfying :

- The binary operation is commutative, we will write it $+$.
- S is finitely generated. In other words, there is a fine set $\mathcal{A} \subset S$ such that $S = \mathbb{N}\mathcal{A} = \{\sum_{m \in \mathcal{A}} a_m m | a_m \in \mathbb{N}\}$.
- S can be embedded in a lattice M.

Example 3.8. $\mathbb{N}^n \subset \mathbb{Z}^n$ is an affine semigroup. It is generated by $\mathcal{A} = \{e_1, ..., e_n\}$.

Definition 3.9. Given an affine semigroup S, the *semigroup algebra* $\mathbb{C}[S]$ is the $\mathbb{C}$-vector space with S as basis and with multiplication induced by the semigroup structure of S. To make this more explicit, we think of the lattice M in which S is embedded as the character lattice of some torus T_N. Then if $\mathcal{A} = \{m_1, ..., m_s\}$ is a generating set of S we define $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, ..., \chi^{m_s}]$.

Example 3.10.

- If $S = \mathbb{N}^n \subset \mathbb{Z}^n$, then $\mathbb{C}[S] \cong \mathbb{C}[x_1, ..., x_n]$.
- If $S = \mathbb{Z}^n = \mathbb{N}\mathcal{A}$ for $\mathcal{A} = \{\pm e_1, ..., \pm e_n\}$, then $\mathbb{C}[S] \cong \mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}] \cong \mathbb{C}[T_N]$.

Proposition 3.11. *Let $S \subset M$ be an affine semigroup. Then*

(a) $\mathbb{C}[S]$ *is a domain and is finitely generated as a* $\mathbb{C}$*-algebra.*

(b) $\operatorname{Spec}(\mathbb{C}[S])$ *is an affine toric variety whose torus has character lattice $\mathbb{Z}S$, and if $S = \mathbb{N}\mathcal{A}$ for a finite set $\mathcal{A} \subset M$, then* $\operatorname{Spec}(\mathbb{C}[S]) = Y_{\mathcal{A}}$.

Proof. (a) Since we have $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, ..., \chi^{m_s}]$, it is indeed finitely generated. The embedding $S \subset M$ implies $\mathbb{C}[S] \subset \mathbb{C}[M]$. By the previous example, we know that the latter is a domain, thus so is $\mathbb{C}[S]$.
(b) Suppose $S = \mathbb{N}\mathcal{A}$ with $\mathcal{A} = \{m_1, ..., m_s\} \subset S \subset M$. We define the morphism

$$\pi : \mathbb{C}[x_1, ..., x_s] \to \mathbb{C}[M]$$

by $x_i \mapsto \chi^{m_i}$. It corresponds to the morphism

$$\Phi_{\mathcal{A}} : T_n \to \mathbb{C}^s,$$

that is, we have $\pi = (\Phi_{\mathcal{A}})^*$. Exercise 3.2 gives that $\ker \pi$ is the toric ideal $\mathbf{I}(Y_{\mathcal{A}})$. The image of π is $\mathbb{C}[\chi^{m_1}, ..., \chi^{m_s}] = \mathbb{C}[S]$. Computing the coordinate ring of $Y_{\mathcal{A}} \subset \mathbb{C}^s$ yields the following :

$$\begin{aligned}\mathbb{C}[Y_{\mathcal{A}}] &= \mathbb{C}[x_1, ..., x_s]/\mathbf{I}(Y_{\mathcal{A}}) \\ &= \mathbb{C}[x_1, ..., x_s]/\ker \pi \\ &\cong \operatorname{Im} \pi = \mathbb{C}[S].\end{aligned}$$

This implies $\operatorname{Spec}(\mathbb{C}[S]) = Y_{\mathcal{A}}$, so $\operatorname{Spec}(\mathbb{C}[S])$ is an affine toric variety. From Proposition 3.3, its character lattice is $\mathbb{Z}\mathcal{A}$. Since $\mathbb{N}\mathcal{A} = S$, we have $\mathbb{Z}S = \mathbb{Z}\mathcal{A}$. This concludes the proof. □

3.5 Equivalence of constructions

First, we will study the action of T_N on $\mathbb{C}[M]$. The action of T_N on itself induces an action on $\mathbb{C}[M]$ as follows : for $t \in T_N$ and $f \in \mathbb{C}[M]$, define $t \cdot f \in \mathbb{C}[M]$ by $p \mapsto f(t^{-1}p)$ for $p \in T_N$.

Lemma 3.12. *Let $A \subset \mathbb{C}[M]$ be a subspace stable under the action of T_N. Then*

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m.$$

Proof. Let $A' = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m$. The inclusion $A' \subset A$ is immediate.
For the other inclusion, pick $f \in A \backslash \{0\} \subset \mathbb{C}[M]$. We can write

$$f = \sum_{m \in \mathcal{B}} c_m \chi^m$$

where $\mathcal{B} \subset M$ is finite and $c_m \neq 0$ for all $m \in \mathcal{B}$. Define

$$B = \operatorname{Span}_{\mathbb{C}}(\chi^m \mid m \in \mathcal{B}) \subset \mathbb{C}[M].$$

Let's evaluate $t \cdot \chi^m$. It is given by $p \mapsto \chi^m(t^{-1} \cdot p) = \chi^m(t^{-1})\chi^m(p)$, so we can write $t \cdot \chi^m = \chi^m(t^{-1})\chi^m$. It follows that B is stable under the action of T_N. Since A is stable under the action of T_N, we get that $B \cap A$ is as well. It is also a finite dimensional vector space, and the above results show that T_N acts on it linearly. Then a previous proposition implies that $B \cap A$ decomposes as the direct sum

$$B \cap A = \bigoplus_{m \in M} B_m$$

with

$$B_m = \{b \in B \cap A \,|\, t \cdot b = \chi^m(t) b \ \ \forall t \in T_N\}.$$

Take $m \in M$ and $b = \sum_{l \in \mathcal{B}} c_l \chi^l \in B \cap A$. Suppose b is nonzero and $b \in B_m$. This means that for every $t \in T_N$:

$$\begin{aligned}\sum_{l \in \mathcal{B}} c_l \chi^m(t) \chi^l &= \chi^m(t) b \\ &= t \cdot b \\ &= t \cdot \left(\sum_{l \in \mathcal{B}} c_l \chi^l \right) \\ &= \sum_{l \in \mathcal{B}} c_l \chi^l(t^{-1}) \chi^l.\end{aligned}$$

This implies that for each component χ^l, for every $t \in T_N$ we have

$$c_l \chi^m(t) = c_l \chi^l(t^{-1}) = c_l \chi^{-l}(t).$$

So either $c_l = 0$ or $m = -l$. Since b is nonzero, $c_l \neq 0$ for some (unique) l, then $m = -l$ and $b = c_l \chi^l$. Thus the B_m contain either only 0 or a character and its multiples. This means that $B \cap A$ is spanned by characters. Then the expression for $f \in B \cap A$ implies $\chi^m \in A$ for every $m \in \mathcal{B}$. Therefore f is in A', which proves $A = A'$. □

3.6 Exercises

Exercise 3.1. Let $L \subseteq \mathbb{Z}^s$ be a sublattice. Prove that

$$\langle x^{\ell_+} - x^{\ell_-} \mid \ell \in L \rangle = \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s, \alpha - \beta \in L \rangle.$$

Note that when $\ell \in L$, the vectors $\ell_+, \ell_- \in \mathbb{N}^s$ have disjoint support (i.e., no coordinate is positive in both), while this may fail for arbitrary $\alpha, \beta \in \mathbb{N}^s$ with $\alpha - \beta \in L$.

Exercise 3.2. Fix an affine variety V. Then elements $f_1, \ldots, f_s \in \mathbb{C}[V]$ give a polynomial map $\Phi : V \to \mathbb{C}^s$, which on coordinate rings is given by

$$\Phi^* : \mathbb{C}[x_1, \ldots, x_s] \longrightarrow \mathbb{C}[V], \quad x_i \longmapsto f_i.$$

Let $Y \subseteq \mathbb{C}$ be the Zariski closure of the image of Φ.

(i) Prove that $\mathbf{I}(Y) = \mathrm{Ker}(\Phi^*)$.

(ii) Explain how this applies to the proof of proposition which tells us that semigroup algebras give rise to affine toric varieties.

Exercise 3.3. Prove that $\mathbf{I} = \langle x^2 - 1, xy - 1, yz - 1 \rangle$ is the lattice ideal for the lattice

$$L = \{(a, b, c) \in \mathbb{Z}^3 \mid a + b + c \equiv 0 \bmod 2\} \subseteq \mathbb{Z}^3.$$

Exercise 3.4. Suppose that $\varphi : M \to M$ is a group isomorphism. Fix a finite set $\mathcal{A} \subseteq M$ and let $\mathcal{B} = \varphi(\mathcal{A})$. Prove that the toric varieties $Y_\mathcal{A}$ and $Y_\mathcal{B}$ are equivariantly isomorphic, meaning that the isomorphism respects the torus action.

Chapter 4. Convex polyhedral cones

Chapter written by Zichen Gao after the talk of Juan Felipe Celis Rojas and Emma Marie Billet

Theorem 4.1. *Let V be an affine variety. The following are equivalent:*

(a) V is toric.

(b) $V = Y_{\mathcal{A}}$ for some $\mathcal{A} \subseteq M$ finite.

(c) V is defined by a toric ideal.

(d) $V = \mathrm{Spec}(\mathbb{C}[S])$ for S an affine semigroup.

Proof. (b)$\Rightarrow$ (a) and $(c) \Leftrightarrow (b) \Leftrightarrow (d)$ was already proved in Chapter 3. It remains to show that $a \Rightarrow (d)$: Let V be a toric variety with torus T_N, and let M be the character lattice of T_N. Then the inclusion $T_N \subset V$ induces a homomorphism $\varphi : \mathbb{C}[V] \to \mathbb{C}[T_N] = \mathbb{C}[M]$. This map is injective, since T_N is dense in V. Recall from Chapter 3 that we already know

$$\mathbb{C}[V] = \oplus_{\chi^m \in \mathbb{C}[V]} \mathbb{C}\chi^m.$$

Set $S = \{m \in M | \chi^m \in \mathbb{C}[V]\}$, then we have $\mathbb{C}[V] = \mathbb{C}[S]$, and hence $V = \mathrm{Spec}(\mathbb{C}[V]) = \mathrm{Spec}(\mathbb{C}[S])$. To show that S is an affine semigroup, what is left is to show that it has a finite generating set. As $\mathbb{C}[V]$ is finitely generated, $\mathbb{C}[V] = \mathbb{C}[f_1, \ldots, f_s]$ for finitely many f_i's. For each f_i we write a factorization of it under χ^m's. The set of all χ^m that appear is a finite generating set of S. Therefore, S is indeed an affine semigroup. □

Example 4.2. Let $V = \mathbb{V}(xy - zw)$. It is the closure of the image of the map

$$\begin{aligned} (\mathbb{C}^*)^3 &\to V \\ (t_1, t_2, t_3) &\mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \end{aligned}$$

The lattice points used in this map can be represented as the column of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

4.1 Convex Polyhedral Cones

Fix a pair of dual vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$.

Definition 4.3. A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}(S) = \{\sum_{u \in S} \lambda_u u | \lambda_u \geq 0\} \subseteq N_{\mathbb{R}},$$

where $S \subseteq N_{\mathbb{R}}$ is finite.

Definition 4.4. A *polytope* in $N_{\mathbb{R}}$ is a set of the form

$$P = \mathrm{Conv}(S) = \{\sum_{u \in S} |\lambda_u \geq 0, \sum_{u \in S} \lambda_u = 1\} \subseteq N_{\mathbb{R}},$$

where $S \subseteq N_{\mathbb{R}}$ is finite.

Example 4.5. Let $S = \{e_1, e_2, e_1 + e_3, e_2 + e_3\}$. The following figure shows the convex polyhedral cone and the polytope determined by S.

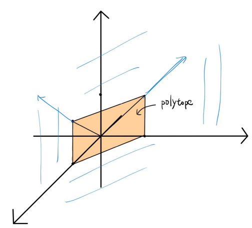


Figure 1: Cone and polytope corresponding to $e_1, e_2, e_1 + e_3, e_2 + e_3$

4.2 Dual Cones and Faces

Definition 4.6. Let $\sigma \in N_{\mathbb{R}}$ be a cone. We define its dual by

$$\sigma^\vee = \{m \in M_{\mathbb{R}} | \langle m, u \rangle \geq 0 \ \forall u \in \sigma\}.$$

Remark 4.7. $(\sigma^\vee)^\vee = \sigma$.

Example 4.8. Let $M_{\mathbb{R}} = N_{\mathbb{R}} = \mathbb{R}^3$, and suppose their pairing is given by the inner product $\langle (m_1, m_2, m_3), (n_1, n_2, n_3) \rangle = m_1 n_1 + m_2 n_2 + m_3 n_3$. Let $\sigma \in N_{\mathbb{R}}$ be defined by $\sigma = \mathrm{Cone}(e_1, e_2) \subseteq \mathbb{R}^3$, then for any $m = (x, y, z) \in M_{\mathbb{R}}$, $m \in \sigma^\vee$ if and only if $\langle (x, y, z), (1, 0, 0) \rangle \geq 0$ and $\langle (x, y, z), (0, 1, 0) \rangle \geq 0$, if and only if $x \geq 0$ and $y \geq 0$. Therefore, $\sigma^\vee = \{(x, y, z) \in \mathbb{R}^3 | x \geq 0, y \geq 0\}$.

Remark 4.9. $\sigma^\vee$ is a convex polyhedral cone.

Definition 4.10. Let $m \in M_{\mathbb{R}}$, $m \neq 0$. The hyperplane defined by m is defined to be $H_m = \{u \in N_{\mathbb{R}} | \langle m, u \rangle = 0\}$. And the closed half-space is defined to be

$$H_m^+ = \{u \in N_{\mathbb{R}} | \langle m, u \rangle \geq 0\} \subseteq N_{\mathbb{R}}.$$

H_m is a *supporting hyperplane* of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ if $\sigma \subseteq H_m^+$, and H_m^+ is a *supporting half-space*. Note that H_m is a supporting hyperplane of σ if and only if $m \in \sigma^\vee \setminus \{0\}$.

Definition 4.11. A *face* of the polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee \setminus \{0\}$. We denote this by $\tau \preceq \sigma$. A *facet* is a face of codimension 1. An *edge* of a cone is a face of dimension 1.

Remark 4.12. From now on, the dimension of a set will mean the dimension of the vector space generated by this set. And the codimension here means the difference between the dimensions of the subspace and the total space.

This relation has the following basic properties:

Lemma 4.13.

(1) If $\tau \preceq \sigma$, then τ is a cone.

(2) If $\tau_1, \tau_2 \preceq \sigma$, then $\tau_1 \cap \tau_2 \preceq \sigma$.

(3) If $\rho \preceq \tau \preceq \sigma$, then $\rho \preceq \sigma$.

Definition 4.14 (Dual face)**.** Given $\tau \leq \sigma$, where $\sigma \subseteq N_{\mathbb{R}}$ is a polyhedral cone. We define

$$\tau^\perp = \{m \in M_{\mathbb{R}} | \langle m, u \rangle = 0 \ \forall u \in \tau\}$$
$$\tau^* = \{m \in \sigma^\vee | \langle m, u \rangle = 0 \ \forall u \in \tau\} = \sigma^\vee \cap \tau^\perp.$$

We call τ^* the *dual face* of τ.

Example 4.15. The Figure 2 shows $\sigma = \mathrm{Cone}(e_1, e_2)$ and its dual $\sigma^\vee \subseteq \mathbb{R}^3$.

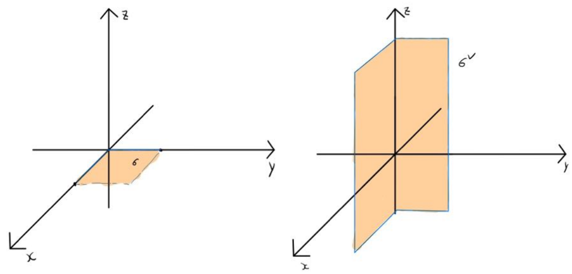


Figure 2: A 2-dimensional cone $\sigma \subseteq \mathbb{R}^3$ and its dual $\sigma^\vee \subseteq \mathbb{R}^3$

Definition 4.16. The *relative interior* $\text{Relint}(\sigma)$ of a cone $\sigma \subseteq N_\mathbb{R}$ is the interior of σ in W, where W is the span of σ. It can be charaterized by the dual space:

$$u \in \text{Relint}(\sigma) \Leftrightarrow \langle m, u \rangle > 0 \quad \forall m \in \sigma^\vee \setminus \sigma^\perp.$$

Definition 4.17. A cone $\sigma \in N_\mathbb{R}$ is *strongly convex* if $\{0\}$ is a face of σ.

Example 4.18. In Example 4.15, the cone $\sigma = \text{Cone}(e_1, e_2)$ is strongly convex, but its dual cone $\sigma^\vee$ is not.

Lemma 4.19 (Separation Lemma)**.** *Let σ_1, σ_2 be polyhedral cones in $N_\mathbb{R}$ that meet along a common face $\tau = \sigma_1 \cap \sigma_2$. Then*

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$$

for any $m \in \text{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$. In particular such an m does exist.

Definition 4.20. A cone $\sigma \subset N_\mathbb{R}$ is *rational* if $\sigma = \text{Cone}(S)$ for $S \subseteq N$ finite. Recall that $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and N is a lattice.

Remark 4.21. Faces and dual of a rational cone are rational.

A strongly convex rational polyhedral cone σ has a canonical generating set, constructed as follows. Let ρ be an edge of σ. Since σ is strongly convex, ρ is a *ray*, i.e. a half-line, and since ρ is rational, the semigroup $\rho \cap N$ is generated by a unique element u of the intersection. We call u the *ray generator* of ρ. The following Figure 3 shows the ray generator of a rational ray ρ in the plane. The points are the lattice $N = \mathbb{Z}^2$ and the white ones are $\rho \cap N$.

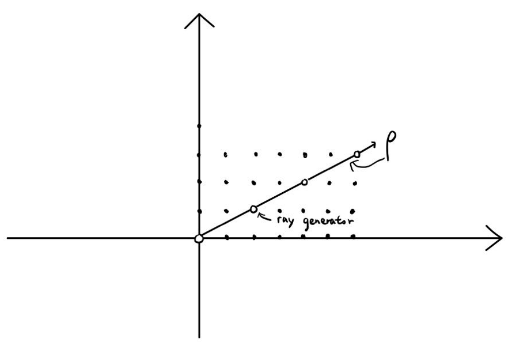


Figure 3: A rational ray $\rho \subseteq \mathbb{R}^2$ and its unique ray generator

Lemma 4.22. *A strongly convex rational polyhedral cone is generated by the ray generators of its edges.*

4.3 Semigroup Algebras and Affine Toric Varieties

Given a rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, the lattice points

$$S_\sigma = \sigma^\vee \cap M \subseteq M$$

form a semigroup. In fact, it is finitely generated.

Lemma 4.23 (Gordan's Lemma). *$S_\sigma = \sigma^\vee \cap M$ is finitely generated and hence is an affine semigroup.*

Proof. Firstly, the dual cone of σ can be written as $\sigma^\vee = \text{Cone}(T)$ for a finite $T \subseteq M$, $T = \{m_1, \ldots, m_r\}$. Let $K = \{\sum_{i=1}^r \delta_i m_i | \delta_i \in [0,1]\} \subseteq M_{\mathbb{R}}$. So K is a bounded region. Since M is discrete, we have that $K \cap M$ is finite. We claim that $T \cup (K \cap M) \subseteq S_\sigma$ generate S_σ. In fact, for any element $w \in S_\sigma$,

$$\begin{aligned} w &= \sum_{i=1}^r \lambda_i m_i \\ &= \sum_{i=1}^r \lfloor \lambda_i \rfloor m_i + \sum_{i=1}^r \{\lambda_i\} m_i \end{aligned}$$

where the latter term lies in $K \cap M$. □

Theorem 4.24. *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a rational cone with an affine semigroup S_σ. Then*

(1) $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma])$ is an affine toric variety.

(2) $\dim U_\sigma = n \Leftrightarrow$ the torus of U_σ is $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \Leftrightarrow \sigma$ is strongly convex.

Proof. (1) This is just a corollary of some previous result.
(2) U_σ is an affine variety whose torus has character lattice $\mathbb{Z}S_\sigma = S_\sigma - S_\sigma = \{m_1 - m_2 | m_1, m_2 \in S_\sigma\}$ First we prove that $M/\mathbb{Z}S_\sigma$ is torsion-free. Let $m \in M$ be such that $km \in \mathbb{Z}S_\sigma$, and $k > 1$, we need to show that $m \in \mathbb{Z}S_\sigma$. Write $km = m_1 - m_2$ where $m_1, m_2 \in S_\sigma$, then $m + m_2 = \frac{m_1}{k} + \frac{k-1}{k} m_2 \in S_\sigma$, so $m = (m + m_2) - m_2 \in \mathbb{Z}S_\sigma$. Therefore,

$$\text{The torus of } U_\sigma \text{ is } T_N \Leftrightarrow \mathbb{Z}S_\sigma = M \Leftrightarrow \text{rank } \mathbb{Z}S_\sigma = n.$$

The first equivalence follows from the fact that the torus of U_σ has character lattice $\mathbb{Z}S_\sigma$ (Proposition 1.1.14(b) of [CLS]). Since $\mathbb{Z}S_\sigma \subseteq M$, and M is the character lattice of T_N, the torus of U_σ is T_N if and only their character lattices are the same, i.e. $\mathbb{Z}S_\sigma = M$. Moreover, note that M is a finitely generated abelian group, and $\mathbb{Z}S_\sigma \subseteq M$ is a subgroup. Since we have just proved that $M/\mathbb{Z}S_\sigma$ is torsion-free, we know that $\mathbb{Z}S_\sigma = M$ if and only if their ranks are equal. So the second equivalence holds. Proposition 1.2.12 of [CLS] tells us that σ is strongly convex if and only $\dim \sigma^\vee = n$, so

$$\dim U_\sigma = n \Leftrightarrow \text{rank}\mathbb{Z}S_\sigma = n \Leftrightarrow \dim \sigma^\vee = n \Leftrightarrow \sigma \text{ is strongly convex.}$$

The first equivalence comes from the fact that the dimension of an affine toric variety is the dimension of its torus, which is the rank of its character lattice. The second equivalence is from Exercise 1.2.6 of [CLS]. □

Example 4.25. Take the cone $\sigma = \text{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq N_{\mathbb{R}} = \mathbb{R}^3$ with $N = \mathbb{Z}^3$. Then its dual cone is $\sigma^\vee = \text{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3) \subseteq \mathbb{R}^3$, and the lattice points in this cone are generated by the matrix corresponding to the affine toric variety $\mathbb{V}(xy - zw)$. Therefore $U_\sigma = \mathbb{V}(xy - zw)$, which has dimension 3. And it's clear that in this case σ is strongly convex. See Figure 4.

Definition 4.26. Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. σ is *smooth* if its ray generators form a part of a $\mathbb{Z}$-basis of N.

For example, if $\sigma = \text{Cone}(e_1, \ldots, e_r) \subseteq \mathbb{R}^r$, then

$$\sigma^\vee = \text{Cone}(e_1, \ldots, e_r, \pm e_{r+1}, \ldots, \pm e_n),$$

and $U_\sigma = \text{Spec}(\mathbb{C}[x_1, \ldots, x_r, x_{r+1}^{\pm}, \ldots, x_n^{\pm}]) \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$. This cone σ is smooth. And as we will see in future lectures, if $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ is a smooth cone of dimension r, then $U_\sigma \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$.

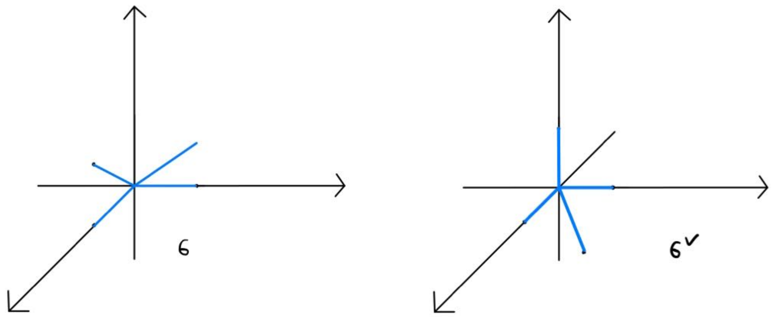


Figure 4: A strongly convex cone $\sigma \subseteq \mathbb{R}^3$ with $\dim U_\sigma = 3$

4.4 Exercises

Exercise 4.1. Let $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ be a polyhedral cone. Then:

$$\begin{aligned} \sigma \text{ is strongly convex} &\iff \{0\} \text{ is a face of } \sigma \\ &\iff \sigma \text{ contains no positive-dimensional subspace of } N_{\mathbb{R}} \\ &\iff \sigma \cap (-\sigma) = \{0\} \\ &\iff \dim \sigma^\vee = n. \end{aligned}$$

Exercise 4.2. Let $\sigma \subseteq N_{\mathbb{R}}$ be strongly convex of maximal dimension and let $S_\sigma = \sigma^\vee \cap M$. Then $\mathscr{H} = \{m \in S_\sigma |\ m \textit{ is irreducible}\}$ has the following properties:

(a) $\mathscr{H}$ is finite and generates S_σ.

(b) $\mathscr{H}$ contains the ray generators of the edges of $\sigma^\vee$.

(c) $\mathscr{H}$ is the minimal generating set of S_σ with respect to inclusion.

Hints:

(a) Using Exercise 4.1 we see that it exists $u \in \sigma \cap N \backslash 0$ such that $\langle m, u \rangle \in \mathbb{N}$ for all $m \in S_\sigma$ and $\langle m, u \rangle = 0 \iff m = 0$.

(b) Show that the ray generators of the edges of $\sigma^\vee$ are irreducible in S_σ. Given an edge ρ of $\sigma^\vee$, it will help to pick $u \in \sigma \cap N \backslash \{0\}$ such that $\rho = H_u \cap \sigma^\vee$.

Exercise 4.3. Consider the cone $\sigma = Cone(3e_1 - 2e_2, e_1) \subseteq \mathbb{R}^2$.

(a) Describe $\sigma^\vee$ and find generators of $\sigma^\vee \cap \mathbb{Z}^2$. Draw a picture of the dual cone.

(b) Compute the toric ideal of the affine variety U_σ.

Exercise 4.4. Consider the cone $\sigma = Cone(e_1, e_2, e_1 + e_2 + 2e_3) \subseteq \mathbb{R}^3$.

(a) Describe $\sigma^\vee$ and find generators of $\sigma^\vee \cap \mathbb{Z}^3$. Draw a picture of the dual cone.

(b) Compute the toric ideal of the affine variety U_σ.

Chapter 5. Smooth and normal affine toric varieties

Chapter written by Joel Hakavuori after the talk of Clotilde Freydt and Julia Morin

5.1 Points of Affine Toric Varieties

Proposition 5.1. *Let $V = Spec(\mathbb{C}[S])$ be the affine toric variety corresponding to an affine semigroup S. Then there is a bijective correspondence between*

(i) Points $p \in V$

(ii) Maximal ideals in $\mathfrak{m} \subset \mathbb{C}[V]$

(iii) Semigroup homomorphisms $S \to \mathbb{C}$, where $\mathbb{C}$ is considered as a semigroup under multiplication.

Proof. The correspondence between (i) and (ii) is the standard correspondence following from Hilbert's Nullstellensatz. For (iii), given a point $p \in V$ we define the corresponding semigroup homomorphism as $m \mapsto \chi^m(p) \in \mathbb{C}$. Conversely, let $\gamma : S \to \mathbb{C}$ be a semigroup homomorphism. By Proposition 3.11, the characters $\{\chi^m\}_{m\in S}$ for a basis for the algebra $\mathbb{C}[S]$. The map γ induces a $\mathbb{C}$-algebra homomorphism $\bar{\gamma} : \mathbb{C}[S] \to \mathbb{C}$ mapping $\chi^m \mapsto \gamma(m)$. Note that $\bar{\gamma}(1) = 1$ as $\gamma(0) = 1$, so by $\mathbb{C}-$linearity we see that this map is surjective, and hence $\mathbb{C}[S]/\ker(\bar{\gamma}) \cong \mathbb{C}$. Recall that an ideal $I \subset R$ is maximal if and only R/I is a field, so we see that $\ker(\bar{\gamma})$ is a maximal ideal of $\mathbb{C}[S] = \mathbb{C}[V]$, and thus corresponds to a point $p \in V$. Concretely, p can be expressed as $p = (\gamma(m_1), \ldots, \gamma(m_s)) \in \mathbb{C}^s$, where $\{m_1, \ldots, m_s\}$ is a generating set of S. □

This result allows us to describe the torus action on V intrinsically. Earlier we saw that the action of the torus T_N of an affine toric variety $Y_{\mathcal{A}} \subseteq \mathbb{C}^s$ is induced by the usual action of $(\mathbb{C}^*)^s$ on $\mathbb{C}^s$. However, this action requires an embedding into $\mathbb{C}^s$. To describe the action intrinsically, you will show in exercise 1 of week 5 that the point $t \cdot p$ for $t \in T_N$ and $p \in V$ corresponds to the semigroup homomorphism $m \mapsto \chi^m(t)\gamma(t)$, where γ is the homomorphism corresponding to $p \in V$ described by Proposition 5.1.

Definition 5.2. An affine semigroup S is *pointed* if $S \cap (-S) = \{0\}$, i.e., if 0 is the only invertible element of S.

Proposition 5.3. *Let V be an affine toric variety and S an affine semigroup.*

(i) If $V = Spec(\mathbb{C}[S])$, then the torus action has a fixed point if and only if S is pointed. In this case, the unique fixed point is given by the semigroup homomorphism defined by

$$m \mapsto \begin{cases} 1, & m = 0 \\ 0, & m \neq 0. \end{cases} \tag{1}$$

(ii) If $V = Y_{\mathcal{A}}$ for some $\mathcal{A} \subseteq S \setminus \{0\}$, then the torus action has a fixed point if and only if $0 \in Y_{\mathcal{A}}$, in which case the unique fixed point is 0.

Proof. For part *i*), let $\gamma : S \to \mathbb{C}$ be the semigroup homomorphism corresponding to $p \in V$. Then p is fixed by the torus action if and only if $\chi^m(t)\gamma(m) = \gamma(m)$ for all $t \in T$ and $m \in S$. As $\chi^0(t) = 1$ and $\gamma(0) = 1$, $m = 0$ satisfies the equation for all t. If $m \neq 0$, there always exists some t for which $\chi^m(t) \neq 0$, so we require that $\gamma(m) = 0$. Thus, if a fix point exists, it is unique and given by (1). From the correspondence between points and semigroup homomorphisms we get that a fixed point exists if and only if S is pointed, as only then the map (1) is a semigroup homomorphism.
For part *ii*), assume that $Y_{\mathcal{A}} \subseteq \mathbb{C}^s$ has a fixed point, so $S = \mathbb{N}\mathcal{A}$ is pointed and the unique fixed point p is given by (1). From the concrete description of p corresponding to the map (1) given in the proof of Proposition 5.1, and the fact that $0 \notin \mathcal{A}$, we see that p is the origin in $\mathbb{C}^s$, and hence $0 \in Y_{\mathcal{A}}$. Conversely, $0 \in Y_{\mathcal{A}}$ is fixed by $(\mathbb{C}^*)^s$, and hence by $T = Y_{\mathcal{A}} \cap (\mathbb{C}^*)^s \subseteq (\mathbb{C}^*)^s$. □

Here is useful corollary, whose proof is left as an exercise.

Corollary 5.4. *Let U_σ be the affine toric variety of a strongly convex rational polyhedral come $\sigma \subseteq N_\mathbb{R}$. Then the torus action has a fixed point if and only if $\dim \sigma = \dim N_\mathbb{R}$, in which case the fixed point is given by the maximal ideal*

$$\langle \chi^m |\ m \in S_\sigma \setminus \{0\}\rangle \subseteq \mathbb{C}[S_\sigma].$$

5.2 Normality and saturation

Recall that an affine variety V is normal if and only if its coordinate ring $C[V]$ is normal, i.e., $\mathbb{C}[V]$ is integrally closed in its field of fractions. In this section we study the conditions for an affine toric variety to be normal.

Definition 5.5. An affine semigroup $S \subseteq M$ is *saturated* if for all $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$, $km \in S$ implies $m \in S$.

Theorem 5.6. *Let V be an affine toric variety with torus T. Then the following are equivalent:*

(i) V is normal.

(ii) $V = Spec(\mathbb{C}[S])$, where $S \subseteq M$ is a saturated affine semigroup.

(iii) $V = Spec(\mathbb{C}[S_\sigma]) = U_\sigma$, where $S_\sigma = \sigma^\vee \cap M$ and $\sigma \subseteq N_\mathbb{R}$ is a strongly convex rational polyhedral cone.

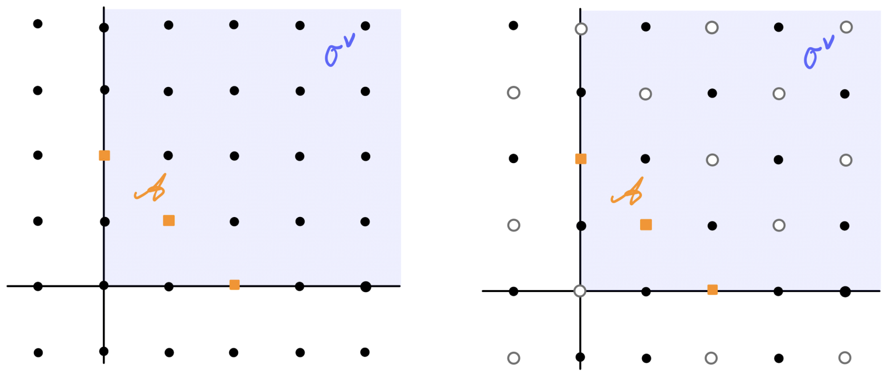
Proof. (i) ⇒ (ii): Assume V is normal, so $\mathbb{C}[S] = \mathbb{C}[V]$ is integrally closed in its field of fractions. Suppose $km \in S$ for some $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$. We want to show that $m \in S$, i.e., that S is saturated. The character χ^m can be considered as a rational function on V, and as $km \in S$, we also have $\chi^{km} \in \mathbb{C}[S]$. We observe that χ^m is a root of the monic polynomial $x^k - \chi^{km}$, which has coefficients in $\mathbb{C}[S]$. As $\mathbb{C}[S]$ is normal, we get that $\chi^m \in \mathbb{C}[S]$, and hence $m \in S$.
(ii) ⇒ (iii): Let $\mathcal{A} \subseteq S$ be a finite generating set of S, so $\mathrm{Cone}(\mathcal{A}) \subseteq M_\mathbb{R}$ and $\mathrm{rank}(\mathbb{Z}\mathcal{A}) = n$. Then dim $\mathrm{Cone}(\mathcal{A}) = n$, as the dimension of the cone of $\mathcal{A}$ is equal to the dimension of the span of S when $\mathcal{A}$ generates S. Hence $\sigma = (\mathrm{Cone}(\mathcal{A}))^\vee \subseteq N_\mathbb{R}$ is a strongly convex rational polyhedral cone, so $S \subseteq \sigma^\vee \cap M$. The other inclusion is proved in Exercise 5.4, from which we get $S = S_\sigma$.
(iii) ⇒ (i): Suppose $\sigma \subseteq N_\mathbb{R}$ is a strongly convex rational polyhedral cone. Let $\rho_1, \dots, \rho_r$ be the rays of σ, so $\sigma^\vee = \cap_{i=1}^r \rho_i^\vee$. Intersecting $\sigma^\vee$ gives $S_\sigma = \sigma^\vee \cap M = \cap_{i=1}^r S_{\rho_i}$, which in turn implies $\mathbb{C}[S_\sigma] = \cap_{i=1}^r \mathbb{C}[S_{\rho_i}]$. By exercise 4b) of week 1, $\mathbb{C}[S_\sigma]$ is normal if each $\mathbb{C}[S_{\rho_i}]$ is normal, so it suffices to show that $\mathbb{C}[S_\rho]$ is normal for a rational ray $\rho \subset N_\mathbb{R}$. Let u_ρ be the ray generator of ρ. As u_ρ is primitive, $\frac{1}{k}u_\rho \notin N_\mathbb{R}$ for all $k > 1$. Then there exists a basis $\{e_1, \dots, e_n\}$ of N with $e_1 = u_\rho$, so we may assume that $\rho = \mathrm{Cone}(e_1)$. The $\mathbb{C}[S_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$. This is the localization $\mathbb{C}[x_1, \dots, x_n]_{x_2 \cdots x_n}$, which is normal by Exercise 1.4, as $\mathbb{C}[x_1, \dots, x_n]$ is normal. □

Example 5.7. Consider again the rational normal cone $\hat{C}_d \subseteq \mathbb{C}^{d+1}$. As we have seen earlier, this is the affine toric variety of a strongly convex rational polyhedral cone, and hence normal. Looking at the $d = 2$ case, we have $\Phi_\mathcal{A}(s, t) = (s^2, st, d^2)$ for $\mathcal{A} = \{(2, 0), (1, 1), (0, 2)\}$ and $\sigma^\vee = \mathrm{Cone}(e_1, e_2)$. In Figure 5 (a) the semigroup generated by $\mathcal{A}$ does not seem to be saturated: for example $2 \cdot (1, 1) \in \mathbb{N}\mathcal{A}$ but $(1, 1) \notin \mathbb{N}\mathcal{A}$ while $(1, 1) \in \sigma^\vee$. However, recall that we have to use the lattice $\mathbb{Z}\mathcal{A}$, plotted in white in Figure 5 (b), from which we see that $\mathcal{A}$ is saturated in this lattice, which is what we expected as $\hat{C}_2$ is normal.

5.3 Normalization of affine toric varieties

Let $V = \mathrm{Spec}(\mathbb{C}[S])$ for an affine semigroup S with character lattice $M = \mathbb{Z}S$. By Theorem 5.6, V will not be normal when S is not saturated, in which case we may want to normalize V. For affine toric varieties, normalization has a particularly simple description. Let $\mathrm{Cone}(S)$ denote the cone of a finite generating set of S, and set $\sigma = \mathrm{Cone}(S)^\vee \subseteq N_\mathbb{R}$. Then we have that

Proposition 5.8. *The cone $\sigma = Cone(S)^\vee$ is a strongly convex rational polyhedral cone in $N_\mathbb{R}$, and the inclusion $\mathbb{C}[S] \hookrightarrow \mathbb{C}[\sigma^\vee \cap M]$ induces a morphism $U_\sigma \to V$ that is the normalization map of V.*



(a) (b)

Figure 5

Example 5.9. Let $\mathcal{A} = \{(4,0),(3,1),(1,3),(0,4)\} \subseteq \mathbb{Z}^2$. Then $\Phi_{\mathcal{A}}(s,t) = (s^4, s^3t, st^3, t^4)$ is a parametrization of the surface $Y_{\mathcal{A}} \subseteq \mathbb{C}$. Observe that $2 \cdot (2,2) = (4,0)+(0,4) \in \mathbb{N}\mathcal{A}$ but $(2,2) \notin \mathbb{N}\mathcal{A}$. However $(3,1)-(1,3)+(0,4) = (2,2) \in \mathbb{Z}\mathcal{A}$, so $\mathbb{N}\mathcal{A}$ is not saturated, and hence $Y_{\mathcal{A}}$ is not normal. However, using the above proposition, we can normalize by taking the cone of $S = \mathbb{N}\mathcal{A}$, which gives the same cone as considering $\mathcal{A}' = \{(4,0),(3,1),(2,2),(1,3),(0,4)\}$, corresponding to the rational normal cone $\hat{C}_4$. Hence, $\hat{C}_4$ is the normalization of $Y_{\mathcal{A}}$, with the normalization map being the projection $\mathbb{C}^5 \to \mathbb{C}^4$.

5.4 Smooth affine toric varieties

Next, we study when affine toric varieties are smooth. Recall that smoothness implies normality, so it suffices to consider toric varieties U_σ coming from strongly convex rational polyhedral cones

Definition 5.10. The set $\mathcal{H} = \{m \in S_\sigma \mid m \text{ is irreducible}\}$ is the *Hilbert basis* of the cone σ.

When $\sigma \subseteq N_{\mathbb{R}}$ is strongly convex and of maximal dimension, the Hilbert basis $\mathcal{H}$ of S_σ is finite, minimally generates S_σ and contains the ray generators of $\sigma^\vee$.
Now, let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone of maximal dimenstion and $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ the corresponding affine toric variety. By 5.4 the torus action has a unique fixed point $p_\sigma \in U_\sigma$. In this case we can relate the dimension of the Zariski tangent space of U_σ at p_σ as follows.

Lemma 5.11. *Let σ, U_σ and p_σ as above, and let $\mathcal{H}$ be the Hilbert basis of S_σ. Then $\dim T_{p_\sigma}(U_\sigma) = |\mathcal{H}|$.*

Proof. By 5.4, the maximal ideal corresponding to p_σ is given by $\mathfrak{m} = \langle \chi^m \mid m \in S_\sigma \setminus \{0\}\rangle$. As these characters form a basis of $\mathbb{C}[S_\sigma]$, we have

$$\mathfrak{m} = \bigoplus_{m \neq 0} \mathbb{C}\chi^m = \Big(\bigoplus_{m \text{ irreducible}} \mathbb{C}\chi^m\Big) \oplus \Big(\bigoplus_{m \text{ reducible}} \mathbb{C}\chi^m\Big) = \Big(\bigoplus_{m \in \mathcal{H}} \mathbb{C}\chi^m\Big) \oplus \mathfrak{m}^2.$$

Quotienting out by $\mathfrak{m}^2$ we see that $\dim \mathfrak{m}/\mathfrak{m}^2 = |\mathcal{H}|$. Mapping to the local ring $\mathcal{O}_{U_\sigma,p_\sigma}$ with maximal ideal $\mathfrak{m}_{U_\sigma,p_\sigma}$ gives an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \mathfrak{m}_{U_\sigma,p_\sigma}/\mathfrak{m}^2_{U_\sigma,p_\sigma}.$$

As $\dim T_{p_\sigma}(U_\sigma)$ is the dual of the $\mathcal{O}_{U_\sigma,p_\sigma}/\mathfrak{m}_{U_\sigma,p_\sigma}$-vector space $\mathfrak{m}_{U_\sigma,p_\sigma}/\mathfrak{m}^2_{U_\sigma,p_\sigma}$, we see that $\dim T_{p_\sigma}(U_\sigma) = |\mathcal{H}|$. ☐

Remark 5.12. The Hilbert basis $\mathcal{H}$ gives an embedding of U_σ as $Y_{\mathcal{H}} \subseteq \mathbb{C}^s$, where $s = |\mathcal{H}|$. If $U_\sigma \hookrightarrow \mathbb{C}^l$ is an embedding, we always have $\dim T_p(U_\sigma) \leq l$ for all points $p \in U_\sigma$, so the cardinality of the Hilbert basis gives a lower bound on the dimension of the affine space we are embedding our variety in. The above lemma shows that we get the most efficient embedding in terms of dimension using the Hilbert basis of S_σ when σ is a strongly convex rational polyhedral cone of maximal dimension.

Recall that a rational polyhedral cone is *smooth* if it has a generating set which is a subset of a basis of the lattice.

Theorem 5.13. *A strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ is smooth if and only if U_σ is smooth. Furthermore, all smooth affine toric varieties are of this form.*

Proof. As we saw in the previous lecture, if σ is smooth, then $U_\sigma \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$, which shows one direction. For the converse, we first consider the case when σ has dimension n. Recall that $\dim T_{p_\sigma}(U_\sigma) = |\mathcal{H}|$ for the Hilbert basis $\mathcal{H}$ of S_σ. We have $n = |\mathcal{H}| \geq |\{\text{edges } \rho \subseteq \sigma^\vee\}| \geq n$, where the first inequality follows from the fact that each edge $\rho \subseteq \sigma^\vee$ contains an element of $\mathcal{H}$ (σ is strongly convex and of maximal dimension, so $\mathcal{H}$ contains the ray generators of $\sigma^\vee$), and the second inequality holds as $\dim \sigma^\vee = n$. Hence σ has n edges, and as $M = \mathbb{Z}S_\sigma$, the ray generators of $\sigma^\vee$ generate M, and hence form a basis. Thus $\sigma^\vee$ is smooth, so $(\sigma^\vee)^\vee = \sigma$ is smooth.
Next, we prove the case where $n > \dim \sigma = r$ by reducing to the above case. Let $N_1 \subseteq N$ be the smallest saturated sublattice containing the generators of σ. Then N/N_1 is torsion free, which implies that there exists a sublattice N_2 such that $N = N_1 \bigoplus N_2$. This induces a decomposition of $M = M_1 \bigoplus M_2$ and semigroups $S_{\sigma,N_1} \subseteq M_1$ and $S_{\sigma,N} \subseteq M$ respectively, which in turn implies that $S_{\sigma,N} = S_{\sigma,N_1} \bigoplus M_2$. For the corresponding semigroup algebras we get

$$\mathbb{C}[S_{\sigma,N}] \simeq \mathbb{C}[S_{\sigma,N_1}] \bigotimes\nolimits_{\mathbb{C}} \mathbb{C}[M_2].$$

The right-hand side is the coordinate ring of $U_{\sigma,N_1} \times T_{N_2}$, so $U_{\sigma,N} \simeq U_{\sigma,N_1} \times T_{N_2}$ and hence $U_{\sigma,N} \simeq U_{\sigma,N_1} \times (\mathbb{C}^*)^{n-r} \subseteq U_{\sigma,N_1} \times \mathbb{C}^{n-r}$. By assumption $U_{\sigma,N}$ is smooth, so $U_{\sigma,N_1} \times \mathbb{C}^{n-r}$ is smooth at any point $(p,q) \in U_{\sigma,N_1} \times (\mathbb{C}^*)^{n-r}$. The variety U_{σ,N_1} will be smooth at any p where (p,q) is smooth in $U_{\sigma,N_1} \times (\mathbb{C}^*)^{n-r}$, and choosing $p = p_\sigma \in U_{\sigma,N_1}$ gives, by the previous case, that σ is smooth in N_1, as $\dim \sigma = \dim(N_1)_{\mathbb{R}} = r$. Hence σ is also smooth in $N_1 \bigoplus N_2 = N$, as a subset of a basis of N_1 is a subset of a basis of N. $\square$

5.5 Exercises

Exercise 5.1. Consider the affine toric variety $Y_{\mathcal{A}} = Spec(\mathbb{C}[\mathrm{S}])$ where $\mathcal{A} = \{m_1, ..., m_s\}$ and $S = \mathbb{N}\mathcal{A}$. Let $\gamma : S \to \mathbb{C}$ be a semigroup homomorphism. In class we mentioned that $p = (\gamma(m_1), ..., \gamma(m_s))$ lies in $Y_{\mathcal{A}}$.

(i) Prove that the maximal ideal $\{f \in \mathbb{C}[\mathrm{S}] : f(p) = 0\}$ is the kernel of the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\mathrm{S}] \to \mathbb{C}$ induced by γ.

(ii) The torus T_N of $Y_{\mathcal{A}}$ has character lattice $M = \mathbb{Z}\mathcal{A}$ and fix $t \in T_N$. Prove that the semigroup homomorphism $m \mapsto \chi^m(t)\gamma(m)$ corresponds to the point

$$(\chi^{m_1}(t), ..., \chi^{m_s}(t)) \cdot ((\gamma(m_1), ..., \gamma(m_s)))$$

coming from the action of $t \in T_N \subseteq (\mathbb{C}^*)^s$ on $p \in Y_{\mathcal{A}} \subseteq \mathbb{C}^s$.

Exercise 5.2. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex polyhedral cone. Then the torus action on U_σ has a fixed point if and only if $\dim \sigma = \dim N_{\mathbb{R}}$, in which case the fixed point is unique and is given by the maximal ideal

$$\langle \chi^m \mid m \in \mathrm{S}_\sigma \backslash \{0\} \rangle \subseteq \mathbb{C}\left[\mathrm{S}_\sigma\right],$$

where as usual $\mathrm{S}_\sigma = \sigma^\vee \cap M$.

Exercise 5.3. In an example, we saw that the rational normal cone $\hat{\mathbb{C}}_d \subseteq \mathbb{C}^{d+1}$ is the toric variety associated to $\sigma = cone(de_1 - e_2, e_2) \subseteq \mathbb{R}^2$.
Compute the Hilbert basis of the semigroup S_σ.
What is the smallest affine space in which we can embed $\hat{\mathbb{C}_d}$? (Use Lemma 1.0.6 from [CLS])

Exercise 5.4. Let $\mathcal{A} \subseteq M$ be a finite set.

(i) Prove that the semigroup $\mathbb{N}\mathcal{A}$ is saturated in M if and only if $\mathbb{N}\mathcal{A} = Cone(\mathcal{A}) \cap M$.
Hint: Apply eq. (1.2.2), page 29 [CLS], to $\mathrm{Cone}(\mathcal{A}) \subseteq M_{\mathbb{R}}$.

Chapter 6. Toric morphisms and projective varieties

Chapter written by Emma Billet after the talk of Coppin and Schuller

6.1 Toric maps

Definition 6.1. Let $V_1 = Spec(\mathbb{C}[S_1]), V_2 = Spec(\mathbb{C}[S_2])$ for S_1, S_2 affine semigroups. A morphism $\varphi : V_1 \to V_2$ is called toric if the corresponding $\varphi^* : \mathbb{C}[S_2] \to \mathbb{C}[S_1]$ is induced by a semigroup homomorphism $\hat{\varphi} : S_2 \to S_1$. In other words, this means that $\varphi^*(\chi^m) = \chi^{\hat{\varphi}(m)}$ for all $m \in S_2$.

Example 6.2. The map $\varphi : \mathbb{C} \to \mathbb{C}, t \mapsto t^2$ is a toric morphism as $\varphi^* : \mathbb{C}[X] \to \mathbb{C}[X], X \mapsto X^2$ is induced by $\hat{\varphi} : \mathbb{N} \to \mathbb{N} : 1 \mapsto 2$ which is a well defined semi-group homomorphism. We also have as a counter-example $\varphi : \mathbb{C} \to \mathbb{C} : t \mapsto t + 1$, we may see it is in fact not toric using the following proposition.

Proposition 6.3. *Let T_{N_1}, T_{N_2} be the tori of the affine toric varieties V_1, V_2 respectively.*
(a) A morphism $\varphi : V_1 \to V_2$ is toric if and only if $\varphi(T_{N_1}) \subseteq T_{N_2}$ and $\varphi_{|T_{N_1}} : T_{N_1} \to T_{N_2}$ is a group homomorphism.
(b) $\varphi : V_1 \to V_2$ is toric $\implies \forall t \in T_{N_1}, p \in V_1 : \varphi(t \cdot p) = \varphi(t) \cdot \varphi(p)$. Such a morphism is called equivariant.

Proof: We can write $V_1 = Spec(\mathbb{C}[S_1]), V_2 = Spec(\mathbb{C}[S_2])$ for some affine semigroups S_1, S_2
(a)" $\implies$ " Suppose $\varphi : V_1 \to V_2$ comes from $\hat{\varphi} : S_2 \to S_1$, it thus can be extended to $\hat{\varphi} : M_2 \to M_1$, where $M_i = \mathbb{Z}S_i$ is the character lattice of T_{N_i} for $i = 1, 2$ and this gives the following diagram

$$\begin{array}{ccc} \mathbb{C}[S_2] & \xrightarrow{\varphi^*} & \mathbb{C}[S_1] \\ \downarrow & & \downarrow \\ \mathbb{C}[M_2] & \longrightarrow & \mathbb{C}[M_1] \end{array}$$

From which, applying Spec we obtain the following one:

$$\begin{array}{ccc} V_2 & \xleftarrow{\varphi} & V_1 \\ \uparrow & & \uparrow \\ T_{N_2} & \longleftarrow & T_{N_1} \end{array}$$

This proves $\varphi(T_{N_1}) \subseteq T_{N_2}$ and we are left to prove the group homomorphism statement. In order to do this we assume $M_2 = \mathbb{Z}^n$ and T_{N_2} is embedded in $\mathbb{C}^n$ then $\hat{\varphi} : M_2 \to M_1$ is determined by $\hat{\varphi}(e_i) = m_i$ $\forall i = 1, ..., n \implies \varphi^* : \mathbb{C}[M_2] \to \mathbb{C}[M_1] : \chi^{e_i} \mapsto \chi^{m_i} \implies \varphi_{|T_{N_1}} : t \mapsto (\chi^{m_1}(t), ..., \chi^{m_n}(t))$ is a group homomorphism as desired.

" $\impliedby$ " Suppose $\varphi(T_{N_1}) \subseteq T_{N_2}$ and $\varphi_{|T_{N_1}}$ is a group homomorphism. Then we have the same diagram:

$$\begin{array}{ccc} V_2 & \xleftarrow{\varphi} & V_1 \\ \uparrow & & \uparrow \\ T_{N_2} & \longleftarrow & T_{N_1} \end{array}$$

which induces

$$\begin{array}{ccc} \mathbb{C}[S_2] & \xrightarrow{\varphi^*} & \mathbb{C}[S_1] \\ \downarrow & & \downarrow \\ \mathbb{C}[M_2] & \longrightarrow & \mathbb{C}[M_1] \end{array}$$

where the bottom map is $\varphi^* : \mathbb{C}[M_2] \to \mathbb{C}[M_1], \chi^m \mapsto \chi^m \circ \varphi$. So φ^* sends characters to characters $\implies$ it comes from a group homomorphism $\hat{\varphi} : M_2 \to M_1$. Using this together with the fact that $\varphi^*(\mathbb{C}[S_2]) \subseteq \mathbb{C}[S_1] \implies \hat{\varphi}(S_2) \subseteq S_1$. This finishes part (a).
(b) Denote by $\varphi_1 : T_{N_1} \times V_1 \to V_1$ the action of T_{N_1} on V_1, similarly for φ_2 on V_2. Then we define

$$T_{N_1} \times V_1 \xrightarrow[\varphi_1]{} V_1 \xrightarrow[\varphi]{} V_2$$

$$T_{N_1} \times V_1 \xrightarrow[\varphi\times\varphi]{} T_{N_2} \times V_2 \xrightarrow[\varphi_2]{} V_2$$

It suffices to prove that these compositions of morphisms are equal in order to prove that φ is equivariant. But using that $\varphi_{|T_{N_1}}$ is a group homomorphism, these are equal on $T_{N_1} \times T_{N_1}$ which is a Zariski dense subset of $T_{N_1} \times V_1$ so they agree everywhere and we proved the equivariance. □

Note that on lattices $N_1, N_2, \bar{\varphi} : N_1 \to N_2$ gives a group morphism $\varphi : T_{N_1} \to T_{N_2}$ by tensoring, since $T_{N_i} \cong N_i \otimes_{\mathbb{Z}} \mathbb{C}^*$ $i = 1, 2$, we could also tensor with $\mathbb{R}$ to obtain $\bar{\varphi}_{\mathbb{R}} : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$.

Proposition 6.4. *Let σ_1, σ_2 be strongly convex rational polyhedral cones and $\bar{\varphi} : N_1 \to N_2$. Then $\varphi : T_{N_1} \to T_{N_2}$ extends to a morphism of affine toric varieties $\varphi : U_{\sigma_1} \to U_{\sigma_2}$ if and only if $\bar{\varphi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$.*

Proof: Exercise 6.1. □

6.2 Faces of cones and affine open subsets

Consider as usual, $\sigma \subseteq N_{\mathbb{R}}$ a strongly convex rational polyhedral cone in some vector space, and $\tau \preccurlyeq \sigma$ a face of this cone, then recall that by definition of a face $\tau = H_m \cap \sigma$ where $\sigma \subseteq H_m^+$, for some $m \in \sigma^\vee \cap M$. This allow us to relate semigroup algebras of σ and τ as follows.
Recall that a face of a cone being itself a cone, it allows us to talk about $S_\tau = \tau^\vee \cap M$

Proposition 6.5. *The semigroup algebra $\mathbb{C}[S_\tau]$ is the localization of $\mathbb{C}[S_\sigma]$ at $\chi^m \in \mathbb{C}[S_\sigma]$.*

Proof:
$\tau \subseteq \sigma \implies S_\sigma \subseteq S_\tau$, but for the m defined above we have $\langle m, u\rangle = 0$.
$\forall u \in \tau \implies \pm m \in \tau^\vee \implies S_\sigma + \mathbb{Z}m \subseteq S_\tau$, if we prove the reverse inclusion we have $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma, \chi^{-m}] = \mathbb{C}[S_\sigma]_{\chi^m}$. The proof of the other inclusion is left to the reader (see also page 43 of [CLS]). □

Example 6.6.

- Consider $\sigma = Cone(e_1, e_2) \subseteq N_{\mathbb{R}} = \mathbb{R}^2$ and $\tau = Cone(e_2)$ a face of σ. Then $\tau^\vee = Cone(\pm e_1, e_2)$ and we have the following $S_\sigma = \mathbb{N}^2 \implies \mathbb{C}[S_\sigma] = \mathbb{C}[X, Y]$ and $U_\sigma = \mathbb{C}^2$. On another hand using the previous proposition $S_\tau = \mathbb{N}\{\pm e_1, e_2\} \implies \mathbb{C}[S_\tau] = \mathbb{C}[X^{\pm 1}, Y] = \mathbb{C}[X, Y]_X$ and $U_\tau = \mathbb{C}^2 \backslash V(X)$.
- $U_\tau = Spec(\mathbb{C}[S_\tau]) = Spec(\mathbb{C}[S_\sigma]_{\chi^m}) = Spec(\mathbb{C}[S_\sigma])_{\chi^m} = (U_\sigma)_{\chi^m} \supseteq U_\sigma$.
- Using the last point, for two cones σ, σ', that meet along a common face τ that is $\sigma \cap \sigma' = \tau \implies U_\sigma \subseteq U_\tau \supseteq U_{\sigma'}$.

6.3 Projective varieties

6.3.1 Background notions

Definition 6.7. The n-dimensional projective space is $\mathbb{P}^n = (\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^*$ and a point in $\mathbb{P}^n$ is denoted as $[z_0 : ... : z_n]$.

Definition 6.8. $V \subseteq \mathbb{P}^n$ is a projective variety if

$$V = \mathbb{V}_p(f_1, ..., f_r) = \{[x_0 : ... : x_n] \in \mathbb{P}^n \;\mid\; f_i(x_0, ..., x_n) = 0 \;\; \forall i \in \{1, ..., r\}\}$$

where $f_1, ..., f_r$ are some homogeneous polynomials in $\mathbb{C}[X_0, ..., X_n]$.

Definition 6.9. The homogeneous coordinate ring of V is $\mathbb{C}[V] = \mathbb{C}[X_0, ..., X_n]/\mathbb{I}(V)$ where $\mathbb{I}(V)$ is the ideal generated by

$$\{f \in \mathbb{C}[X_0, ..., X_n] \mid f \text{ homogeneous}, f(v) = 0 \ \forall v \in V\}$$

Remark 6.10. $\mathbb{C}[V]_d = \mathbb{C}[X_0, ..., X_n]_d/\mathbb{I}(V)_d$, that is to say the coordinate ring inherits a grading from the grading on $\mathbb{C}[X_0, ..., X_n]$ where $\mathbb{C}[X_0, ..., X_n]_d$ is the vector space of homogeneous polynomials of degree d.

Definition 6.11. For a projective variety $V \subseteq \mathbb{P}^n$, we can define its corresponding affine cone $\hat{V} := \mathbb{V}_a(\mathbb{I}(V)) \subseteq \mathbb{C}^{n+1}$ which has the following properties: $V = (\hat{V}\backslash\{0\})/\mathbb{C}^*$ (equality as sets for now); and $\mathbb{C}[\hat{V}] = \mathbb{C}[V]$.

Example 6.12.

- Consider the ideal $I = (\{X_iX_{j+1} - X_{i+1}X_j | 0 \leq i < j \leq d-1\}) \subseteq \mathbb{C}[X_0, ..., X_d]$. I being an homogeneous ideal, it defines a projective variety $C_d = \mathbb{V}_p(I) \subseteq \mathbb{P}^d$ which is the image of the following map $\Phi : \mathbb{P}^1 \to \mathbb{P}^d, [s : t] \mapsto [s^d : s^{d-1}t : ... : st^{d-1} : t^d]$.

  Also the corresponding affine cone is $\mathbb{V}_a(I) = \hat{C}_d$ that we discussed several times. Moreover C_d is a curve (via Φ) called the rational normal curve.

- $\hat{V} := \mathbb{V}_a(XY - ZW) \subseteq \mathbb{C}^4$ is the affine cone of the projective variety $V = \mathbb{V}_p(XY - ZW) \subseteq \mathbb{P}^3$. In addition $V \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Remark 6.13. Note that we're still working with the Zariski topology, that is for V a projective variety the closed sets of V are subvarieties (i.e. projective varieties of $\mathbb{P}^n$ contained in V).

Consider $f, g \in \mathbb{C}[X_0, ..., X_n]$ homogeneous, $g \neq 0$, such that deg(g)=deg(f)= m. We motivate the next definition with the following formula which proves that we could have a well-defined notion of "functions" on $\mathbb{P}^n$ (under some conditions).

$$\lambda \in \mathbb{C}^* : \frac{f(\lambda x)}{g(\lambda x)} = \frac{\lambda^m f(x)}{\lambda^m g(x)}$$

Definition 6.14. Let $f, g \in \mathbb{C}[X_0, ..., X_n]$ homogeneous of the same degree such that $g \neq 0$. Then $\frac{f}{g}$: $\mathbb{P}^n\backslash\mathbb{V}(g) \to \mathbb{C}$ is well defined and called a rational function on $\mathbb{P}^n$. More generally consider V an irreducible projective variety, $f, g \in \mathbb{C}[V]$ homogeneous of same degree such that $g \neq 0$, and recall a proprety of affine cones that $\mathbb{C}[\hat{V}] = \mathbb{C}[V]$. Then f, g define functions on the affine cone $\hat{V}$ and therefore an element $\frac{f}{g} \in \mathbb{C}(\hat{V})$. Then we can define

$$\mathbb{C}(V) = \{\frac{f}{g} \in \mathbb{C}(\hat{V}) | f, g \in \mathbb{C}[V] \text{ homogeneous of same degree, } g \neq 0\},$$

this set is also denoted as $(\mathbb{C}(\hat{V}))_0 = (\mathbb{C}(V))_0$ because it corresponds to degree 0 elements of $\mathbb{C}(\hat{V})$. In addition it induces $\frac{f}{g} : V \dashrightarrow \mathbb{C}$, where the dashed arrow means that it is define on some open of V.

Proposition 6.15. *We define $U_i := \mathbb{P}^n\backslash\mathbb{V}_p(X_i)$ to be the affine charts of $\mathbb{P}^n$. Then $\{U_i\}_{i\geq 1}$ is an open cover of $\mathbb{P}^n$. Furthermore $U_i \cong \mathbb{C}^n$.*

Proof:
The affine variety isomorphism is given by

$$\varphi : U_i \to \mathbb{C}^n, [a_0 : ... : a_n] \mapsto (\frac{a_1}{a_i}, ..., \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, ..., \frac{a_n}{a_i})$$

and

$$\varphi^{-1} : \mathbb{C}^n \to U_i, (a_1, ..., a_n) \mapsto [a_1 : ...a_{i-1} : 1 : a_{i+1} : ... : a_n].$$

□

Let $V \subseteq \mathbb{P}^n$ be a projective variety, $\{V \cap U_i\}_{i\geq 1}$ covers V and it maps via the previous map to the affine variety in $\mathbb{C}^n$ defined by the equation $f(z_1, ..., 1, ..., z_n) = 0$ for all f homogeneous polynomials in $\mathbb{I}(V)$.

Lemma 6.16. $\mathbb{C}[V \cap U_i] \cong (\mathbb{C}[V]_{\bar{X}_i})_0$.

Proof:
$\mathbb{C}[V]_{\bar{X}_i} \cong \frac{\mathbb{C}[X_0,...,X_n]_{X_i}}{\mathbb{I}(V)_{X_i}}$ since localization is exact. Taking elements of degree 0 we have $(\mathbb{C}[V]_{\bar{X}_i})_0 \cong \frac{(\mathbb{C}[X_0,...,X_n]_{X_i})_0}{(\mathbb{I}(V)_{X_i})_0}$.
Also $(\mathbb{C}[X_0,...,X_n]_{X_i})_0 = \mathbb{C}[\frac{X_0}{X_i},...,\frac{X_{i-1}}{X_i},\frac{X_{i+1}}{X_i},...,\frac{X_n}{X_i}]$. Let $f \in \mathbb{I}(V)$ homogeneous of degree k $\implies \frac{f}{X_i^k} = f(\frac{X_0}{X_i},...,\frac{X_{i-1}}{X_i},1,\frac{X_{i+1}}{X_i},...,\frac{X_n}{X_i}) \in (\mathbb{I}(V)_{X_i})_0$. But using the equation defining $V \cap U_i$ we see that $(\mathbb{I}(V)_{X_i})_0$ maps to $\mathbb{I}(V \cap U_i)$. Finally we prove this map is surjective, so consider an element $g(\frac{X_0}{X_i},...,\frac{X_{i-1}}{X_i},1,\frac{X_{i+1}}{X_i},...,\frac{X_n}{X_i}) \in \mathbb{I}(V \cap U_i)$. It exists $k >> 0$ clearing the denominators such that $X_i^k g = h(X_0,...,X_n)$ is homogeneous of degree k. Then looking at $X_i h$ we note that it vanishes on $V \cap U_i$ because of g and it vanishes on the complementary of U_i because of $X_i \implies X_i h \in \mathbb{I}(V) \implies \frac{X_i h}{X_i^{k+1}} \in (\mathbb{I}(V)_{X_i})_0$ and this maps to g. □

6.3.2 Product of projective spaces

Definition 6.17. A polynomial $f \in \mathbb{C}[X_0,...,X_n,Y_0,...,Y_m]$ is called homogeneous of bidegree (a,b) if it is homogeneous of degree a in $(\mathbb{C}[Y_0,...,Y_m])[X_0,..,X_n]$ and vice versa.

Definition 6.18. A variety $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is the vanishing locus of finitely many bihomogeneous polynomials.

Definition 6.19.

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$$
$$([a_i]_{i=0}^n,[b_j]_{j=0}^m) \mapsto [a_i b_j]_{1 \le i \le n; 1 \le j \le m}$$

is called the Segree embedding and is in fact an embedding.
$\sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{V}(I)$ where $I = (\{z_{ij}z_{kl} - z_{il}z_{kj} | 0 \le i,k \le n, 0 \le j,l \le m\}])$.

Remark 6.20. These two definitions give a priori two different notions of the product of two projective varieties. The following proposition shows that they in fact agree.

Proposition 6.21. *For $V \subseteq \mathbb{P}^n, W \subseteq \mathbb{P}^m$ subvarieties, $V \times W \subset \mathbb{P}^n \times \mathbb{P}^m$ is a projective variety i.e. $\sigma_{n,m}(V \times W)$ is a projective subvariety of $\mathbb{P}^{nm+n+m}$.*

Proof: Exercise 6.4. □

6.4 Exercises

Exercise 6.1. Suppose we have strongly convex rational polyhedral cones $\sigma_1 \subset (N_1)_\mathbb{R}$ and $\sigma_2 \subset (N_2)_\mathbb{R}$, and a homomorphism $\overline{\varphi} : N_1 \to N_2$. Recall that by tensoring with $\mathbb{C}^*$ this gives a group homomorphism $\varphi : T_{N_1} \to T_{N_2}$ of tori and by tensoring with $\mathbb{R}$ it gives a map $\overline{\varphi}_\mathbb{R} : (N_1)_\mathbb{R} \to (N_2)_\mathbb{R}$. Prove that $\varphi : T_{N_1} \to T_{N_2}$ extends to a map of affine toric varieties $\varphi : U_{\sigma_1} \to U_{\sigma_2}$ if and only if $\overline{\varphi}_\mathbb{R}(\sigma_1) \subset \sigma_2$. Also argue that in that case, the extended φ is a toric morphism.

Exercise 6.2. (Maps to projective space) Let $V \subseteq \mathbb{P}^n$ a projective variety and $f_0, \dots, f_m$ be polynomials of degree d such that $V \cap \mathbf{V}(f_1, \dots, f_m) = \emptyset$. Show that the map

$$(a_0, \dots, a_n) \mapsto (f_1(a_0, \dots, a_n), \dots, f_m(a_0, \dots, a_n))$$

induces a well-defined map $V \to \mathbb{P}^m$.

Exercise 6.3. Show that the Segree embedding $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ defined by $([a_i],[b_j]) \mapsto [a_i bj]$ is indeed an embedding. Furthermore, show that $\sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) = \mathbf{V}(I)$ where I is the ideal generated by

$$\{z_{ij}z_{kl} - z_{il}z_{kj} \mid 0 \le i,k \le n \; ; \; 0 \le j,l \le m\}.$$

Exercise 6.4. Let $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ defined by $f_l(x,y) = 0$ where f_l is bihomogeneous of bidegree (a_l, b_l) for $l = 0, \dots, s$. The goal of this exercise is to show that V can be viewed as a projective variety of $\mathbb{P}^{nm+n+m}$ via the Segree embedding.

(i) For each l, consider $d_l \geq \max(a_l, b_l)$ and $g_l = f_l(x, y) \prod_{i,j} x_i^{d_l - a_l} y_j^{d_l - b_l}$. Show that V is the vanishing locus of the g_l's.

(ii) Deduce from (a) that $\sigma_{n,m}(V)$ is a projective subvariety of $\mathbb{P}^{nm+n+m}$.

Chapter 7. Projective toric varieties

Chapter written by Julia Morin after the talk of Julie Bannwart and Louis Gognia

Definition 7.1. A projective irreducible variety X over $\mathbb{C}$ is called *toric* if :

- there exists $T \subseteq X$ with $T \simeq (\mathbb{C}^*)^n$
- T is open in X
- the action of T on itself extends to X

Proposition 7.2. *$\mathbb{P}^n$ is toric with torus $T_{\mathbb{P}^n} := \mathbb{P}^n \setminus \mathbf{V}_p(x_0...x_n)$.*

Proof. $T_{\mathbb{P}^n} := \mathbb{P}^n \setminus \mathbf{V}_p(x_0...x_n) = \{[a_0 : ... : a_n] \in \mathbb{P}^n \mid a_i \neq 0\} = \{[1 : \tilde{a}_1 : ... : \tilde{a}_n] \in \mathbb{P}^n \mid \tilde{a}_i \neq 0\} \simeq (\mathbb{C}^*)^n$. $T_{\mathbb{P}^n}$ is Zariski open in $\mathbb{P}^n$ and its action on itself clearly extends to an action on $\mathbb{P}^n$. □

Remark 7.3. There is a short exact sequence of tori

$$1 \xrightarrow{\iota} \mathbb{C}^* \longrightarrow (\mathbb{C}^*)^{n+1} \xrightarrow{\pi} T_{\mathbb{P}^n} \longrightarrow 1$$

which induces a s.e.s of character lattices (Exercise 7.1) :

$$0 \longleftarrow \mathbb{Z} \xleftarrow{-\circ\iota} \mathbb{Z}^{n+1} \longleftarrow \mathscr{M}_n \longleftarrow 0$$

As this sequence is exact $\mathscr{M}_n$ is the kernel of the map $(a_0, ..., a_n) \mapsto \sum a_i$ thus $\mathscr{M}_n = \{(a_0, ..., a_n) \in \mathbb{Z}^{n+1} \mid \sum a_i = 0\}$. If we dualize again we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota\circ-} \mathbb{Z}^{n+1} \longrightarrow \mathscr{N}_n \longrightarrow 0$$

and the lattice of one-parameter subgroups $\mathscr{N}_n$ is the quotient

$$\mathscr{N}_n = \mathbb{Z}^{n+1}/\mathbb{Z}(1, \dots, 1).$$

Question 1. *How can we construct new projective varieties ?*

7.1 Lattice points and projective toric varieties

Let T_N be a torus with lattices M and N as usual. Let $\mathscr{A} = \{m_1, ..., m_s\} \subseteq M$, and with

$$\Phi_{\mathscr{A}} : T_N \to \mathbb{C}^s$$
$$t \mapsto (\chi^{m_1}(t), ..., \chi^{m_s}(t))$$

we have $Y_{\mathscr{A}} := \mathrm{cl}_{\mathbb{C}^s}(\mathrm{im}\, \Phi_{\mathscr{A}})$. Let us consider the composition:

$$\pi \circ \Phi_{\mathscr{A}} : T_N \to \mathbb{C}^s \xrightarrow{\pi} \mathbb{P}^{s-1}$$
$$t \mapsto [\chi^{m_1}(t) : ... : \chi^{m_s}(t)].$$

Now we define $X_{\mathscr{A}} := \mathrm{cl}_{\mathbb{P}^{s-1}}(\mathrm{im}(\pi \circ \Phi_{\mathscr{A}}))$

Proposition 7.4. *$X_{\mathscr{A}}$ is toric, with torus $T_{X_{\mathscr{A}}} = X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}}$.*

Proof. The image $\mathrm{im}(\pi \circ \Phi_{\mathscr{A}})$ of the group homomorphism $\pi \circ \Phi_{\mathscr{A}}$ is a torus that is closed in $T_{\mathbb{P}^{s-1}}$, let call it $T_{X_{\mathscr{A}}}$. It follows that $T_{X_{\mathscr{A}}}$ is Zariski open in $X_{\mathscr{A}}$, and the action of $T_{X_{\mathscr{A}}}$ extends to $X_{\mathscr{A}}$ (arguments are the same as in the affine case, see Proposition 3.4). The inclusion $T_{X_{\mathscr{A}}} \subseteq X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}}$ is trivial. Now

$$X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}} = \mathrm{cl}_{\mathbb{P}^{s-1}}(T_{X_{\mathscr{A}}}) \cap T_{\mathbb{P}^{s-1}} = \mathrm{cl}_{T_{\mathbb{P}^{s-1}}}(T_{X_{\mathscr{A}}}) = T_{X_{\mathscr{A}}}$$

. □

Remark 7.5. In the following examples, since $M = \mathbb{Z}^n$ and $\mathscr{A} = \{m_1, ..., m_s\} \subseteq M$, we will see $\mathscr{A}$ as an $n \times s$ matrix A so that $m_i = \mathrm{col}_i(A)$.

Example 7.6. Let $M = \mathbb{Z}^2$, $T_N = (\mathbb{C}^*)^2$ and

$$A = \begin{pmatrix} d & d-1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & d-1 & d \end{pmatrix}$$

Recall that the rational normal curve C_d is the image of the map $\Phi : \mathbb{P}^1 \to \mathbb{P}^s, [s : t] \mapsto [s^d : s^{d-1}t : ... : st^{d-1} : t^d]$. This map corresponds to the map $\pi \circ \Phi_{\mathscr{A}}$. It means that C_d is a projective toric variety.

Let us now understand the link between the affines and projective toric varieties $Y_{\mathscr{A}}$ and $X_{\mathscr{A}}$.

7.2 Affine cones and projective toric varieties

Recall the short exact sequence of character lattices :

$$0 \to L \to \mathbb{Z}^s \xrightarrow{\hat{\Phi}_{\mathscr{A}}} M \tag{2}$$

where $\hat{\Phi}_{\mathscr{A}}(e_i) = m_i$ is the map induced on character lattices by $\Phi_{\mathscr{A}}$, with $\mathscr{A} = \{m_1, ..., m_s\}$. $L = \ker \Phi_{\mathscr{A}}$ and we proved that the toric ideal $\mathbf{I}_a(Y_{\mathscr{A}}) = I_L = \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L\rangle$ (see Proposition 3.6). Then we have the following result:

Proposition 7.7. *Given $Y_{\mathscr{A}}, X_{\mathscr{A}}$ and I_L as above, the following are equivalent:*

(i) $\widehat{X}_{\mathscr{A}} = Y_{\mathscr{A}}$

(ii) $I_L = \mathbf{I}_p(X_{\mathscr{A}})$.

(iii) *I_L is homogeneous.*

(iv) *There is $u \in N$ and $k > 0$ in $\mathbb{N}$ such that $\langle m_i, u\rangle = k$ for $i = 1, \ldots, s$ (i.e A lies in an affine hyperplane of $M_{\mathbb{Q}}$ not containing 0).*

Proof.

- *1.* $\Rightarrow$ *2.*
  $\mathbf{I}_p(X_{\mathscr{A}}) = \mathbf{I}_a(\widehat{X}_{\mathscr{A}}) = \mathbf{I}_a(\widehat{Y}_{\mathscr{A}}) = I_L$
- *2.* $\Rightarrow$ *3.* by definition
- *3.* $\Rightarrow$ *4.*
  Assume that I_L is homogeneous and take $x^\alpha - x^\beta \in I_L$. If x^α and x^β had different degrees, then x^α and x^β would lie in $I_L = \mathbf{I}_a(Y_{\mathscr{A}})$ i.e would vanish on $Y_{\mathscr{A}}$ but this is impossible since $(1, \ldots, 1) \in Y_{\mathscr{A}}$. Hence x^α and x^β have same degree.
  Given $\ell = (\ell_1, \ldots, \ell_s) \in L$, set

  $$\ell_+ = \sum_{\ell_i > 0} \ell_i e_i \quad \text{and} \quad \ell_- = -\sum_{\ell_i < 0} \ell_i e_i.$$

  Note that $\ell = \ell_+ - \ell_-$ and that $\ell_+, \ell_- \in \mathbb{N}^s$. $x^{\ell_+} - x^{\ell_-} \in I_L$ therefore $\sum_{\ell_i > 0} \ell_i = \sum_{\ell_i < 0} \ell_i$ which implies $\ell \cdot (1, ..., 1) = 0$, $\forall \ell \in L$.

  Now we tensor (1) above with $\mathbb{Q}$ and take $\mathrm{Hom}_{\mathbb{Q}}(-, \mathbb{Q})$ to obtain an exact sequence

  $$N_{\mathbb{Q}} \xrightarrow{\alpha} \mathbb{Q}^s \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{Q}}(L \otimes \mathbb{Q}, \mathbb{Q}) \longrightarrow 0$$

  where $\alpha(\tilde{u}) = (\langle m_i, \tilde{u}\rangle)_i$ and $\beta(r_1, ..., r_s) = ((\ell \otimes s) \mapsto s \sum \ell_i r_i)$ (Exercise 7.1).
  Therefore, $(1, ..., 1) \in \mathbb{Q}^s$ is sent to 0 and $\exists \tilde{u} \in N_{\mathbb{Q}}$ such that $\langle m_i, \tilde{u}\rangle = 1$ for all $1 \leq i \leq s$.

- *4.* ⇒ *1.* We already have that $Y_{\mathscr{A}} \subseteq \widehat{X}_{\mathscr{A}}$ and since $\widehat{X}_{\mathscr{A}}$ is irreducible, if we show $\widehat{X}_{\mathscr{A}} \cap (\mathbb{C}^*)^s \subseteq Y_{\mathscr{A}}$ it will follow that $\widehat{X}_{\mathscr{A}} \subseteq Y_{\mathscr{A}}$. So we just need to show this first inclusion, let $p \in \widehat{X}_{\mathscr{A}} \cap (\mathbb{C}^*)^s$.

  Then $\pi(p) \in X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}} = T_{X_{\mathscr{A}}}$. Therefore $\exists \mu \in \mathbb{C}^*$ and $t \in T_N$ such that $p = \mu \cdot (\chi^{m_1}(t), ..., \chi^{m_s}(t))$. Let $u \in N$ be as in the hypothesis *4*. This gives a oneparameter subgroup of T_N, which we write as $\tau \mapsto \lambda^u(\tau)$ for $\tau \in \mathbb{C}^*$. Then

$$\Phi_{\mathscr{A}}(\lambda^u(\tau)t) = (\chi^{m_1}(\lambda^u(\tau)t), ..., \chi^{m_s}(\lambda^u(\tau)t))$$

$$= (\tau^{\langle m_1, u\rangle}\chi^{m_1}(t), ..., \tau^{\langle m_s, u\rangle}\chi^{m_s}(t)) = \tau^k \cdot (\chi^{m_1}(t), ..., \chi^{m_s}(t))$$

  Using $k > 0$, we can choose τ so that $p = \Phi_{\mathscr{A}}(\lambda^u(\tau)t) \in Y_{\mathscr{A}}$ and then we have $p \in \operatorname{im} \Phi_{\mathscr{A}} \subseteq Y_{\mathscr{A}}$.

□

7.3 The affine cone of $X_{\mathscr{A}}$

Example 7.8. In Example 7.6 we worked with

$$A = \begin{pmatrix} d & d-1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & d-1 & d \end{pmatrix}$$

and found $X_{\mathscr{A}} = C_d$. Let us check the conditions of Proposition 1.8. $I_L = \langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i \leq j \leq d-1 \rangle$. We saw that $I_L = \mathbf{I}_p(X_{\mathscr{A}})$ and I_L is homogeneous. Concerning point *4.*, notice that the affine hyperplane of $\mathbb{Z}^2$ containing $\mathscr{A}$ consists of all points (a, b) such that $a + b = d$. Therefore, taking $u = (1, 1)$ and $k = d$ we have that $\langle m_i, u \rangle = d$, $\forall i$. Therefore $Y_{\mathscr{A}} = \widehat{C_d}$.

Example 7.9. Let $M = \mathbb{Z}$ and

$$\mathscr{B} = \begin{pmatrix} 0 & 1 & \cdots & d-1 & d \end{pmatrix}$$

For all t in $\mathbb{C}$, we have $\pi \circ \Phi_{\mathscr{B}}(t) = [1 : t : ... : t^d] = [s^d : s^{d-1}st : ... : s^0(st)^d] = \pi \circ \Phi_{\mathscr{A}}(s, \tilde{t})$ with $\tilde{t} = st$. Therefore $X_{\mathscr{B}} = X_{\mathscr{A}} = C_d$. However $\widehat{X}_{\mathscr{B}} \neq Y_{\mathscr{B}}$ because $\mathbf{I}(Y_{\mathscr{B}}) \in \mathbb{C}[x_0, ..., x_n]$ is not homogeneous. For example $x_2 - x_1^2 \in \mathbf{I}(Y_{\mathscr{B}})$ (because it vanishes at $(1, t, ..., t^{d-1}, t^d)$ for all $t \in \mathbb{C}^*$) but $x_2 \notin \mathbf{I}(Y_{\mathscr{B}})$ because $(1, ..., 1) \in Y_{\mathscr{B}}$.

Question 2. *We can ask ourselves how to change $\mathscr{A}$ so that $\widehat{X}_{\mathscr{A}}$ stays the same, but the conditions of Proposition 7.7 are met. This means we want to construct $\mathscr{A}'$ from $\mathscr{A}$ such that $\widehat{X}_{\mathscr{A}} = \widehat{X}_{\mathscr{A}'}$ and $\widehat{X}_{\mathscr{A}'} = Y_{\mathscr{A}'}$*

We claim that we can use $\mathscr{A}' = \mathscr{A} \times \{1\}$. Indeed, $\forall t \in T_N$, $\mu \in \mathbb{C}^*$, we then have :

$$\pi \circ \Phi_{\mathscr{A} \times \{1\}}(t, \mu) = [\chi^{m_1}(t)\mu : ... : \chi^{m_s}(t)\mu] = [\chi^{m_1}(t) : ... : \chi^{m_s}(t)] = \pi \circ \Phi_{\mathscr{A}}(t)$$

$$\implies X_{\mathscr{A}} = X_{\mathscr{A} \times \{1\}}$$

Since $X_{\mathscr{A} \times \{1\}}$ lies in an affine hyperplane not containing the origin, $\widehat{X}_{\mathscr{A}} = \widehat{X}_{\mathscr{A} \times \{1\}} = Y_{\mathscr{A} \times \{1\}}$ by Proposition 7.7!

7.4 Torus and character lattice of $X_{\mathscr{A}}$

Let T_N be a torus with character lattice M as usual, and again set $\mathscr{A} = \{m_1, ..., m_s\} \subseteq M$.

Definition 7.10. $\mathbb{Z}'\mathscr{A} = \{\sum_{i=1}^s a_i m_i \in \mathbb{Z}\mathscr{A} \mid a_i \in \mathbb{Z}\ \forall i, \sum_{i=1}^s a_i = 0\}$

Proposition 7.11. *(i) $\mathbb{Z}'\mathscr{A}$ is the character lattice of the torus of $X_{\mathscr{A}}$, in particular* $\dim X_{\mathscr{A}} = \operatorname{rank} \mathbb{Z}'\mathscr{A}$.

(ii) The dimension of $X_{\mathscr{A}}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathscr{A}$ and

$$\operatorname{rank} \mathbb{Z}'\mathscr{A} = \begin{cases} \operatorname{rank} \mathbb{Z}\mathscr{A} - 1 & \text{if } \exists u \in N, k \in \mathbb{N}^* \text{ s.t } \langle m_i, u \rangle = k\ \forall i \leq s, \\ \operatorname{rank} \mathbb{Z}\mathscr{A} & \text{else.} \end{cases}$$

Proof. Use Exercise 7.2 to show *2.*
Now for part *1.*, let M' denote the character lattice of $T_{X_{\mathscr{A}}}$, and consider the following commutative diagram:

$$\begin{array}{ccccc} T_N & \longrightarrow & T_{\mathbb{P}^{s-1}} & \hookrightarrow & \mathbb{P}^{s-1} \\ & \searrow^{\pi\circ\Phi_{\mathscr{A}}} & \uparrow & & \\ & & T_{X_{\mathscr{A}}} & & \end{array}$$

Dualizing, we obtain (Exercise 7.1):

$$\begin{array}{ccc} M & \xleftarrow{\theta} & M_{s-1} \\ & \nwarrow & \downarrow \psi \\ & & M' \end{array}$$

where $M_{s-1} = \{(a_1, ..., a_s) \in \mathbb{Z}^s \mid \sum_{i=1}^s a_i = 0\}$. θ is induced by $\hat{\Phi}_{\mathscr{A}} : \mathbb{Z}^s \to M$, $e_i \mapsto m_i$. Then $M' = \operatorname{im}\psi \simeq \operatorname{im}\theta = \{\sum_{i=1}^s a_i m_i \in \mathbb{Z}\mathscr{A} \mid \sum_{i=1}^s a_i = 0\}$.

□

Example 7.12. Let $\mathscr{A} = \{e_1, e_2, e_1 + 2e_2, 2e_1 + e_2\} \subseteq \mathbb{Z}^2$. Since $\mathbb{Z}^2 = \mathbb{Z}e_1 + \mathbb{Z}e_2 \subseteq \mathbb{Z}\mathscr{A}$ we have $\mathbb{Z}\mathscr{A} = \mathbb{Z}^2$. $\mathbb{Z}'\mathscr{A} = \left\{(a, b) \in \mathbb{Z}^2 \mid a + b \equiv 0 \bmod 2\right\}$. Thus $[\mathbb{Z}\mathscr{A} : \mathbb{Z}'\mathscr{A}] = 2$. This means that $Y_{\mathscr{A}} \neq \widehat{X}_{\mathscr{A}}$ and that the map of tori

$$T_{Y_{\mathscr{A}}} \longrightarrow T_{X_{\mathscr{A}}}$$

is two-to-one, i.e its kernel has order 2.

7.5 Affine pieces and semi-groups

We know that $\mathbb{P}^{s-1} = \bigcup_i U_i$, where the affine open set $U_i = \mathbb{P}^{s-1} \backslash \mathbf{V}(x_i)$. Moreover $X_{\mathscr{A}} = \bigcup_i X_{\mathscr{A}} \cap U_i$. We have the following result:

Proposition 7.13. *$X_{\mathscr{A}} \cap U_i$ is an affine toric variety.*

Proof.

$$T_{X_{\mathscr{A}}} = X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}} \subseteq T_{\mathbb{P}^{s-1}} \subseteq U_i$$

Since $X_{\mathscr{A}} = \mathrm{cl}_{\mathbb{P}^{s-1}}(T_{X_{\mathscr{A}}})$, it follows that $X_{\mathscr{A}} \cap U_i = \mathrm{cl}_{U_i}(T_{X_{\mathscr{A}}})$. Therefore, $X_{\mathscr{A}} \cap U_i$ is an affine toric variety.

□

Given $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M_{\mathbb{R}}$, let us determine the affine semigroup associated to $X_{\mathscr{A}} \cap U_i$.
Using that the isomorphism $U_i \simeq \mathbb{C}^{s-1}$ is given by

$$(a_1, \ldots, a_s) \longmapsto (a_1/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_s/a_i).$$

we can see that $X_{\mathscr{A}} \cap U_i$ is the Zariski closure of the image of the map

$$T_N \longrightarrow T_{X_{\mathscr{A}}} \longrightarrow U_i \simeq \mathbb{C}^{s-1}$$

$$t \longmapsto \left(\chi^{m_1 - m_i}(t), \ldots, \chi^{m_{i-1}-m_i}(t), \chi^{m_{i+1}-m_i}(t), \ldots, \chi^{m_s - m_i}(t)\right).$$

If we set $\mathscr{A}_i = \mathscr{A} - m_i = \{m_j - m_i \mid j \neq i\}$, it follows that

$$X_{\mathscr{A}} \cap U_i \simeq Y_{\mathscr{A}_i} = \operatorname{Spec}(\mathbb{C}[\mathbb{N}\mathscr{A}_i]).$$

Remark 7.14. Later on in the course, we will be interested to determine what is $X_{\mathscr{A}} \cap U_i \cap U_j$, when $i \neq j$. For now, let us notice that $U_i \cap U_j$ consists of points of $X_{\mathscr{A}} \cap U_i$ where $x_j/x_i \neq 0$. Thus

$$X_{\mathscr{A}} \cap U_i \cap U_j = Y_{\mathscr{A}_i} \setminus \mathbf{V}_a(\chi^{m_j - m_i}) = \operatorname{Spec}(\mathbb{C}[\mathbb{N}\mathscr{A}_i])_{\chi^{m_j - m_i}}$$

$$= \operatorname{Spec}\left(\mathbb{C}[\mathbb{N}\mathscr{A}_i]_{\chi^{m_j - m_i}}\right).$$

Remark 7.15. The set of vertices of a polytope is its minimal generating set. This means that if $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M$, $P = \mathrm{Conv}(\mathscr{A}) \subseteq M_{\mathbb{R}}$, and V is the set of vertices of P, then $P = \mathrm{Conv}(V)$ and $V \subseteq \mathscr{A}$ is minimal with this property.

Proposition 7.16. *Let $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M$, $P = \mathrm{Conv}(\mathscr{A}) \subseteq M_{\mathbb{R}}$ and $J = \{1 \leq j \leq s \mid m_j$ is a vertex of $P\}$. Then*

$$X_{\mathscr{A}} = \bigcup_{j \in J} X_{\mathscr{A}} \cap U_j.$$

Proof. We will prove that if $i \in \{1, \dots, s\}$, then $X_{\mathscr{A}} \cap U_i \subseteq X_{\mathscr{A}} \cap U_j$ for some $j \in J$.
Remark 1.15 above implies that $\forall\ 1 \leq i \leq s$, there are $\lambda_j \in \mathbb{Q}^+$ such that

$$m_i = \sum_{j \in J} \lambda_j m_j \text{ with } \sum_{j \in J} \lambda_j = 1 \tag{3}$$

Now for all $j \in J$, write $\lambda_j = \frac{p_j}{q_j}$ with $p_j \in \mathbb{N}$, $q_j \in \mathbb{N}^*$. Multiplying (2) by $\prod_{j \in J} q_j$ we have :

$$\sum_{j \in J} k_j m_j = k m_i$$

with all k_j and k being integers, and $\sum_{j \in J} k_j = k$. Thus we can rewrite this as:

$$\sum_{j \in J} k_j (m_j - m_i) = 0$$

Now fix $j \in J$, we have:

$$m_i - m_j = \sum_{l \in J, l \neq j} k_l (m_l - m_i) + (k_j - 1)(m_j - m_i)$$

which shows that $m_i - m_j \in \mathbb{N}\mathscr{A}_i$. Therefore $\chi^{m_j - m_i} \in \mathbb{C}[\mathbb{N}\mathscr{A}_i]$ is invertible, so $\mathbb{C}[\mathbb{N}\mathscr{A}_i]_{\chi^{m_j - m_i}} = \mathbb{C}[\mathbb{N}\mathscr{A}_i]$. We then have $X_{\mathscr{A}} \cap U_i \cap U_j = \mathrm{Spec}(\mathbb{C}[\mathbb{N}\mathscr{A}_i]) = X_{\mathscr{A}} \cap U_i$, showing that $X_{\mathscr{A}} \cap U_i \subseteq X_{\mathscr{A}} \cap U_j$. □

7.6 Exercises

Let T_N be a torus with character lattice M, and consider a finite subset

$$\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M.$$

Exercise 7.1.

(i) Let $0 \longrightarrow T \longrightarrow T' \longrightarrow T'' \longrightarrow 0$ be an exact sequence of tori and algebraic group homomorphisms. Show that it induces an exact sequence of their character lattices, by showing in particular that an injection, respectively a surjection, of tori induces a surjection, respectively an injection, of their character lattices.

Hint: Show that, given the following diagram of algebraic group homomorphisms, with α injective, χ extends to $(\mathbb{C}^*)^m$:

$$\begin{array}{ccc} \mathbb{C}^* & & \\ \uparrow \chi & \nwarrow & \\ (\mathbb{C}^*)^n & \overset{\alpha}{\hookrightarrow} & (\mathbb{C}^*)^m \end{array}$$

To do this, you can represent the map α by a matrix.

(ii) Prove the claim used during the lecture that the exact sequence

$$\begin{aligned} 0 \longrightarrow L \longrightarrow \mathbb{Z}^s &\longrightarrow M \\ e_i &\longmapsto m_i \end{aligned}$$

induces, by tensoring with $\mathbb{Q}$ and taking duals, an exact sequence:

$$N_{\mathbb{Q}} \longrightarrow \mathbb{Q}^s \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(L_{\mathbb{Q}}, \mathbb{Q}) \longrightarrow 0$$

Exercise 7.2. Prove the claims we used during the course about

$$\mathbb{Z}'\mathscr{A} := \left\{ \sum_{i=1}^{s} a_i m_i \;\middle|\; a_i \in \mathbb{Z}\ \forall i \leq s,\ \sum_{i=1}^{s} a_i = 0 \right\} :$$

(i) $\mathbb{Z}'\mathscr{A}$ is a lattice.

(ii) Its rank is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathscr{A}$.

(iii) $\mathsf{rank}\ \mathbb{Z}'\mathscr{A} = \begin{cases} \mathsf{rank}\ \mathbb{Z}\mathscr{A} - 1 & \text{if } \exists u \in N, k \in \mathbb{N}^*,\ \langle m_i, u \rangle = k\ \forall i \leq s, \\ \mathsf{rank}\ \mathbb{Z}\mathscr{A} & \text{else.} \end{cases}$

Exercise 7.3. Given $m \in M$, let $\mathscr{A} + m = \{m' + m \mid m' \in \mathscr{A}\}$.

(i) Prove that $\mathscr{A}$ and $\mathscr{A} + m$ give rise to the same *projective* toric variety: $X_{\mathscr{A}} = X_{\mathscr{A}+m}$.

(ii) Show by an example that they do not necessarily give rise to the same *affine* toric variety in general: $Y_{\mathscr{A}} \neq Y_{\mathscr{A}+m}$.

Exercise 7.4. Let $M = \mathbb{Z}^{3\times 3}$ be the lattice of 3×3 integer matrices and let $\mathscr{P}_3$ be the set of the six 3×3 permutation matrices, i.e.

$$\mathscr{P}_3 = \{(\delta_{j=\sigma(i)})_{i,j} \mid \sigma \in S_3\} \subseteq \mathbb{Z}^{3\times 3}.$$

Also let $\mathbb{P}_5$ have homogeneous coordinates x_{ijk} indexed by triples such that $\binom{1\ \ 2\ \ 3}{i\ \ j\ \ k}$ is a permutation in S_3.

(i) Prove that three of the permutation matrices sum to the other three and use this to explain why $x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213} \in \mathbf{I}(X_{\mathscr{P}_3})$.

(ii) Show that $\dim X_{\mathscr{P}_3} = 4$ by computing $\mathbb{Z}'\mathscr{P}_3$.

(iii) Conclude that $\mathbf{I}(X_{\mathscr{P}_3}) = \langle x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213} \rangle$.

Chapter 8. Polytopes

Chapter written by Louis Gogniat after the talk of Matthew Dupraz and Zichen Gao

8.1 Definitions and basic properties of polytopes

Let us recall that a *polytope* $P \subseteq M_{\mathbb{R}}$ is the convex hull of a finite set $S \subseteq M_{\mathbb{R}}$, that is, $P = \text{Conv}(S) = \{\sum_{m \in S} \lambda_m m \mid \lambda_m \geq 0, \ \sum_{m \in S} \lambda_m = 1\}$. Following this, we provide some elementary definitions regarding polytopes.

Definition 8.1. The *dimension* of a polytope $P \subseteq M_{\mathbb{R}}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing P. It is then said that P has *full dimension* if $\dim P = \dim_{\mathbb{R}} M_{\mathbb{R}}$.

Definition 8.2. Let $u \in N_{\mathbb{R}} \backslash 0$, and $a \in \mathbb{R}$. We denote $H_{u,a}$ as the affine hyperplane defined by $H_{u,a} := \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle = a\}$. Similarly, $H_{u,a}^+$ represents the closed half-space defined by $H_{u,a}^+ := \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq a\}$.

Definition 8.3. Let P be a polytope. We say that $Q \subseteq P$ is a *face* of P, denoted as $Q \preceq P$, if there exists an affine hyperplane $H_{u,a}$ such that $Q = P \cap H_{u,a}$ and $Q \subseteq H_{u,a}^+$. In this case, $H_{u,a}$ is called a *supporting affine hyperplane*.

By convention, P is considered as its own face, i.e., $P \preceq P$, even if it may not necessarily satisfy the condition of Definition 8.3.

It is not difficult to observe that any face of a polytope is still a polytope. In fact, if $P = \text{Conv}(S)$ and $Q \preceq P$ with a supporting affine hyperplane H, then $Q = \text{Conv}(S \cap H)$. In particular, each face Q possesses a dimension as defined in Definition 8.1. If P is a polytope of dimension n, we refer to *vertices*, *edges*, and *facets* to denote a face $Q \preceq P$ of dimension 0, 1, and $n - 1$ respectively.

Below is an example provided for illustration.

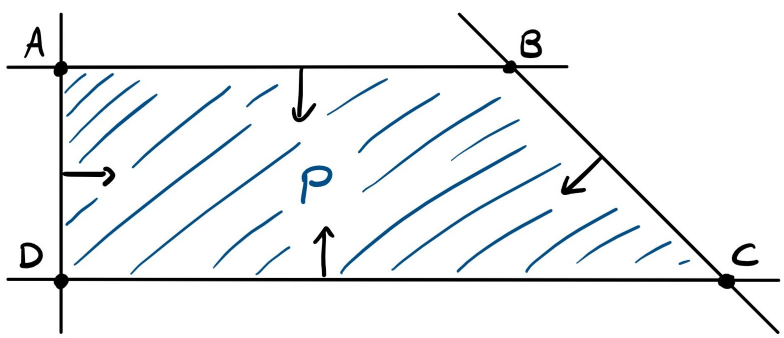


Figure 6: A polytope $P \subseteq \mathbb{R}^2$ with its four supporting hyperplanes. The points A, B, C, and D are the four vertices of P, while the segments AB, BC, CD, and DA are the edges, which are also facets of P (in dimension 2).

We now state without proof some useful results about polytopes that will be helpful for what follows.

Proposition 8.4. *Let $P = \text{Conv}(S) \subseteq M_{\mathbb{R}}$ be a polytope. Then:*

(i) $P = \text{Conv}(\{v \in P \mid v \text{ is a vertex}\})$, *which means that P is the convex hull of its vertices.*

(ii) Every vertex of P belongs to S.

(iii) The relation $\preceq$ among the faces of P is transitive.

(iv) If $Q \prec P$ is a proper face of P, then $Q = \bigcap_{F \in \mathcal{F}_Q} F$, where $\mathcal{F}_Q$ is the set of all facets of P containing Q.

It is worth noting, moreover, that any polytope P can be obtained as the finite intersection of closed half-spaces. Furthermore, when P is full-dimensional, each facet $F \preceq P$ is contained in a unique hyperplane $H_F = H_{u_F,-a_F}$ with $(u_F, -a_F) \in N_{\mathbb{R}} \times \mathbb{R}$, unique up to multiplication by a positive real number (see Exercise 8.1). In this case, we can represent P as follows:

$$P = \bigcap_{F \in \mathcal{F}} H^+_{u_F,-a_F}, \tag{4}$$

where $\mathcal{F}$ denotes the the set of all facets of P.

Remark 8.5. Note that conversely, if $P \subseteq M_{\mathbb{R}}$ is bounded with $P = \bigcap_{i=1}^l H_i^+$ for some closed half-sapces H_i^+, then P is a polytope. In other words, any bounded finite intersection of closed half-spaces is a polytope.

We now introduce various types of specific polytopes.

Definition 8.6. Let $P \subseteq M_{\mathbb{R}}$ be a polytope of dimension d.

(i) P is called a (d-)*simplex* if it has exactly $d + 1$ vertices.

(ii) In $\mathbb{R}^n$, the *standard n-simplex* is the polytope $\Delta_n = \text{Conv}(0, e_1, \dots, e_n)$, where the e_i's denote the canonical vectors basis of $\mathbb{R}^n$.

(iii) P is said to be *simplicial* if each of its facets is a simplex.

(iv) P is called *simple* if each vertex of P lies in exactly d facets of P.

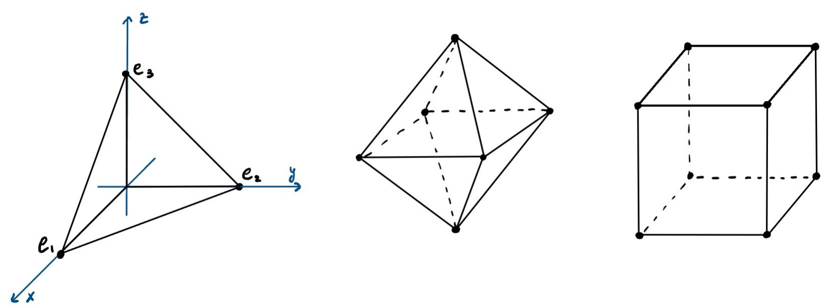


Figure 7: From left to right: The standard 3-simplex in $\mathbb{R}^3$, an octahedron (simplicial), and a cube (simple).

Definition 8.7. Two polytopes $P, Q \subseteq M_{\mathbb{R}}$ are said to be *combinatorially equivalent* if there exists a bijection between the set of faces of P and that of Q that preserves intersections, the inclusion relations $\preceq$, and the dimensions of the faces.

It is not difficult to see that every simplex of dimension d is combinatorially equivalent to the standard d-simplex. Similarly, every convex polygon with $n \geq 3$ vertices are combinatorially equivalent to the regular convex n-gone.

We now introduce some "algebraic" operations on polytopes.

Definition 8.8. For $A_1, A_2 \subseteq M_{\mathbb{R}}$ two finite subsets, the *Minkowski sum* of A_1 and A_2 is defined to be $A_1 + A_2 := \{a_1 + a_2 \mid a_1 \in A_1,\ a_2 \in A_2\}$.

Definition 8.9. Let $P_1 = \text{Conv}(A_1)$, $P_2 = \text{Conv}(A_2)$ be two polytopes, and $r \in \mathbb{R}_{\geq 0}$. We then define new polytopes:

(i) $r \cdot P_1 := \text{Conv}(r \cdot A_1)$,

(ii) $P_1 + P_2 := \text{Conv}(A_1 + A_2)$.

For $r, s \in \mathbb{R}_{\geq 0}$ and $P \subseteq M_{\mathbb{R}}$ a polytope, note that the operations defined above satisfy $rP + sP = (r+s)P$.

Definition 8.10. For $A \subseteq M_{\mathbb{R}}$, we define the *dual* of A, denoted by A°, as

$$A^\circ = \{u \in N_{\mathbb{R}} \mid \langle m, u\rangle \geq -1,\ \forall m \in A\} = \bigcap_{m \in A} H^+_{m,-1}.$$

From this definition, we observe that for any $A \subseteq M_{\mathbb{R}}$, we have $A^\circ = \text{Conv}(A)^\circ$. In particular, for a polytope $P = \text{Conv}(S)$ where $S \subseteq M_{\mathbb{R}}$ is finite, we have

$$P^\circ = \text{Conv}(S)^\circ = S^\circ = \bigcap_{m \in S} H^+_{m,-1}.$$

Thus, the dual of a polytope is a finite intersection of closed half-spaces, and according to Remark 8.5, we conclude that P° is a polytope if and only if $\bigcap_{m\in S} H^+_{m,-1}$ is bounded. It is not difficult to see that this last condition is satisfied if and only if 0 is an interior point of P. Therefore, for any (full-dimensional) polytope P containing 0 as an interior point, we conclude that the dual of P is also a polytope.

The dual of a polytope P has the following additional properties.

Proposition 8.11. *A full-dimensional polytope $P \subseteq M_{\mathbb{R}}$ containing the origin as an interior point satisfies:*

(i) $P^\circ = \text{Conv}(\frac{1}{a_F} u_F \mid F \in \text{ is a facet})$, *if* $P = \bigcap_{F\in} H^+_{u_F, a_F}$,

(ii) $(P^\circ)^\circ = P$,

(iii) if P is simplicial, then P° is simple and vice versa.

Proof. See Exercise 8.2. □

Below is an example of a polytope in $\mathbb{R}^2$ and its dual.

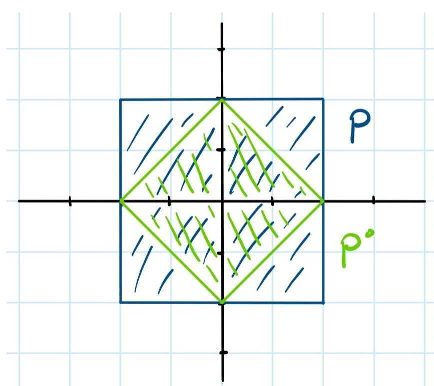


Figure 8: The polytope $P = \text{Conv}(2e_1 + 2e_2, 2e_1 - 2e_2, -2e_1 + 2e_2, -2e_1 - 2e_2)$ in blue and its dual P° in green represented in the same space $M_{\mathbb{R}} = N_{\mathbb{R}} = \mathbb{R}^2$.

8.2 Lattice, normal and very ample polytopes

Now, consider M and N as two dual lattices with associated vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$.

Definition 8.12. A *lattice polytope* is a polytope $P = \text{Conv}(S)$, where $S \subseteq M$ is a set of lattice points.

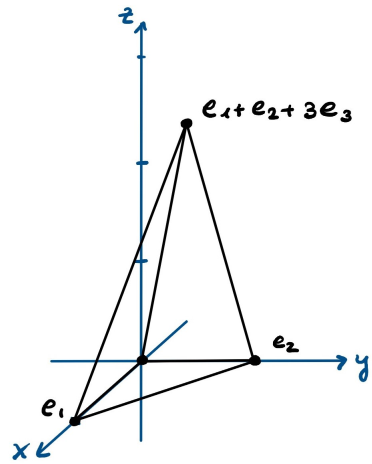


Figure 9: The lattice polytope $P = \text{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$.

For example, the standard simplices as well as the square illustrated in Figure 8 are lattice polytopes.

When $P \subseteq M_{\mathbb{R}}$ is full-dimensional, let us recall that by (4), we have a decomposition $P = \bigcap_{F\in\mathcal{F}} H^+_{u_F,-a_F}$ where $\mathcal{F}$ is the set of facets of P, and where the pairs $(u_F, -a_F)$ are unique up to multiplication by positive real number. When P is moreover a lattice polytope, each u_F defines a rational ray, and we may choose in the above decomposition u_F to be the ray generator of this ray. The decomposition (4) then becomes unique, and in that case the coefficients a_F are integers since for any vertices v of the facet $F \preceq P$, we have $-a_F = \langle v, u_F\rangle \in \mathbb{Z}$.

A new example of a lattice polytope is provided to illustrate this decomposition.

Example 8.13. The 3-simplex $P = \text{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$ is a lattice polytope (see Figure 9). The decomposition of P as an intersection of closed half-spaces, given by the above comment, is then

$$P = H^+_{e_3,0} \cap H^+_{3e_1-e_3,0} \cap H^+_{3e_2-e_3,0} \cap H^+_{-3e_1-3e_2+e_3,-3}.$$

However, notice that $\frac{1}{3}P$ has a similar decomposition with integer coefficients but is not a lattice polytope. Indeed, we have

$$\frac{1}{3}P = H^+_{e_3,0} \cap H^+_{3e_1-e_3,0} \cap H^+_{3e_2-e_3,0} \cap H^+_{-3e_1-3e_2+e_3,-1},$$

but it is not a lattice polytope since $\frac{1}{3}P = \text{Conv}\left(0, \frac{1}{3}e_1, \frac{1}{3}e_2, \frac{1}{3}e_1 + \frac{1}{3}e_2 + e_3\right)$.

Later on, we will see how to construct toric varieties $X_{P\cap M}$ from a polytope P. We will observe that this construction works quite well when the polytope in question has "sufficiently" many lattice points. In the following, we introduce two types of polytopes that fulfill this role, namely, *normal* and *very ample* polytopes.

Definition 8.14. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is said to be *normal* if for any $k, l \in \mathbb{N}$ we have

$$(kP \cap M) + (lP \cap M) = (k + l)P \cap M.$$

Note that equivalently, P is normal if

$$\underbrace{P \cap M + \cdots + P \cap M}_{k \text{ times}} = kP \cap M \text{ for all } k \in \mathbb{N}.$$

We remark additionally, that in both cases, the left-to-right inclusion is always satisfied.

Example 8.15. The standard n-simplex $\Delta_n = \text{Conv}(0, e_1, \ldots, e_n) \subseteq \mathbb{R}^n$ is normal. Indeed, for any $k \in \mathbb{N}$, note that

$$k\Delta_n = \{\lambda_0 \cdot 0 + \sum_{i=1}^{n} \lambda_i e_i \mid \sum_{i=0}^{n} \lambda_i = k,\ \lambda_i \geq 0\}.$$

Thus, if $x \in k\Delta_n \cap \mathbb{Z}^n$, we get

$$x = \lambda_0 \cdot 0 + \sum_{i=1}^{n} \lambda_i e_i, \text{ where } \lambda_i \in \mathbb{N},\ \sum_{i=0}^{n} \lambda_i = k.$$

We can then rewrite x as follows:

$$x = \underbrace{0 + \cdots + 0}_{\lambda_0 \text{ times}} + \cdots + \underbrace{e_n + \cdots + e_n}_{\lambda_n \text{ times}} \in \underbrace{\Delta_n \cap \mathbb{Z}^n + \cdots + \Delta_n \cap \mathbb{Z}^n}_{k \text{ times}}.$$

Example 8.16. The 3-simplex $P = \text{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3)$ from Example 8.13 is a non-normal lattice polytope. Indeed, it is not difficult to observe that its only lattice points are its vertices, i.e., $P \cap \mathbb{Z}^3 = \{0, e_1, e_2, e_1 + e_2 + 3e_3\}$. Thus, $e_1 + e_2 + e_3$ does not belong to $P \cap \mathbb{Z}^3 + P \cap \mathbb{Z}^3$. However, $e_1 + e_2 + e_3 \in 2P \cap \mathbb{Z}^3$ since

$$e_1 + e_2 + e_3 = \frac{1}{6} \cdot 0 + \frac{1}{3} \cdot 2e_1 + \frac{1}{3} \cdot 2e_2 + \frac{1}{6} \cdot (2e_1 + 2e_2 + 6e_3).$$

Below is presented a theorem on normality.

Theorem 8.17. *Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional lattice polytope of dimensions $n \geq 2$. Then kP is normal for all $k \geq n - 1$.*

Proof. We will prove this theorem in four parts.

Step 1: Let us assume initially that P satisfies

$$(k+1)P \cap M \subseteq kP \cap M + P \cap M \quad \forall k \geq n - 1. \tag{$\star$}$$

Then, for any integer $l \geq 2$, we have

$$lkP \cap M \subseteq (lk-1)P \cap M + P \cap M \subseteq \cdots \subseteq kP \cap M + \underbrace{P \cap M + \cdots + P \cap M}_{(l-1)k \text{ times}},$$

where we have successively used the inclusion given by ($\star$). Since clearly we have $\underbrace{P \cap M + \cdots + P \cap M}_{k \text{ times}} \subseteq kP \cap M$, we then find that

$$lkP \cap M \subseteq kP \cap M + \underbrace{kP \cap M + \cdots + kP \cap M}_{(l-1) \text{ times}} = \underbrace{kP \cap M + \cdots + kP \cap M}_{l \text{ times}}.$$

Therefore, in this case, we conclude that kP is normal for all $k \geq n - 1$. Thus, to prove the theorem, it is sufficient to verify that P satisfies the condition ($\star$).

Step 2: Let us now assume that P decomposes into a union of polytopes $P = \bigcup_{i=1}^{n} P_i$ such that each polytope P_i satisfies ($\star$). For any $k \geq n - 1$, we then have

$$\begin{aligned}(k+1)P \cap M &= ((k+1)\bigcup_{i=1}^{n} P_i) \cap M = \bigcup_{i=1}^{n} ((k+1)P_i \cap M) \\ &\subseteq \bigcup_{i=1}^{n} (kP_i \cap M + P_i \cap M) \subseteq kP \cap M + P \cap M,\end{aligned}$$

where the first inclusion follows from the assumption that ($\star$) holds for each P_i. Consequently, we observe in this case that P also satisfies the condition ($\star$). Thus, to prove the theorem, it suffices to find a decomposition of P into a union of polytopes for which $(\star)$ is verified.

Step 3: We now demonstrate that every lattice polytope of dimension n decomposes into a finite union of n-dimensional lattice simplices without interior lattice points. To accomplish this, let us recall the following theorem[4]:

Theorem 8.18 (Carathéodory)**.** *For a finite set $A \subseteq M_{\mathbb{R}}$, the convex hull of A can be decomposed as $\mathrm{Conv}(A) = \bigcup_{i=1}^{l} \mathrm{Conv}(B_i)$, where $B_i \subseteq A$ are subsets of A such that $\mathrm{Card}(B_i) = \dim \mathrm{Conv}(A) + 1$ and $\dim \mathrm{Conv}(B_i) = \dim \mathrm{Conv}(A)$.*

For $P = \mathrm{Conv}(S)$, full-dimensional of dimension n, Carathéodory's Theorem implies that $P = \bigcup_{i=1}^{l} \mathrm{Conv}(B_i)$, where $\mathrm{Conv}(B_i)$ is an n-simplex for all i. Furthermore, note that any n-simplex Q with interior lattice point can be decomposed into a union of n-simplices without interior lattice points. Specifically, if $Q = \mathrm{Conv}(w_0, \ldots, w_n)$ with an interior lattice point v, then $Q = \bigcup_{i=0}^{n} Q_i$, where $Q_i = \mathrm{Conv}(w_0, \ldots, \hat{w_i}, \ldots, w_n, v)$. Each Q_i is then an n-simplex with fewer lattice points, and as Q is bounded, we can repeat the same argument until obtaining the desired decomposition (see Figure 10). By combining Carathéodory's Theorem with the above argument, we conclude that every polytope P can be decomposed into a finite union of n-simplices without interior points.

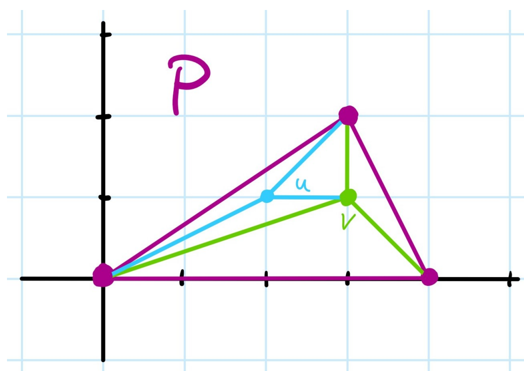


Figure 10: Decomposition of the 2-simplex $P = \mathrm{Conv}(0, 4e_1, 3e_1 + 2e_2)$ (containing the interior lattice points u and v) into smaller 2-simplices that do not contain any interior lattice points.

Step 4: By combining the results from Step 2 and Step 3, we note that it suffices to demonstrate that $(\star)$ holds for simplices without interior lattice points.
So let us consider $P = \mathrm{Conv}(m_0, \ldots, m_n)$, where $m_i \in M$ and P has no interior lattice points. Let $m \in (k+1)P \cap M$. Then,

$$m = \sum_{i=0}^{n} \mu_i (k+1) m_i,$$

where each $\mu_i \geq 0$ and $\sum_{i=0}^{n} \mu_i = 1$. Setting $\lambda_i = (k+1)\mu_i$, we have

$$m = \sum_{i=0}^{n} \lambda_i m_i, \text{ with } \lambda_i \geq 0, \text{ and } \sum_{i=0}^{n} \lambda_i = k+1.$$

We distinguish two cases:

(i) If there exists a $\lambda_i \geq 1$, we easily observe that $m - m_i \in kP \cap M$. Therefore, $m = (m - m_i) + m_i \in kP \cap M + P \cap M$.

[4]For a proof of this theorem, see, for example, Prop. 1.1.15 in "Lectures On Polytopes" by Ziegler, Günter M.

(ii) If, on the other hand, $\lambda_i < 1$ for all indices i, then $\sum_{i=0}^n < n+1$. Since by hypothesis $k \geq n-1$, we obtain in that case that

$$k+1 = \sum_{i=0}^n \lambda_i < n+1 \leq k+2,$$

and therefore $n = k+1$. We then define $\tilde{m} = m_0 + \cdots + m_n - m$, and observe that

$$\tilde{m} = \sum_{i=0}^n m_i - \sum_{i=0}^n \lambda_i m_i = \sum_{i=0}^n (1-\lambda_i) m_i.$$

Since $\sum_{i=0}^n (1-\lambda_i) = (n+1) - n = 1$ and $m_0 + \cdot + m_n - m \in M$, we find that $\tilde{m} \in P \cap M$. Furthermore, noting that $1 > 1 - \lambda_i > 0$ for all i, we see that $\tilde{m}$ is an interior lattice point of P. Since, by assumption, P does not have any interior lattice points, we conclude that this second case cannot occur.

We conclude that only the first case is possible, and consequently, that P satisfies the condition $(\star)$. This completes the proof.

□

For a lattice polytope P, its *cone* is defined as $C(P) = \mathrm{Cone}(P \times \{1\}) \subseteq M_\mathbb{R} \times \mathbb{R}$. Explicitly, we have $C(P) = \{(rx, r) \mid r \geq 0, x \in P\}$, and we will think of the parameter $r \geq 0$ as a height.

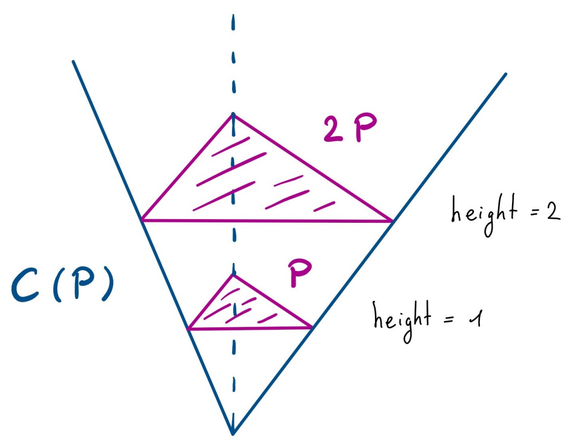


Figure 11: The cone $C(P)$ of the polytope P sliced at heights 1 and 2.

The following lemma allows us to interpret the normality of a polytope in terms of its cone.

Lemma 8.19. *Let $P \subseteq M_\mathbb{R}$ be a lattice polytope. Then, P is normal if and only if $(P \cap M) \times \{1\}$ generates the semigroup $C(P) \cap (M \times \mathbb{Z})$.*

Proof. Exercise 8.3. □

The lemma above allows for another proof of the fact that the polytope $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3)$ from Example 8.13 cannot be normal. Indeed, it can be shown that the Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$ contains elements of height 2 (see Exercise 8.4), and thus, by Lemma 8.19, we conclude that P is not normal.

Definition 8.20. A lattice polytope $P \subseteq M_\mathbb{R}$ is called *very ample* if for every vertex $m \in P$, the semigroup $S_{P,m} = \mathbb{N}(P \cap M - m)$ is saturated in M.

Proposition 8.21. *A normal polytope P is very ample.*

Proof. Let m_0 be a vertex of P and $m \in M$ such that $km \in S_{P,m_0}$ for some integer $k \geq 1$. Then we have

$$km = \sum_{m' \in P \cap M} a_{m'}(m' - m_0) = \sum_{m' \in P \cap M} a_{m'} m' - \sum_{m' \in P \cap M} a_{m'} m_0, \quad a'_m \in \mathbb{N}.$$

Take $d \in \mathbb{N}$ such that $kd \geq \sum_{m' \in P \cap M} a_{m'}$, then

$$km + kdm_0 = \sum_{m' \in P \cap M} a_{m'} m' + (kd - \sum_{m' \in P \cap M} a_{m'}) m_0 \in kdP.$$

Dividing the above equation by k gives

$$m + dm_0 = \sum_{m' \in P \cap M} \frac{1}{k} a_{m'} m' + \left(d - \frac{1}{k} \sum_{m' \in P \cap M} a_{m'} \right) m_0 \in dP.$$

Due to the normality of P, we therefore have $m + dm_0 = \sum_{i=1}^{d} m_i$, for some $m_i \in P \cap M$. Consequently,

$$m = \left(\sum_{i=1}^{d} m_i \right) - dm_0 = \sum_{i=1}^{d} (m_i - m_0) \in S_{P, m_0},$$

as desired. □

By combining Theorem 8.17 with Proposition 8.21, we obtain the following corollary.

Corollary 8.22. *Let $P \subseteq M_{\mathbb{R}} \cong \mathbb{R}^n$ be a full-dimensional lattice polytope. Then, kP is very ample for all $k \geq n - 1$.*

Note that a very ample polytope need not to be normal, so the converse of the previous corollary does not necessarily hold.

8.3 Exercises

Exercise 8.1. Let P be a polytope. Show that each facet of P has a unique supporting affine hyperplane if and only if P is of maximal dimension.

Exercise 8.2. Let $P \subseteq M_{\mathbb{R}}$ be a polytope of maximal dimension d with the origin as an interior point.

(i) Write $P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \text{ for all facets } F\}$. Prove that $a_F > 0$ for all F and that $P^\circ = \mathrm{Conv}((1/a_F) u_F \mid F \text{ is a facet})$.[5] Deduce that $(P^\circ)^\circ = P$.

(ii) Show there is a bijective, inclusion reversing correspondence between the faces of P and the faces of P°, through which the faces of dimension n correspond to faces of dimension $d - n - 1$. Deduce that the dual of a simplicial polytope is simple and vice versa.

(iii) Show that $(rP)^\circ = (1/r)P^\circ$ for all $r > 0$. Use this to construct an example of a lattice polytope whose dual is not a lattice polytope.

Exercise 8.3. Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope. Prove that P is normal if and only if $(P \cap M) \times \{1\}$ generates the semigroup $C(P) \cap (M \times \mathbb{Z})$.

Exercise 8.4. Let $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$ be the simplex mentioned in the lecture.

(a) Show that the only lattice points of P are its vertices.

(b) Show that the toric variety $X_{P \cap \mathbb{Z}^3}$ is $\mathbb{P}^3$.

(c) Show that the Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$ is

$$(0,1), (e_1, 1), (e_2, 1), (e_1 + e_2 + 3e_3, 1), (e_1 + e_2 + e_3, 2), (e_1 + e_2 + 2e_3, 2).$$

Combining with the previous exercise, show that P is not normal.

[5] You may use that if $C \subset M_{\mathbb{R}}$ is a convex subset, $p \notin C$ a point, then p and C are separable by a hyperplane. In other words, there exists some $u \in N_{\mathbb{R}}$ and $a \in \mathbb{R}$, such that $\langle u, p \rangle \leq a$ and for all $m \in C$, $\langle u, m \rangle \geq a$.

Chapter 9. Normal fans

Juan Felipe Celis after the talk of Clotilde Freydt and Julia Morin

Our goal is to define a projective toric variety from a lattice polytope.

9.1 Very ample polytopes

Consider the following setting. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional very ample lattice polytope with $\dim P = n$, $P \cap M = \{m_1, \dots, m_s\}$. Recall that the toric variety $X_{P\cap M}$ is the Zariski closure of the map

$$T_N \to \mathbb{P}^{s-1}, t \mapsto [\chi^{m_1}(t) : \cdots : \chi^{m_s}(t)].$$

Now fix homogeneous coordinates $x_1, \dots, x_s$ for $\mathbb{P}^{s-1}$. Then we have $U_i = \mathbb{P}^{s-1} \backslash \mathbb{V}(x_i)$. We examine the variety $X_{P\cap M}$. Remember that

$$S_i = \mathbb{N}(P \cap M - m_i)$$

and that

$$X_{P\cap M} \cap U_i \simeq \operatorname{Spec}(\mathbb{C}[S_i]),$$
$$X_{P\cap M} = \cup_{i=1}^{s} X_{P\cap M} \cap U_i,$$

since $(U_i)_{i=1}^{s}$ form an affine open cover of $\mathbb{P}^{s-1}$.

Theorem 9.1. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional very ample lattice polytope with $\dim P = n$. Then*

(i) For all $m_i \in P \cap M$ we have

$$X_{P\cap M} \cap U_i = U_{\sigma_i} = \operatorname{Spec}(\mathbb{C}[\sigma_i]^\vee \cap M)$$

where $\sigma_i \subseteq N_{\mathbb{R}}$ is the strongly convex rational dual cone of $\operatorname{Cone}(P \cap M - m_i)$, and $\dim \sigma_i = n$.

(ii) The torus of $X_{P\cap M}$ is T_N.

Proof. (i) Let $C_i = \operatorname{Cone}(P \cap M - m_i)$ and observe that $\sigma_i = C_i^\vee$. Take $H_{u,a}$ a supporting hyperplane of m_i such that $P \subseteq H_{u,a}^+$ and $P \cap H_{u,a} = \{m_i\}$. Now we use Exercise 9.1 which says the that $H_{u,0}$ is a supporting hyperplane of $0 \in C_i$ and $\dim C_i = \dim P$.

This already proves that $\dim \sigma_i = n$. Observe we have the inclusion

$$S_i \subseteq C_i \cap M = \sigma_i^\vee \cap M$$

where both S_i and $\sigma_i^\vee \cap M$ are generated by $P \cap M - m_i$. As P is very ample S_i is saturated so this inclusion is in fact an equality. See Exercise 1.3.4 in [CLS]. This concludes the proof as

$$X_{P\cap M} \cap U_i = U_{\sigma_i} = \operatorname{Spec}(\mathbb{C}[S_i]) = \operatorname{Spec}(\mathbb{C}[\sigma_i]^\vee \cap M).$$

(ii) Notice that for all $i \in \{1, \dots, s\}$ we have

$$T_N \subseteq U_{\sigma_i} = X_{P\cap M} \cap U_i \subseteq X_{P\cap M}.$$

Then T_N is the torus of $X_{P\cap M}$ as it is an open subset, thus dense, of $X_{P\cap M}$.

□

9.2 Normal fans

For $P \subseteq M_{\mathbb{R}}$ we denote facets by F, faces by Q and vertices by v. Recall from previous sections that

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \,\forall F \preceq P \text{ facet}\}$$

and from a vertex $v \in P$ we can define cones

$$C_v = \operatorname{Cone}(P \cap M - v) \subseteq M_{\mathbb{R}}$$

and

$$\sigma_v = C_v^{\vee} \subseteq N_{\mathbb{R}}.$$

We can see there is a bijective correspondence between faces of P containing v and faces of C_v. More explicitly,

$$\begin{aligned} \{v \in Q \preceq P\} &\leftrightarrow \{R \preceq C_v\} \\ Q &\mapsto Q_v = \operatorname{Cone}(Q \cap M - v) \\ (R + v) \cap P &\mapsfrom R. \end{aligned}$$

To have some intuition on this bijection we can rely on fig. 12 as a small example.

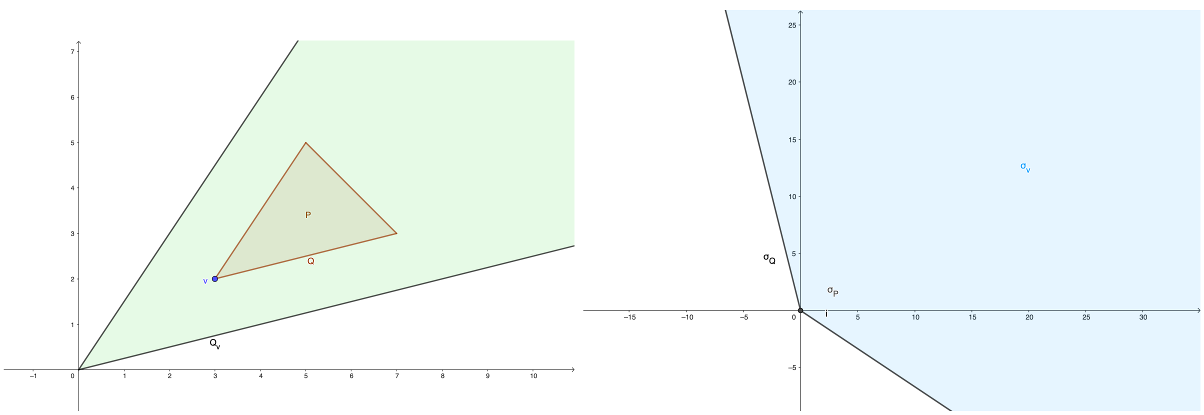


Figure 12: Bijective correspondence of faces

Notice that

$$C_v = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq 0 \,\forall F \succeq v\}$$

and by duality

$$\begin{aligned} \sigma_v &= \operatorname{Cone}(u_F \mid F \succeq v) \\ \sigma_Q &= \operatorname{Cone}(u_F \mid F \succeq Q). \end{aligned}$$

Observe that $\sigma_P = \{0\}$ because there are no facets containing P.

Theorem 9.2. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope and set*

$$\Sigma_P = \{\sigma_Q \mid Q \preceq P\}.$$

Then:

(i) For all $\sigma_Q \in \Sigma_P$, each face of σ_Q is also in Σ_P.

(ii) For any two faces $Q, Q' \preceq P$, the intersection $\sigma_Q \cap \sigma_{Q'}$ in Σ_P is a face of each.

Definition 9.3 (Fan)**.** A collection of strongly convex rational polyhedral cones satisfying (i) and (ii) is called a *fan*. Moreover if this collection comes from a polytope P as in the theorem we say that it is the *normal fan* of P.

To prove this theorem we will first state and prove some useful lemmas and propositions. Then the theorem will follow as a consequence of these.

Lemma 9.4. *Let $Q \preceq P$ and let $H_{u,b}$ be a supporting hyperplane of P. Then $u \in \sigma_Q$ iff $Q \subseteq H_{u,b} \cap P$.*

Proof. Left as an exercise to the reader. To see the full proof see Lemma 2.3.3 in [CLS], page 77. □

Corollary 9.5. *If $Q \preceq P$ and $F \preceq P$ is a facet. Then $u_F \in \sigma_Q$ iff $Q \preceq F$.*

Proof. Assume $Q \preceq F$ then by definition of σ_Q it is clear that $u_F \in \sigma_Q$.
Now suppose that $u_F \in \sigma_Q$. Then by our lemma $H_{u_F,-a_F}$ is a supporting hyperplane of F such that its intersection with P contains Q. Moreover

$$Q \subseteq H_{u_F,-a_F} \cap P = F,$$

which finishes the proof. □

Proposition 9.6. *Let Q, Q' be faces of a full dimensional polytope $P \subseteq M_{\mathbb{R}}$. Then:*

(i) $Q \subseteq Q'$ iff $\sigma_{Q'} \subseteq \sigma_Q$.

(ii) If $Q \subseteq Q'$ then $\sigma_{Q'}$ is a face of σ_Q and all faces of σ_Q are of this form.

(iii) We have $\sigma_Q \cap \sigma_{Q'} = \sigma_{Q''}$ where Q'' is the smallest face of P containing both Q and Q'.

Proof. (i) First suppose that $Q \subseteq Q'$. Then by definition of σ_Q and $\sigma_{Q'}$ we get $\sigma_{Q'} \subseteq \sigma_Q$ because all facets containing Q' also contain Q.

Now suppose $\sigma_{Q'} \subseteq \sigma_Q$. Then for all $F \succeq Q'$ we have $u_F \in \sigma_{Q'} \subseteq \sigma_Q$ using the previous corollary. So $F \succeq Q$. We can conclude

$$Q' = \bigcap_{F \succeq Q'} F \supseteq Q.$$

(ii) Let $v \in P$ be a vertex such that $v \in Q$. Then recall that $Q_v^* := \sigma_v \cap Q_v^\perp$ is a face of $C_v^\vee = \sigma_v$. And

$$\begin{aligned} Q_v^* &= C_v^\vee \cap Q_v^\perp \\ &= \sigma_v \cap Q_v^\perp \\ &= \{m \in \sigma_v \mid \langle m, u \rangle = 0 \, \forall \, u \in Q_v\} \\ &= \mathrm{Cone}(u_F \mid F \ni v, Q_v \subseteq H_{u_F,0}) \end{aligned}$$

So for $v \in Q$, $Q_v \subseteq H_{u_F,0}$ iff $Q \subseteq H_{u_F,-a_F}$ iff $Q \subseteq F$. Then

$$Q_v^* = \mathrm{Cone}(u_F \mid F \ni v, Q \subseteq F) = \sigma_Q$$

Thus σ_Q is a face of σ_v. Moreover if $Q \subseteq Q'$, $\sigma_{Q'} \subseteq \sigma_Q$ we have $\sigma_{Q'} \preceq \sigma_v$.

Now if $\tau \preceq \sigma_{Q'} \preceq \sigma_Q$ then $\tau = \sigma_{Q''}$ for $Q'' \preceq P$.

(iii) Let Q'' be the smallest face of P containing Q and Q'. Then by part (i) we have $\sigma_{Q''} \subseteq \sigma_Q$ and $\sigma_{Q''} \subseteq \sigma_{Q'}$. Thus $\sigma_{Q''} \subseteq \sigma_Q \cap \sigma_{Q'}$.

Now we consider two cases. If $\sigma_Q \cap \sigma_{Q'} = \{0\} = \sigma_P$ then $Q'' = P$. Otherwise there is $u \in (\sigma_Q \cap \sigma_{Q'}) \setminus \{0\}$ and define

$$b = \min\{\langle v, u \rangle \mid v \in P \text{ vertex}\}.$$

Then $P \subseteq H_{u,b}^+$ so $H_{u,b}$ is a supporting hyperplane of P. Moreover $Q \subseteq H_{u,b} \cap P$ and $Q' \subseteq H_{u,b} \cap P$ thus $H_{u,b} \cap P$ is a face containing Q, and Q'. Then $Q'' \subseteq H_{u,b} \cap P$ and it follows that $u \in \sigma_{Q''}$. This concludes the proof.

□

Remark 9.7. This proposition proves the theorem. Indeed parts (i) and (ii) from this proposition imply (i) from the theorem, and part (iii) of the proposition implies part (ii) of the theorem.

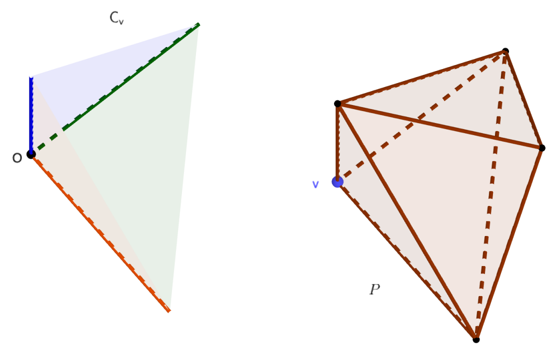


Figure 13: Cone C_v for a vertex $v \in P$

Example 9.8. In fig. 13 we can see a polytope P with a vertex v and its cone C_v.

Proposition 9.9. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope of dimension n. Then:*

(i) For all faces Q of P, $\sigma_Q \in \Sigma_P$ and
$$\dim \sigma_Q + \dim Q = n.$$

(ii) Moreover
$$N_{\mathbb{R}} = \bigcup_{v \in P} \sigma_v = \bigcup_{\sigma_Q \in \Sigma_P} \sigma_Q$$

Proof. (i) For $v \in Q$ we have
$$\dim Q + \dim \sigma_Q = \dim Q_v + \dim Q_v^* = n.$$

(ii) Let $u \in N_{\mathbb{R}}$ be non-zero. Take
$$b = \min\{\langle v, u \rangle \mid v \in P \text{ vertex}\}.$$

Then $H_{u,b}$ is a supporting hyperplane of P. There is at least one $v \in P$ such that $v \in H_{u,b}$. Thus $u \in \sigma_v$.

□

Proposition 9.10. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. For all lattice points $m \in M$, and any integer $k \geq 1$, $m + P$ and kP have the same normal fan as P.*

Proof. Exercise 9.2. □

Example 9.11 (Normal fan)**.** In fig. 14 we see a lattice hexagon with its normal fan.

Example 9.12. In fig. 15 we can see a cube P and its dual an octahedron. Here want to understand the relation between the dual polytope P° and the normal fan Σ_P. Notice that the cone of a face of P° is an element of Σ_P. For more details in this correspondence see Exercise 9.3.

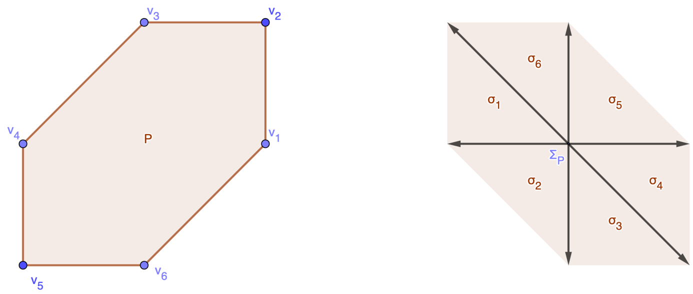


Figure 14: Lattice hexagon P with its normal fan

9.3 Toric variety of a polytope

Now consider

$$X_{P\cap M} \cap U_v \cap U_w$$

and let $P \cap M = \{m_1, \ldots, m_s\}$. If $v \in P \cap M$, there is some $i \in \{1, \ldots, s\}$ such that $v = m_i$ and $U_v = X_{P\cap M} \cap U_i \simeq \operatorname{Spec}(\mathbb{C}[S_i])$.

Proposition 9.13. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional very ample lattice polytope. Let $v, w \in P$ be two distinct vertices, and $Q \preceq P$ be the smallest face containing v and w. Then*

$$X_{P\cap M} \cap U_v \cap U_w = U_{\sigma_Q} = \operatorname{Spec}(\mathbb{C}[\sigma_Q^\vee \cap M]).$$

Proof. We have

$$\begin{aligned} X_{P\cap M} \cap U_v \cap U_w &= \operatorname{Spec}(\mathbb{C}[\sigma_v^\vee \cap M]) \backslash \mathbb{V}_p(\chi^{w-v}) \\ &= (U_{\sigma_v})_{\chi^{w-v}} \end{aligned}$$

and similarly

$$X_{P\cap M} \cap U_v \cap U_w = (U_{\sigma_w})_{\chi^{v-w}}.$$

So it is enough to show that

$$(U_{\sigma_v})_{\chi^{w-v}} = U_{\sigma_Q}.$$

Now observe that $w - v \in C_v = \sigma_v^\vee$ thus $\tau = H_{w-v} \cap \sigma_v \preceq \sigma_v$. We get

$$(U_{\sigma_v})_{\chi^{w-v}} = U_\tau$$

so to prove the proposition it suffices to show $\tau = \sigma_Q$. Equivalently we need to show that $H_{w-v} \cap \sigma_v = \sigma_v \cap \sigma_w$, because theorem 9.6(iii) yields $\sigma_Q = \sigma_v \cap \sigma_w$.
Let $u \in H_{w-v} \cap \sigma_v$. If $u \neq 0$ then there is a supporting hyperplane $H_{u,b}$ of P. Thus by lemma 9.4 $v \in H_{u,b}$ and as $u \in H_{w-v}$ we deduce that $w \in H_{u,b}$ and $u \in \sigma_w$. Then $H_{w-v} \cap \sigma_v \subseteq \sigma_v \cap \sigma_w$.
Now let $u \in \sigma_v \cap \sigma_w$, $u \neq 0$. Then there is a supporting hyperplane $H_{u,b}$ of P containing v and w. Again by lemma 9.4 we conclude that $u \in H_{w-v}$. Whence $H_{w-v} \cap \sigma_v = \sigma_v \cap \sigma_w$ and the proposition follows. ☐

Remark 9.14. This proposition alongside the theorem about normal fans prove that the normal fan Σ_P completely determines $X_{P\cap M}$.

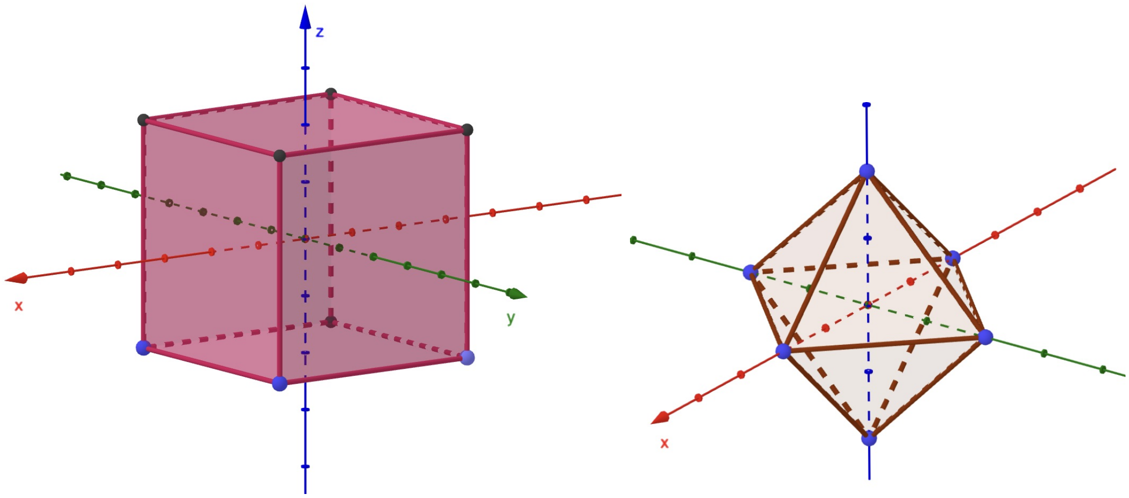


Figure 15: A cube and its dual octahedron

Now we can define a toric variety from a polytope.

Definition 9.15 (Toric variety associated to a polytope)**.** Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. Then we define the *toric variety of* P to be

$$X_P = X_{(kP)\cap M}$$

where k is any positive integer such that kP is very ample.

9.4 Exercises

Exercise 9.1. Let $P \subseteq M_{\mathbb{R}}$, be a full dimensional very ample polytope.

(i) Let $H_{u,a}$ be a supporting hyperplane of a vertex $m \in P$. Prove that $H_{u,0}$ is a supporting hyperplane of $0 \in C = \text{Cone}\,(P \cap M - m)$.

(ii) Prove that $\dim C = \dim P$.

Exercise 9.2. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. Then for any lattice point $m \in M$ and any integer $k \geq 1$, the polytopes $m + P$ and kP have the same normal fan as P.

Exercise 9.3. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope containing 0 as an interior point, and let $P^{\circ} \subseteq N_{\mathbb{R}}$ be its dual polytope. Prove that the normal fan Σ_P consists of the cones over the faces of P°. Hint: Use Exercise 2.2.1 of [CLS].

Exercise 9.4. (i) Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Prove that the normal fan of the standard n-simplex consists of the cones $\text{Cone}(S)$ for all proper subsets $S \subseteq \{e_0, e_1, \ldots, e_n\}$, where $e_0 = -\sum_{i=1}^{n} e_i$. Draw pictures of the normal fan for $n = 1, 2, 3$.

(ii) For an integer $k \geq 1$, show that the variety $X_{k\Delta_n} \subseteq \mathbb{P}^{s_k - 1}$ is given by the map $\nu_k : \mathbb{P}^n \longrightarrow \mathbb{P}^{s_k-1}$ defined using all monomials of total degree k in $\mathbb{C}\,[x_0, \ldots, x_n]$.

Chapter 10. Smooth projective toric varieties and abstract varieties

Isak Sundelius after the talk of Coppin and Schuller.

10.1 Projective toric varieties

Recall: Let $P \subseteq M_{\mathbb{R}}$ denote a full-dimensional and very ample lattice polytope. Denote by $s = |P \cap M|$ the number of lattice points. We then have the projective toric variety $X_{P\cap M}$, given as a subvariety of $\mathbb{P}^{s-1}$. It decomposes as the union

$$X_{P\cap M} = \bigcup_{v \text{ vertex}} X_{P\cap M} \cap U_v$$

with

$$X_{P\cap M} \cap U_v = U_{\sigma_v} = \operatorname{Spec}(\mathbb{C}[\sigma_v^\vee \cap M]).$$

Let $v \neq w$ be vertices and Q the smallest edge containing them. Then

$$X_{P\cap M} \cap U_v \cap U_w = U_{\sigma_Q}.$$

Furthermore,

$$U_{\sigma_v} \supseteq X_{P\cap M} \cap U_v \cap U_w \subseteq U_{\sigma_w}$$

and in particular

$$U_{\sigma_v} \supseteq (U_{\sigma_v})_{\chi^{w-v}} = U_{\sigma_Q} = (U_{\sigma_w})_{\chi^{v-w}} \subseteq U_{\sigma_w}$$

by which we conclude that the normal fan Σ_P determines $X_{P\cap M}$.

10.1.1 The toric variety of a polytope

Definition 10.1. Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional lattice polytope. The *toric variety of* P denoted by X_P is defined as $X_{(kP)\cap M}$, where $k \geq 1$ is chosen such that kP is very ample.

Remark 10.2.

- Such a k as in definition 10.1 exists and satisfies $k \geq n-1$.
- If k and ℓ are two such integers, kP and ℓP have the same normal fan.

Example 10.3. Let $\Delta \subseteq \mathbb{R}^n$ be the standard n-simplex given by $\operatorname{Cone}(0, e_1, \dots, e_n)$. If $k \geq 1$ we denote by $s_k = \binom{n+k}{k}$ the number of lattice points in $k\Delta_n$, which are given by monomials in $\mathbb{C}[t_1, \dots, t_n]$ of total degree $\leq k$. Then there is an embedding

$$X_{\Delta_n} \subseteq \mathbb{P}^{s_k-1}$$

In the case of $k = 1$ we clearly only have the lattice points

$$\Delta_n \cap \mathbb{Z}^n = \{0, e_1, \dots, e_n\}.$$

Then $X_{\Delta_n} = \mathbb{P}^n$.
For general $k \geq 1$ we get the embedding

$$\mathcal{V}_k : \mathbb{P}^n \to \mathbb{P}^{s_k-1}$$

with image defined by using the monomials in $\mathbb{C}[x_0, \dots, x_n]$ of degree k. Setting $n = 1$ and $k = 2$ we get that this is the Veronese embedding

$$\mathcal{V}_2 : \mathbb{P}^1 \to \mathbb{P}^2$$

$$[x_0 : x_1] \mapsto [x_0^2 : x_0x_1 : x_1^2].$$

10.1.2 Normality

Definition 10.4. A projective variety embedded in $\mathbb{P}^n$ is *projectively normal* if its affine cone is normal.

Remark 10.5. We will later see that a variety X is defined to be normal if it is irreducible and the local rings $\mathcal{O}_{X,p}$ are normal for all $p \in X$.

Recall:

- If σ is a strongly convex rational polyhedral cone then the affine toric variety $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$ is normal.
- The affine toric variety $\operatorname{Spec}(\mathbb{C}[S])$ is normal if and only if S is saturated.

Theorem 10.6. *Let $P \subseteq M_\mathbb{R}$ be a full-dimensional lattice polytope. We then have the following:*

(a) X_P is normal;

(b) X_P is projectively normal under the embedding given by kP if and only if kP is normal.

Proof. (a) X_P is toric, hence irreducible. The affine pieces are given by U_{σ_v}, and since these are normal X_P is normal.

(b) Note that

$$X_{(kP)\cap M} = X_{((kP)\cap M)\times\{1\}}.$$

The right hand side may be viewed as the closure of the image of

$$\Phi_{((kP)\cap M)\times\{1\}}(t,\mu) = [\chi^{m_1}(t)\mu : \cdots : \chi^{m_s}(t)\mu] = [\chi^{m_1}(t) : \cdots : \chi^{m_s}(t)].$$

The affine cone of $X_{(kP)\cap M}$ is $Y_\mathcal{A}$, with $\mathcal{A} = ((kP)\cap M)\times\{1\}$. In particular,

$$Y_\mathcal{A} = \operatorname{Spec}(\mathbb{C}[S])$$

where $S = \mathbb{N}\mathcal{A}$. The affine cone $Y_\mathcal{A}$ is normal if and only if S is saturated. We see that $\mathcal{A}$ generates the cone $C(kP) := \operatorname{Cone}((kP)\times\{1\})$. We then have that S is saturated if and only if $C(kP)\cap(M\times\mathbb{Z})$ is generated by $\mathcal{A}$, which in turn is the case if and only if kP is normal.

□

10.1.3 Smoothness

Definition 10.7. Let $P \subseteq M_\mathbb{R}$ be a lattice polytope and let v be a vertex of P. Let E be an edge of P containing v and denote by w_E the first lattice point encountered when moving along E beginning at v, not equal to v. Then we define P to be smooth if for every vertex $v \in P$ the set

$$\{w_E - v \mid v \in E \subseteq P \text{ edge and } w_E \text{ the lattice point given by } v \text{ and } E\}$$

forms a subset of a $\mathbb{Z}$-basis of M.

Recall: A cone is smooth if its ray generators form a subset of a $\mathbb{Z}$-basis of M.

Theorem 10.8. *Let $P \subseteq M_\mathbb{R}$ be a full-dimensional lattice polytope. The following are equivalent:*

(a) X_P is a smooth projective variety;

(b) Σ_P is a smooth fan, i.e., every $\sigma \in \Sigma_P$ is smooth;

(c) P is a smooth polytope.

Proof. (a) $\iff$ (b) Smoothness is a local condition, so X_P is smooth if and only if all of its affine pieces are smooth. Since the affine pieces are given by U_{σ_v}, and these are smooth if and only if σ_v are smooth cones, we have that this is satisfied if and only if Σ_P is smooth.
(b) $\iff$ (c) For a vertex v the cone σ_v is smooth if and only if $\sigma_v^\vee = C_v := \operatorname{Cone}(P\cap M - v)$ is. The ray generators of the cone C_v are $w_E - v$. With this we conclude that σ_v is smooth for every vertex $v \in P$ if and only if P is smooth. □

Proposition 10.9. *Every smooth full-dimensional lattice polytope is very ample.*

Proof. As usual, we let P denote a lattice polytope such as in the statement of the proposition and let $v \in P$ denote a vertex, fixed throughout the proof. We want to show that $S_v = \mathbb{N}(P \cap M - v)$ is saturated.
Since P is smooth we have, by definition, that the $w_E - v$, for varying $E \ni v$, constitute a subset of a basis for M. Due to P being full-dimensional we have that the $w_E - v$, for varying $E \ni v$, in fact constitute a basis for M.
This furthermore gives us that $\{w_E - v \mid E \ni v\}$ generates S_v, since this is a subset of S_v generating M.
Let us now take $km \in S_v$ for $k \geq 1$ and a lattice point m. Since the $w_E - v$ constitute a basis we can write m uniquely as

$$m = \sum_{E \text{ edge}} \lambda_E (w_E - v)$$

where $\lambda_E \in \mathbb{Z}$ for all E. In a similar way we may write

$$km = \sum_{E \text{ edge}} \mu_E (w_E - v)$$

for unique $\mu_E \in \mathbb{N}$, since km is assumed to belong to S_v. Then, since the μ_E and λ_E are unique, we have equality $\mu_E / k = \lambda_E$ for every edge E containing v and so since $k \in \mathbb{N}$, by assumption, we get that $\lambda_E \in \mathbb{N}$. This means that

$$m = \sum_{E \text{ edge}} \lambda_E (w_E - v) \in S_v$$

since $S_v = \mathbb{N}(P \cap M - v)$, so we are done. □

Remark 10.10. A natural question to ask is whether every smooth polytope is normal. However, this is still an open problem.

Example 10.11.

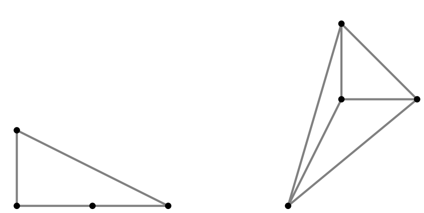
- The standard n-simplex $\Delta_n = \text{Cone}(0, e_1, \ldots, e_n)$ is smooth, since $\{e_1 - 0, \ldots, e_n - 0\}$ constitutes a basis for $M \cong \mathbb{Z}^n$.
- Let $P = \text{Conv}(0, 2e_1, e_2)$. Then $X_P = X_{P \cap \mathbb{Z}^2}$ is given by the image of the morphism

$$(\mathbb{C}^*)^2 \to \mathbb{P}^3$$

$$(s, t) \mapsto [1 : s : s^2 : t]$$

so $X_P = \mathbb{V}(y_0 y_2 - y_1^2) \subseteq \mathbb{P}^2$. Then the intersection with the affine chart $X_P \cap U_3 = \mathbb{V}(y_0 y_2 - y_1^2) \subseteq \mathbb{C}^3$, but the point $(0, 0, 0)$ corresponds to the point $[0 : 0 : 0 : 1] \in \mathbb{P}^3$, which is singular.

We also see that the cone spanned by the basis $\{e_1, -2e_2 - e_1\}$ is not smooth.



Left: The polytope P; Right: The normal fan Σ_P of P with its three cones, the bottom one, the span of $\{e_1, -2e_2 - e_1\}$, being nonsmooth/singular.

10.2 Abstract varieties

We want to study varieties, regardless of if they are affine or projective.

10.2.1 What is a sheaf

Let X be a topological space.

Definition 10.12. A *presheaf* $\mathcal{F}$ of abelian groups (or rings or $\mathbb{C}$-algebras etc.) on X consists of the data

(i) for every open subset $U \subseteq X$ an abelian group $\mathcal{F}(U)$;

(ii) for every inclusion of open subsets $V \subseteq U \subseteq X$ a group homomorphism

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

subject to

(iii) for every open subset $U \subseteq X$, $\rho_{U,U} = \mathrm{Id}_{\mathcal{F}(U)}$;

(iv) for every triple of inclusions of open subsets $W \subseteq V \subseteq U \subseteq X$,

$$\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}.$$

Remark 10.13. A presheaf constitutes a functor

$$\mathcal{F} : \mathrm{Open}(X)^{op} \to \mathrm{AbGrp}.$$

For $s \in \mathcal{F}(U)$ with $V \subseteq U \subseteq X$ open subsets, we denote by $s\,|_V := \rho_{U,V}(s)$ the restriction of s to V.

Definition 10.14. A presheaf $\mathcal{F}$ is said to be a *sheaf* if it satisfies

(i) for $U \subseteq X$ an open subset with open cover $\{U_\alpha\}_\alpha$, and $s \in \mathcal{F}(U)$, then $s\,|_\alpha = 0$ for all α implies that $s = 0$;

(ii) for $U \subseteq X$ an open subset with open cover $\{U_\alpha\}_\alpha$ and given sections $s_\alpha \in \mathcal{F}(U_\alpha)$ for every α satisfying $s_\alpha\,|_{U_\alpha \cap U_\beta} = s_\beta\,|_{U_\alpha \cap U_\beta}$ for all pairs α, β there exists a section (unique, by (i)) $t \in \mathcal{F}(U)$ such that $t\,|_{U_\alpha} = s_\alpha$ for all α.

Definition 10.15. Let $\mathcal{F}$ be a presheaf of abelian groups on X. The *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at $x \in X$ is defined as the direct limit

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U) = \{(U, s) \mid x \in U \text{ open subset of } X, s \in \mathcal{F}(U)\}/\sim,$$

where $(U, s) \sim (V, t)$ if there exists an open neighbourhood W of x such that $W \subseteq U \cap V$ and

$$s\,|_W = t\,|_W .$$

Example 10.16. An example of a sheaf is the sheaf of holomorphic functions on $\mathbb{C}$,

$$\mathcal{O}_{\mathbb{C}}^{hol}(U) = \{f \mid f \text{ holomorphic on } U\}.$$

10.2.2 Sheaf of regular functions

Let $V = \mathrm{Spec}(R)$ be an affine variety.

Proposition 10.17. *(i) For every $f \in R$, $V_f := V \setminus \mathbb{V}(f) = \mathrm{Spec}(R_f)$;*

(ii) For every open subset $U \subseteq V$, $U = \bigcup_{f \in S} V_f$ for a finite subset $S \subseteq R$.

Definition 10.18. Let $U \subseteq V$ be open. A map $\varphi : U \to \mathbb{C}$ is said to be *regular* on U if for every point $p \in U$, there exists an $f_p \in R$, $p \in V_{f_p} \subseteq U$ an open neighbourhood, such that $\varphi\,|_{V_{f_p}} \in R_{f_p}$. We set

$$\mathcal{O}_V(U) = \{\varphi : U \to \mathbb{C} \mid \varphi \text{ regular on } U\}.$$

Proposition 10.19. *(i)* $\mathcal{O}_V(V) = R$;

(ii) For all $f \in R$ *we have that* $\mathcal{O}_V(V_f) = R_f$.

Theorem 10.20. *The sheaf* $\mathcal{O}_V : U \mapsto \mathcal{O}_V(U)$ *is a sheaf of* $\mathbb{C}$*-algebras on* V*. It is called the structure sheaf on* V*:*
If V *is irreducible, the stalk* $(\mathcal{O}_V)_p$ *is actually isomorphic to* $\mathcal{O}_{V,p}$.

Definition 10.21. Let V_1, V_2 be affine varieties and $U_i \subseteq V_i$ open subsets for $i = 1, 2$. A map $\Phi : U_1 \to U_2$ is a morphism of varieties if

$$\Phi^{\#} : \mathcal{O}_{V_2}(U_2) \to \mathcal{O}_{V_1}(U_1)$$
$$f \mapsto f \circ \Phi$$

is a homomorphism of $\mathbb{C}$-algebras.

Remark 10.22. There is a 1:1-correspondence

$$\left\{ \overset{U_1 \to U_2}{\text{morphism}} \right\} \leftrightarrow \left\{ \overset{\mathcal{O}_{V_2}(U_2) \to \mathcal{O}_{V_1}(U_1)}{\mathbb{C}\text{-algebra homomorphism}} \right\}$$

where U_1 and U_2 are affine varieties. We say that a morphism Φ is an isomorphism if it is bijective and its inverse $\Phi^{-1} : U_2 \to U_1$ is also a morphism.

10.2.3 Abstract varieties

Definition 10.23. Let $\{V_\alpha\}_\alpha$ be a finite collection of affine varieties such that for every α, β, there exist open subsets $V_{\alpha\beta} \subseteq V_\alpha$ and $V_{\beta\alpha} \subseteq V_\beta$ and an isomorphism $g_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ that verify

(i) $g_{\beta\alpha} = (g_{\alpha\beta})^{-1}$;

(ii) $g_{\alpha\beta}(V_{\alpha\beta} \cap V_{\beta\alpha}) = V_{\beta\alpha} \cap V_{\alpha\beta}$;

(iii) $g_{\alpha\gamma} = g_{\beta\gamma} \circ g_{\alpha\beta}$ for every α, β, γ.

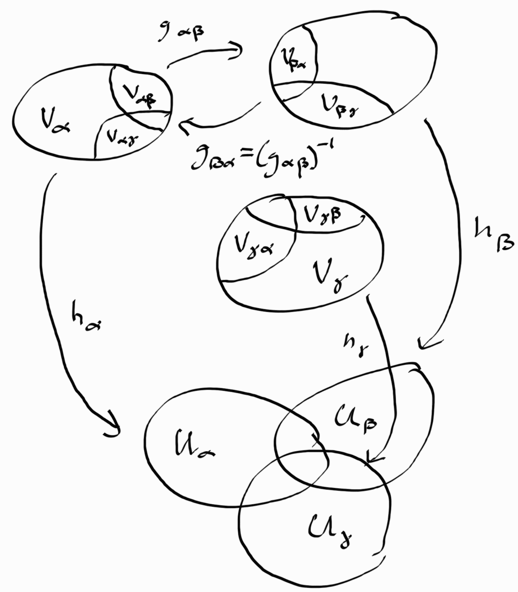
We define

$$Y = \bigsqcup_\alpha V_\alpha / \sim$$

where $a \sim b$ if there exist α, β such that $a \in V_\alpha$, $b \in V_\beta$ and $g_{\alpha\beta}(a) = b$. We introduce

$$h_\alpha : V_\alpha \to U_\alpha := \{[a] \mid a \in V_\alpha\}.$$

This procedure can be illustrated by the following:



The topology on Y is induced by that of the open cover $\{U_\alpha\}_\alpha$. And *abstract variety* is determined by the above data.

Remark 10.24. To show that a topological space X is an abstract variety we need to be able to construct an open cover $X = \bigcup_\alpha U_\alpha$ with $U_\alpha \cong V_\alpha$ affine varieties for all α. We also require that the intersections are not dependent on inside which subset they appear, up to isomorphism, i.e.,

$$(U_1 \cap U_2)_{U_1} \cong (U_1 \cap U_2)_{U_2},$$

where the subscripts denote the intersection as viewed in U_1 and U_2 respectively.

Example 10.25.

(i) An affine variety V is an abstract variety.

(ii) Projective n-space $\mathbb{P}^n$ constitutes an abstract variety;

$$\mathbb{P}^n = \bigcup_i U_i = \bigcup_i \mathbb{P}^n \setminus \mathbb{V}(x_i)$$

where $U_i := \mathbb{P}^n \setminus \mathbb{V}(x_i) \cong \mathbb{C}^n$. Now for i, j,

$$(U_i)_{\frac{x_j}{x_i}} = (U_j)_{\frac{x_i}{x_j}} = U_i \cap U_j.$$

We define

$$g_{ij} : (U_i)_{\frac{x_j}{x_i}} \xrightarrow{\cong} (U_j)_{\frac{x_i}{x_j}}$$

$$g_{ij}^{\#} : \mathbb{C}\left[\frac{x_1}{x_j}, \dots, \frac{x_n}{x_j}\right]_{\frac{x_i}{x_j}} \to \mathbb{C}\left[\frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right]_{\frac{x_j}{x_i}}$$

$$\frac{x_k}{x_j} \mapsto \frac{x_k}{x_i} \Big/ \frac{x_j}{x_i}, \qquad \left(\frac{x_i}{x_j}\right)^{-1} \mapsto \frac{x_j}{x_i}.$$

(iii) Projective varieties are abstract varieties: We have a canonical decomposition $V = \bigcup_i V \cap U_i$ where $V \cap U_i$ is affine for all i.

(iv) We have that $\mathbb{P}^n \times \mathbb{C}^m$ is an abstract variety:
Pick the open cover $\{U_i \times \mathbb{C}^m\}_i$, i.e., so that $U_i \times \mathbb{C}^m \cong \mathbb{C}^{n+m}$.
Note that this variety is neither affine nor projective.

Definition 10.26. Let X be an abstract variety:

(i) A closed subset of X is called a *subvariety*;

(ii) We say that X is *irreducible* if it cannot be written as the union of two proper subvarieties.

Remark 10.27. One can show that an abstract variety X admits a decomposition $X = \bigcup_{\text{finite}} Y_i$ where Y_i are irreducible subvarieties of X, called irreducible components of X.

10.3 Exercises

Exercise 10.1. Consider the polytope $P = (n+1)\Delta_n - (1, ..., 1)$.

(i) Determine the facet presentation of P, show that P is smooth and show that $P^\circ = \mathrm{Conv}(e_0, e_1, ..., e_n)$ where $e_0 = -e_1 - ... - e_n$.

(ii) Determine the facet presentation of P° and show that $\sigma_{e_0} = \mathrm{Cone}(v_1, ..., v_n)$ where $v_i = e_0 + (n+1)e_i$. Hint: you know the vertices of P.

(iii) Show that P° is not smooth for $n \geq 2$.

Exercise 10.2. Let $V = \mathrm{Spec}(R)$ be an affine variety.

(i) Show that every ideal $I \subset R$ can be written in the form $I = \langle f_1, ..., f_s \rangle$, where $f_i \in R$. (This is the Hilbert Basis Theorem in R.)

(ii) Let $W \subset V$ be a subvariety. Show that the complement of W in V can be written as a union of a finite collection of open affine sets of the form V_f.

(iii) Deduce that every open cover of V (in the Zariski topology) has a finite subcover. (This says hat affine varieties are *quasicompact* in the Zariski topology.)

Exercise 10.3. Let X be an irreducible abstract variety.

(i) Let f, g be rational functions on X. Show that $f \sim g$ if $f|_U = g|_U$ for some nonempty open set $U \subset X$ is an equivalence relation.

(ii) Show that the set of equivalence classes of the relation in part (a) is a field.

(iii) Show that if $U \subset X$ is a nonempty open subset of X, then $\mathbb{C}(U) \cong \mathbb{C}(X)$.

Exercise 10.4. In this exercise, we will study the blowup of $\mathbb{C}^n$ at the origin. Write the homogeneous coordinates on $\mathbb{P}^{n-1}$ as $x_0, \dots, x_{n_1}$ and the affine coordinates on $\mathbb{C}^n$ as $y_1, \dots, y_n$. Consider

$$W = \mathrm{Bl}_0(\mathbb{C}^n) = V(x_{i-1}y_j - x_{j-1}y_i \mid 1 \leq i < j \leq n) \subseteq \mathbb{P}^{n-1} \times \mathbb{C}^n.$$

Let U_i, $i = 1, \dots, n$ be the standard affine opens in $\mathbb{P}^{n-1}$:

$$U_{i-1} = \mathbb{P}^{n-1} \setminus V(x_{i-1}).$$

So the $\{U_{i-1} \times \mathbb{C}^n\}_i$ is an open cover of $\mathbb{P}^{n-1} \times \mathbb{C}^n$.

(i) Show that for each i, $W_{i-1} = W \cap (U_{i-1} \times \mathbb{C}^n)$ is isomorphic to

$$\operatorname{Spec}\left(\mathbb{C}\left[\frac{x_0}{x_{i-1}}, \ldots, \frac{x_n}{x_{i-1}}, y_i\right]\right).$$

(ii) Give the gluing data for identifying the subset $W_{i-1} \setminus V(x_{j-1})$ and $W_{j-1} \setminus V(x_{i-1})$.

Chapter 11. Toric varieties from abstract fans

Matthew Dupraz after the talk of Julie Bannwart and Louis Gogniat

Recall that an *abstract variety* is the data of

- a finite collection of affine varieties $\{V_\alpha\}$
- Zariski open subsets $V_{\beta\alpha} \subseteq V_\alpha$ for all α, β
- isomorphisms $g_{\beta\alpha} : V_{\beta\alpha} \to V_{\alpha\beta}$ such that:
  - $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ for all α, β
  - the cocycle condition is satisfied, i.e.

    $$g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$$

    and

    $$g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$$

    on $V_{\beta\alpha} \cap V_{\gamma\alpha}$ for all α, β, γ.

The underlying subspace is the gluing of these affine pieces via the maps $g_{\alpha\beta}$, more precisely,

$$X = \bigsqcup_\alpha V_\alpha / \sim$$

where the equivalence relation $\sim$ is induced by

$$a \sim g_{\beta\alpha}(a) \text{ for all } \alpha, \beta \text{ and } a \in V_\alpha$$

The topology on this space is the quotient topolgy induced from the Zariski topology on the affine pieces. Recall that in this course we consider only affine varieties of finite type over $\mathbb{C}$, so they may be embedded in some $\mathbb{C}^n$, which leads us to the following definition

Definition 11.1. Let X be an abstract variety as above. The *classical topology* on X is the quotient topology obtained by considering the $V_\alpha \subset \mathbb{C}^n$ with the Euclidean topology.

Example 11.2. Consider $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[x, y])$ and $\mathbb{P}^1$ with homogeneous coordinates $[x_0 : x_1]$. The *blowup of* $\mathbb{C}^2$ *at the origin* is the variety

$$V = \mathbb{V}(x_0 y - x_1 x) \subseteq \mathbb{C}^2 \times \mathbb{P}^1.$$

We can cover $\mathbb{C}^2 \times \mathbb{P}^1$ with the affine pieces

$$\mathbb{C}^2 \times U_0 = \mathrm{Spec}(\mathbb{C}[x, y, x_1/x_0])$$

and

$$\mathbb{C}^2 \times U_1 = \mathrm{Spec}(\mathbb{C}[x, y, x_0/x_1]).$$

If we denote $s = x_1/x_0$ and $t = x_0/x_1$, we have that V can be written as the gluing of the two affine pieces

$$V \cap \mathbb{C}^2 \times U_0 = \mathbb{V}(y - sx) \subseteq \mathrm{Spec}(\mathbb{C}[x, y, s])$$

and

$$V \cap \mathbb{C}^2 \times U_1 = \mathbb{V}(ty - x) \subseteq \mathrm{Spec}(\mathbb{C}[x, y, t]).$$

11.1 Morphisms

Definition 11.3. Let $X = \bigcup_\alpha U_\alpha$ and $Y = \bigcup_\beta V_\beta$ be abstract varieties. A map $\varphi : X \to Y$ is a *morphism* if it is Zariski continuous and for all α, β,

$$\varphi|_{U_\alpha \cap \varphi^{-1}(V_\beta)} : U_\alpha \cap \varphi^{-1}(V_\beta) \to V_\beta$$

is a morphism. If $Y = \mathbb{C}$, φ is called a *regular function.*

Definition 11.4. Let X be an abstract variety. The *structure sheaf* $\mathcal{O}_X$ of X is the sheaf given by the data

$$\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \mid f \text{ is regular}\}$$

along with the usual restriction maps.

Definition 11.5. The *local ring at* $p \in X$ is

$$\mathcal{O}_{X,p} = \{f : U \to \mathbb{C} \mid U \text{ an open neighbourhood of } p\}/\sim,$$

where the equivalence relation $\sim$ is given by $(f : U \to \mathbb{C}) \sim (g : V \to \mathbb{C})$ if and only if there exists $p \in W \subseteq U \cap V$ such that $f|_W = g|_W$

Definition 11.6. Let X be an irreducible variety. We define the *function field* to be

$$\mathbb{C}(X) = \{f : U \to \mathbb{C} | f \text{ is regular}, U \neq \emptyset\}/\sim$$

where the equivalence relation $\sim$ is given by $(f : U \to \mathbb{C}) \sim (g : V \to \mathbb{C})$ if and only if $f|_{U\cap V} = g|_{U\cap V}$. The elements of $\mathbb{C}(X)$ are called *rational functions.*

11.2 Normality and smoothness

Definition 11.7. Let X be an irreducible variety. X is called *normal* if for all $p \in X$, $\mathcal{O}_{X,p}$ is integrally closed.

Proposition 11.8. *X is normal if and only if for all α, V_α is normal.*

Proof. For all $p \in X$, there is some α such that $p \in V_\alpha$. We have that $\mathcal{O}_{X,p} \cong \mathcal{O}_{V_\alpha,p}$ as for any $[f : U \to \mathbb{C}] \in \mathcal{O}_{X,p}$, we have that

$$[f : U \to \mathbb{C}] = [f|_{U\cap V_\alpha} : U \cap V_\alpha \to \mathbb{C}],$$

which follows from Definition 11.5, and so this shows that the natural inclusion $\mathcal{O}_{V_\alpha,p} \hookrightarrow \mathcal{O}_{X,p}$ is actually an isomorphism. Exercise 1.4 implies that V_α is normal if and only if $\mathcal{O}_{V_\alpha,p}$ is normal for all $p \in V_\alpha$ and so the statement follows □

Definition 11.9. Let X a variety, then $p \in X$ is a *smooth point* if $\dim T_p(X) = \dim_p X$. Here $T_p(X) = T_p(V_\alpha)$ for some α such that $p \in V_\alpha$ and $\dim_p X = \dim_p V_\alpha$. The variety X is *smooth* if every $p \in X$ is a smooth point. The fact that this is well-defined is shown in Exercise 11.1.

11.3 Products

Definition 11.10. Let $X = \bigcup_\alpha U_\alpha$, $Y = \bigcup_\beta V_\beta$. Define the product $X \times Y$ as the abstract variety with affine pieces given by $\{U_\alpha \times V_\beta\}_{\alpha,\beta}$, where

$$(X \times Y)_{(\alpha,\beta)(\alpha',\beta')} = U_{\alpha\alpha'} \times V_{\beta\beta'}$$

and the glueing maps are given by

$$g_{(\alpha,\beta)(\alpha',\beta')} = g_{\alpha\alpha'} \times g_{\beta\beta'}$$

for all $\alpha, \alpha', \beta, \beta'$.

Proposition 11.11. *The product of two varieties satisfies the universal property of the product, that is for all* $X_1 \xleftarrow{\varphi_1} W \xrightarrow{\varphi_2} X_2$*, there exists a unique morphism* $\varphi : W \to X_1 \times X_2$*, such that the diagram*

$$\begin{array}{ccccc} & & W & & \\ & \swarrow^{\varphi_1} & \downarrow{\scriptstyle \varphi} & \searrow^{\varphi_2} & \\ X_1 & \xleftarrow{\pi_1} & X_1 \times X_2 & \xrightarrow{\pi_2} & X_2 \end{array}$$

commutes.

Example 11.12. As we have seen in Example 11.2, the product $\mathbb{C}^2 \times \mathbb{P}^1$ can be covered by the affine pieces $\widehat{U_0} = \mathbb{C}^2 \times U_0$ and $\widehat{U_1} = \mathbb{C}^2 \times U_1$, where we have

$$\widehat{U}_{10} = \operatorname{Spec}(\mathbb{C}[x, y, s]_s)$$
$$\widehat{U}_{01} = \operatorname{Spec}(\mathbb{C}[x, y, t]_t)$$

and the isomorphism $g_{10} : \widehat{U}_{10} \to \widehat{U}_{01}$ is induced by

$$x \mapsto x, \quad y \mapsto y \quad t \mapsto s^{-1}. \tag{5}$$

11.4 Separatedness

Recall that when X is a topological space, X is Hausdorff if and only if the image of the diagonal map

$$\Delta : X \to X \times X$$
$$x \mapsto (x, x)$$

is closed in $X \times X$ endowed with the product topology.
Separatedness is a property of abstract varieties analogous to that of being Hausdorff for topological spaces.

Definition 11.13. A variety X is *separated* if the image of the diagonal map $\Delta : X \to X \times X$ is Zariski closed in $X \times X$.

In fact the analogy is not vacuous as we have the following theorem.

Theorem 11.14. *A variety is separated if and only if it is Hausdorff when endowed with the classical topology.*

Separatedness is a desirable condition as we have for example the following proposition.

Proposition 11.15. *Suppose X is a separated variety.*

(i) If $f, g : Y \to X$ are two morphisms, then the set

$$\{y \in Y \mid f(y) = g(y)\}$$

is Zariski closed in V.

(ii) If $U, V \subset X$ are open affine subsets, then $U \cap V$ is affine too.

Example 11.16. Any affine variety $V \subseteq \mathbb{C}^n$ is separated. Indeed, $\Delta_V \subseteq V \times V$ is closed because $\Delta_V = (V \times V) \cap \Delta_{\mathbb{C}^n}$, $V \times V$ is closed in $\mathbb{C}^n \times \mathbb{C}^n$ and we have that

$$\Delta_{\mathbb{C}^n} = \mathbb{V}(x_1 - y_1, \dots, x_n - y_n),$$

so being the intersection of two closed subspaces, Δ_V is closed.

We will now give an example of a variety that is not separated.

Example 11.17. Let $U = \mathrm{Spec}(\mathbb{C}[x]) \cong \mathbb{C}$ and $V = \mathrm{Spec}(\mathbb{C}[y]) \cong \mathbb{C}$ glued along $U_x \subseteq U$ and $V_y \subseteq V$ with the map $U_x \to V_y$ induced by

$$\begin{aligned} \mathbb{C}[y]_y &\to \mathbb{C}[x]_x \\ y &\mapsto x \end{aligned}$$

Let X be the resulting variety, X is called the *line with two origins*. We will denote the two origins 0_U and 0_V to distinguish them. This is a standard example for a topological space which is not Hausdorff. To see that X is not separated without using the Theorem 11.14, notice that $X \times X$ is covered by the affine pieces $U \times U$, $U \times V$, $V \times U$ and $V \times V$. The space $X \times X$ may be seen as the plane with doubled axes and four origins. In order for Δ_X to be closed in $X \times X$, it has to be closed in all of those affine pieces, but we have that

$$\Delta_X \cap (U \times V) = \mathbb{V}(x - y) \setminus \{(0_U, 0_V)\},$$

and this is not a Zariski closed subspace of $U \times V \cong \mathbb{C}^2$.

11.5 Fans

Definition 11.18. A fan Σ is a collection of strongly convex rational polyhedral cones $\sigma \subseteq N_\mathbb{R}$ such that

- For all $\sigma \in \Sigma$, if $\tau \preceq \sigma$ then $\tau \in \Sigma$.
- For all $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2 \preceq \sigma_i$ for $i = 1, 2$.

Recall that if $\sigma \subseteq N_\mathbb{R}$ is a strongly convex rational cone, then $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$, where $S_\sigma = \sigma^\vee \cap M$. If $\tau \preceq \sigma$ is a face, then there is some $m \in \sigma^\vee \cap M$ such that $\tau = \sigma \cap H_m$ and $\sigma \subseteq H_m^+$. In this case we get that $S_\tau = S_\sigma + \mathbb{Z}m$ and $U_\tau = (U_\sigma)_{\chi^m}$ as in ([CLS]], Proposition 1.3.16).

Proposition 11.19 (Separation Lemma). *Let $\sigma_1, \sigma_2 \in \Sigma$ and $\tau = \sigma_1 \cap \sigma_2$, then*

$$S_\tau = S_{\sigma_1} + S_{\sigma_2}$$

Proof. We have that $\tau^\vee = (\sigma_1 \cap \sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee$ and hence this implies the inclusion $S_\tau \supseteq S_{\sigma_1} + S_{\sigma_2}$. By (Cox, Lemma 1.2.13) we know that there exists some $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$ such that

$$\sigma_1 \cap H_m = \sigma_2 \cap H_m = \tau.$$

Then from the decomposition $S_\tau = S_{\sigma_1} + \mathbb{Z}m$ we get that for any $p \in S_\tau$, there is some $l \in \mathbb{Z}$ such that $p = q + lm$, but then clearly $p \in S_{\sigma_1} + S_{\sigma_2}$. □

Given a fan Σ in $N_\mathbb{R}$, we may associate to each $\sigma \in \Sigma$ its corresponding affine toric variety U_σ. We can glue these varieties on their intersections as follows. Given $\sigma_1, \sigma_2 \in \Sigma$, and $\tau = \sigma_1 \cap \sigma_2$, we know from above that

$$U_{\sigma_1} \supseteq (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subseteq U_{\sigma_2}$$

So we just take

$$g_{\sigma_2\sigma_1} : (U_{\sigma_1})_{\chi^m} \xrightarrow{\sim} (U_{\sigma_2})_{\chi^{-m}}$$

the idenitity on U_τ. This yields an abstract variety X_Σ.

Theorem 11.20. *For a fan Σ, the associated variety X_Σ is toric, normal and separated.*

Proof. $\boxed{X_\Sigma \text{ is toric}}$ For all $\sigma \in \Sigma$, we have that $\{0\}$ is a face of σ and hence $U_{\{0\}} \subseteq U_\sigma$. We have that

$$T_N := U_{\{0\}} = \mathrm{Spec}(\mathbb{C}[M]) \cong (\mathbb{C}^\times)^n$$

which is a torus. These tori are all identified in X_Σ, so we may see T_N as a torus in X_Σ, which is independent of the chosen σ. To show T_N is dense in X_Σ, if C is the closure of T_N in X_Σ, then for all $\sigma \in \Sigma$, $U_\sigma \cap C$ is closed in U_σ. But since T_N is also the torus of the toric variety U_σ, it is dense in U_σ and so $C \supset U_\sigma$. Since

σ was arbitrary, $C = X_\Sigma$ and hence T_N is dense in X_Σ. This implies that X_Σ is irreducible, as it contains an irreducible torus as a dense subset.
For all σ, T_N acts on U_σ and these actions coincide on intersections, since the glueing map is the identity. So this action extends to all of X_Σ. This action is algebraic since it is so on every affine piece.
$\boxed{X_\Sigma \text{ is normal}}$ Since every cone $\sigma \in \Sigma$ is strongly convex, the affine piece U_σ is normal and so Proposition 11.2 implies that X_Σ is normal.
$\boxed{X_\Sigma \text{ is separated}}$ We want to show that $\Delta : X_\Sigma \to X_\Sigma \times X_\Sigma$ has Zariski closed image, so it suffices to show that for every $\sigma_1, \sigma_2 \in \Sigma$, and $\tau = \sigma_1 \cap \sigma_2$,

$$\Delta|_{U_\tau} : U_\tau \to U_{\sigma_1} \times U_{\sigma_2}$$

has Zariski closed image. This is because $\operatorname{im}\Delta$ is closed in the product only if it's closed in every affine piece covering the product and as we have seen these are exactly the $U_{\sigma_1} \times U_{\sigma_2}$. Furthermore, by noticing that $U_\tau = \Delta^{-1}(U_{\sigma_1} \times U_{\sigma_2})$, we get that $\operatorname{im}\Delta \cap U_{\sigma_1} \times U_{\sigma_2} = \operatorname{im}\Delta|_{U_\tau}$. The map $\Delta|_{U_\tau}$ comes from the $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \Delta^* : \mathbb{C}[S_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma_2}] &\to \mathbb{C}[S_\tau] \\ \chi^m \otimes \chi^n &\mapsto \chi^{m+n} \end{aligned}$$

Now, the separation lemma implies that the map is surjective and so Δ^* induces the isomorphism

$$\mathbb{C}[S_\tau] \cong \mathbb{C}[S_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[S_{\sigma_2}] / \ker \Delta^*,$$

which implies that $U = \mathbb{V}(\ker \Delta^*) \subseteq U_{\sigma_1} \times U_{\sigma_2}$, which is closed in $U_{\sigma_1} \times U_{\sigma_2}$ □

Remark 11.21. We have seen that for every normal affine toric variety X, there exists a strongly convex rational polyhedral cone $\sigma \subset N_\mathbb{R}$, such that $X \cong U_\sigma$. If we write $\langle \sigma \rangle$ to be the smallest fan containing σ (so the fan containg precisely σ and all its faces), then in fact $X_\Sigma \cong X$.

We also have the following result.

Proposition 11.22. *If $P \subseteq M_\mathbb{R}$ is a full-dimensional lattice polytope, then we have that $X_P \cong X_{\Sigma_P}$, where Σ_P is the normal fan of P.*

One may also show that the converse of Theorem 11.20 holds.

Theorem 11.23. *If X is separated normal and toric with torus T_N, then $X \cong X_\Sigma$ for some fan Σ in $N_\mathbb{R}$.*

Example 11.24. Let X be a 1-dimensional separated normal toric variety, then X is isomorphic to either $\mathbb{C}^\times$, $\mathbb{C}$ or $\mathbb{P}^1$.
Indeed, in this case $T_N = \mathbb{C}^\times$, $N = \mathbb{Z}$ and $N_\mathbb{R} = \mathbb{R}$. Then $\tau = \{0\}$, $\sigma_1 = [0, +\infty)$ and $\sigma_2 = (-\infty, 0]$ are the only strongly convex polyhedral cones. Then the only possibilities up to exchanging σ_1 with σ_2 are

- $\Sigma = \{\tau\}$, in which case $X_\Sigma = U_\tau = \operatorname{Spec}(\mathbb{C}[\mathbb{Z}]) \cong \mathbb{C}^\times$.
- $\Sigma = \{\tau, \sigma_1\}$, then $X_\Sigma = U_{\sigma_1} = \operatorname{Spec}(\mathbb{C}[\mathbb{N}]) \cong \operatorname{Spec}(\mathbb{C}[x])) \cong \mathbb{C}$.
- $\Sigma = \{\tau, \sigma_1, \sigma_2\}$, in which case we may take $1 \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap \mathbb{Z} = \mathbb{N}$, and so we get that the affine pieces $U_{\sigma_1} \cong \mathbb{C}$ and $U_{\sigma_2} \cong \mathbb{C}$ glue along the map induced by

$$\begin{aligned} g^*_{\sigma_1\sigma_2} : \mathbb{C}[x]_x &\to \mathbb{C}[x^{-1}]_{x^{-1}} \\ x &\mapsto x \end{aligned}$$

  We see then that $X_\Sigma \cong \mathbb{P}^1$ via the identification $x \mapsto x_0/x_1$, where $[x_0 : x_1]$ are the coordinates in $\mathbb{P}^1$.

Proposition 11.25. *Let Σ_1 be a fan in $(N_1)_\mathbb{R}$ and Σ_2 a fan in $(N_2)_\mathbb{R}$, then $\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 | \sigma_i \in \Sigma_i\}$ is a fan in $(N_1 \times N_2)_\mathbb{R}$ and we have*

$$X_{\Sigma_1 \times \Sigma_2} \cong X_{\Sigma_1} \times X_{\Sigma_2}$$

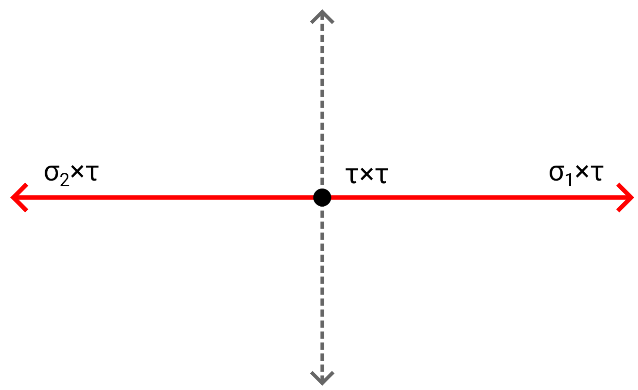


Figure 16: Fan Σ in $\mathbb{R}^2$ corresponding to $\mathbb{P}^1 \times \mathbb{C}^\times$

Example 11.26. Consider the fan $\Sigma_1 = \{\tau, \sigma_1, \sigma_2\}$ from Example 11.24 and the fan $\Sigma_2 = \{\tau\}$. Then the fan $\Sigma = \Sigma_1 \times \Sigma_2$ is a fan in $\mathbb{R}^2$ made of the cones $\tau \times \tau = \{0\}$, $\sigma_1 \times \tau = \mathrm{Cone}((1,0))$ and $\sigma_2 \times \tau = \mathrm{Cone}((-1,0))$. The toric variety corresponding to Σ is $\mathbb{P}^1 \times \mathbb{C}^\times$.

Example 11.27. Consider the cones $\sigma_1 = \mathrm{Cone}(e_1, e_1 + e_2)$ and $\sigma_2 = \mathrm{Cone}(e_2, e_1 + e_2)$. Let $\Sigma = \langle \sigma_1, \sigma_2 \rangle$ be the smallest fan in $\mathbb{R}^2$ containing the two cones. Then X_Σ is isomorphic to the blowup of $\mathbb{C}^2$ at the origin. Indeed, we have

$$S_{\sigma_1} = \mathbb{N}(e_1 - e_2) + \mathbb{N}e_2 \quad \text{and} \quad S_{\sigma_2} = \mathbb{N}(e_2 - e_1) + \mathbb{N}e_1$$

and so we have that

$$U_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[xy^{-1}, y]) \quad \text{and} \quad U_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[yx^{-1}, x]).$$

The glueing map is given by the identity on $U_{\sigma_1 \cap \sigma_2}$, so it's induced from the map

$$\begin{aligned} \mathbb{C}[xy^{-1}, y]_{xy^{-1}} &\to \mathbb{C}[yx^{-1}, x]_{yx^{-1}} \\ xy^{-1} &\mapsto (yx^{-1})^{-1} \\ y &\mapsto (yx^{-1})x. \end{aligned}$$

It should be clear now that this coincides with the description of the blowup in Example 11.2.

11.6 Exercises

Exercise 11.1. Prove the following claims about local rings and smoothness:

(i) If $p \in X$ lies in the intersection of two affine open sets U_α, U_β, then $T_{U_\alpha, p}$ and $T_{U_\beta, p}$ are isomorphic as $\mathbb{C}$-vector spaces.

(ii) The local dimension $\dim_p X$ is a well-defined integer.

(iii) Smoothness is well-defined for abstract varieties.

Exercise 11.2. Prove the following properties of separated varieties (proposition 3.0.18): let X be a separated abstract variety, then:

(i) If $f, g : Y \to X$ are morphisms, then $\{y \in Y \mid f(y) = g(y)\}$ is Zariski closed in Y.

(ii) If U and V are affine open subsets of X, then $U \cap V$ is affine. *Hint*[6].

(iii) Without proving all details, give counterexamples to the two above statements when X is not separated. *Hint*[7].

[6]Show first that $U \cap V$ can be identified with $\Delta(X) \cap (U \times V) \subseteq X \times X$.

[7]Recall the example of non separated variety discussed in class.

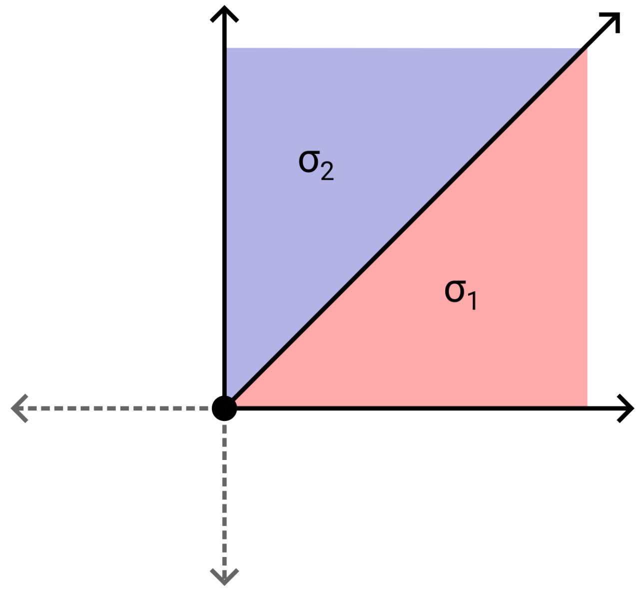


Figure 17: Fan Σ in $\mathbb{R}^2$ corresponding to the blowup of $\mathbb{C}^2$ at the origin

Exercise 11.3. In $N_{\mathbb{R}} = \mathbb{R}^2$, consider the fan Σ with cones $\{0\}$, $\mathrm{Cone}(e_1)$ and $\mathrm{Cone}(-e_1)$. Show that $X_\Sigma \simeq \mathbb{P}^1 \times \mathbb{C}^*$.

Exercise 11.4. Suppose we have fans Σ_1 in $(N_1)_{\mathbb{R}}$ and Σ_2 in $(N_2)_{\mathbb{R}}$. Prove that

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 \mid \sigma_i \in \Sigma_i\}$$

is a fan in $(N_1)_{\mathbb{R}} \times (N_2)_{\mathbb{R}} = (N_1 \times N_2)_{\mathbb{R}}$ and

$$X_{\Sigma_1 \times \Sigma_2} \simeq X_{\Sigma_1} \times X_{\Sigma_2}.$$

Chapter 12. The orbit-cone correspondance

Maxence Coppin after the talk of Emma Billet and Juan Rojas
In this section, we will study the orbits of the action of T_N on a the toric variety X_Σ.
Recall that a 1-parameter subgroup is the data of a homomorphism

$$\begin{array}{cccc} \lambda^u & : & \mathbb{C}^* & \longrightarrow & T_N \\ & & t & \longmapsto & (t^{b_1}, \dots, t^{b_n}) \end{array}$$

where $u = (b_1, \dots, b_n) \in \mathbb{Z}^n = N$.

Example 12.1. Take $X_{\Delta_2} \cong \mathbb{P}^2$. Its torus is $T_N = \{[1 : x : y] \mid x, y \neq 0\}$ and we have

$$T_N \cong (\mathbb{C}^*)^2 \hookrightarrow \mathbb{P}^2.$$

Let $u = (a, b) \in \mathbb{Z}^2$, then $\lambda^u(t) = [1, t^a, t^b]$ for $t \in \mathbb{C}^*$. Now, let's study its limit when t goes to zero in two cases :

- If $u = (a, b) \in \mathbb{Z}^2_{>0}$, we have

$$\lim_{t\to 0} \lambda^u(t) = \lim_{t\to 0} [1 : t^a : t^b] = [1 : 0 : 0].$$

- If $u = (a, a) \in \mathbb{Z}^2_{<0}$, we have

$$\lim_{t\to 0} \lambda^u(t) = \lim_{t\to 0} [1 : t^a : t^a] = \lim_{t\to 0} [t^{-a} : 1 : 1] = [0 : 1 : 1].$$

What are the Orbits of the action $T_N \curvearrowright \mathbb{P}^2$. The action is given on U_1 by

$$((s, t), [1 : x : y]) \longmapsto [1 : sx : ty].$$

We have that

- For $p = [1 : 0 : 0]$, $\mathcal{O}_p = \{[1 : 0 : 0]\}$.
- For $q = [0 : 0 : 1]$, $\mathcal{O}_q = \{[0 : x_1 : x_2] \mid x_1, x_2 \neq 0\} \ni q$.

Doing this for all orbits and finding all possible limits of λ^u, we have a correspondence between cones σ and orbits $\mathcal{O}$ by

$$\sigma \text{ corresponds to } \mathcal{O} \;\Leftrightarrow\; \lim_{t\to 0} \lambda^u(t) \in \mathcal{O},\ \forall u \in \mathrm{Relint}(\sigma)$$

Using the affine toric variety structure of U_σ for a given cone σ, recall that we have 1-1 correspondence between

(i) Maximal ideals of $\mathbb{C}[S_\sigma]$.

(ii) Points $p \in U_\sigma$.

(iii) Semi-group homomorphism $\gamma : S_\sigma \to \mathbb{C}$.

Where $2 \to 3$ is given by $p \mapsto \gamma_p(m) = \lambda^m(p)$.

Definition 12.2. Consider $\gamma : S_\sigma \to \mathbb{C}$ defined by $\gamma(m) = 1$ if $m \in \sigma^\perp \cap M$ and $\gamma(m) = 0$ otherwise. It corresponds to a point $\gamma_\sigma \in U_\sigma$ called the distinguished point of σ.

12.1 Limits of 1-parameter subgroup

Proposition 12.3. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and $u \in N$. Then*

(i) $u \in \sigma$ if and only if $\lim_{t\to 0} \lambda^u(t)$ exists and lies in U_σ.

(ii) If $u \in Relint(\sigma)$, then $\lim_{t\to 0} \lambda^u(t) = \gamma_\sigma$.

Proof. (i) We have the follwong equivalences :

$$\begin{array}{rcl} \lim_{t\to 0} \lambda^u(t) \text{ exists and is in } U_\sigma & \iff & \lim_{t\to 0} \chi^m(\lambda^u(t)) \text{ exists in } \mathbb{C},\ \forall m \in S_\sigma \\ & \iff & t^{\langle m,u\rangle} \text{ exists in } \mathbb{C},\ \forall m \in S_\sigma \\ & \iff & \langle m,u\rangle \geq 0,\ \forall m \in \sigma^\vee \cap M \\ & \iff & u \in (\sigma^\vee)^\vee \end{array}$$

where the first equivalence is proved in the exercise 12.2.

(ii) Suppose that $u \in \mathrm{Relint}(\sigma)$. Then $\langle m,u\rangle > 0$ for all $m \in S_\sigma \setminus \sigma^\perp$ and $\langle m,u\rangle = 0$ for all $m \in S_\sigma \cap \sigma^\perp$ by definition of the relative interior. Now, for $m \in S_\sigma$ we have

$$\begin{array}{rcl} \gamma_u(m) & = & \chi^m(\lim_{t\to 0} \lambda^u(t)) \\ & = & \lim_{t\to 0} \chi^m(\lambda^u(t)) \\ & = & \lim_{t\to 0} t^{\langle m,u\rangle} \\ & = & \begin{cases} 1 & \text{if } m \in \sigma^\perp \cap M \\ 0 & \text{ortherwise} \end{cases} \end{array}$$

□

12.2 Torus Orbits

Lemma 12.4. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. Consider $N_\sigma = \langle \sigma \cap N\rangle \leq N$ and $N(\sigma) = N/N_\sigma$. Then there exists a perfect pairing*

$$\langle \cdot,\cdot\rangle : (\sigma^\perp \cap M) \times N(\sigma) \to \mathbb{Z}$$

Induced by the usual pairing $M \times N \to \mathbb{Z}$. Furthermore, it induces isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \cong T_{N(\sigma)} \cong N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*$$

Proof. Omitted. □

Definition 12.5. Any cone $\sigma \in \Sigma$ corresponds to a distinguished point $\gamma_\sigma \in U_\sigma \subseteq X_\Sigma$. Consider the set

$$\mathcal{O}(\sigma) := T_N \cdot \gamma_\sigma \subseteq X_\Sigma.$$

We know that a point $p \in U_\sigma$ corresponds to a semi-group homomorphism $\gamma : S_\sigma \to \mathbb{C}$. Now, for $t \in T_N$, the point $t \cdot p$ given by the action $T_N \curvearrowright U_\sigma$ corresponds to the semi-group homomorphism γ_t defined by $m \mapsto \chi^m(t) \cdot \gamma(t)$.

Lemma 12.6. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. Then*

$$\mathcal{O}(\sigma) \overset{(1)}{=} \{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \iff m \in \sigma^\perp \cap M\}$$

$$\overset{(2)}{\cong} \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \cong T_{N(\sigma)}$$

Proof. Denote by $\mathcal{O}'$ the set $\{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \iff m \in \sigma^\perp \cap M\}$.

(2) The subspace $\sigma^\perp$ of $M_\mathbb{R}$ is the largest contained in $\sigma^\vee$, hence $\sigma^\perp \cap M \leq S_\sigma$. Let $\gamma \in \mathcal{O}'$, then $\gamma|_{\sigma^\perp \cap M}$ induces $\tilde{\gamma} : \sigma^\perp \cap M \to \mathbb{C}^*$. In the other way, let $\tilde{\gamma} \in \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$, then we can extend it to $\gamma : S_\sigma \to \mathbb{C}$ taking $\gamma(m) = \tilde{\gamma}(m)$ if $m \in \sigma^\perp \cap M$ and $\gamma(m) = 0$ otherwise. These two maps are obviously inverse, thus $\mathcal{O}' \cong \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$.

(1) We have a short exact sequence

$$0 \longrightarrow N_\sigma \longrightarrow N \longrightarrow N(\sigma) \longrightarrow 0$$

By tensoring with $\mathbb{C}^*$, we get a surjection

$$N \otimes_\mathbb{Z} \mathbb{C}^* = T_N \to T_{N(\sigma)} = N(\sigma) \otimes_\mathbb{Z} \mathbb{C}^* \cong \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \cong \mathcal{O}'.$$

The bijections $T_{N(\sigma)} \cong \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \cong \mathcal{O}'$ are compatibles with the T_N-action, hence T_N acts transitively on $\mathcal{O}'$. That is, $\gamma_\sigma \in \mathcal{O}'$ implies that $\mathcal{O}' = T_N \cdot \gamma_\sigma = \mathcal{O}(\sigma)$ by definition of $\mathcal{O}(\sigma)$.

□

Example 12.7. Take the affine toric variety $\mathbb{V}(xy - zw) \subseteq \mathbb{C}^4$. We know that $V = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ where $\sigma^\vee$ is $\mathrm{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3)$. We have that $T_N \cong (\mathbb{C}^*)^3 \hookrightarrow V$ defined by $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$. Let $u = (a, b, c) \in N = \mathbb{Z}^3$, then $\lambda^u(t) = (t^a, t^b, t^c)$ is mapped to $(t^a, t^b, t^c, t^{a+b-c})$ in V. Suppose that $a, b, c \geq 0$ and $a + b \geq c$, then for $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$, we have $\gamma_\sigma = (0, 0, 0, 0)$.

12.3 The Orbit-Cone Correspondence Theorem

Here comes the most important theorem of this section.

Theorem 12.8. *Let Σ be a fan with associated toric variety X_Σ. Then*

(i) We have the 1-1 correspondance :

$$\begin{array}{ccc} \{\sigma \in \Sigma\} & \longleftrightarrow & \{T_N - \text{orbits in } X_\Sigma\} \\ \sigma & \longleftrightarrow & \mathcal{O}(\sigma) \end{array}$$

(ii) let $n = \dim N_\mathbb{R}$. We have $\dim \mathcal{O}(\sigma) = n - \dim \sigma$.

(iii) For $\sigma \in \Sigma$, the affine variety U_σ is the union of orbits

$$U_\sigma = \bigcup_{\tau \preccurlyeq \sigma} \mathcal{O}(\tau).$$

(iv) For $\tau \in \Sigma$, we have

$$\tau \preccurlyeq \sigma \iff \mathcal{O}(\sigma) \subseteq \overline{\mathcal{O}(\tau)} = \bigcup_{\sigma' \succcurlyeq \tau} \mathcal{O}(\sigma')$$

where $\overline{\mathcal{O}(\tau)}$ denotes the closure in both classical topology and Zariski topology.

Proof. (i) Consider the open affine cover $\{U_\sigma\}_{\sigma \in \Sigma}$ of X_Σ which are all T_N-invariants. Furthermore we know that $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2}$. Thus for $\mathcal{O} \subseteq X_\Sigma$ a T_N-orbit, there is a unique $\sigma \in \Sigma$ minimal such that $\mathcal{O} \subseteq U_\sigma$.

We claim that $\mathcal{O} = \mathcal{O}(\sigma)$. Let $\gamma \in \mathcal{O}$ and consider the set $\{m \in S_\sigma \mid \gamma(m) \neq 0$ which is contained in $\sigma^\vee \cap \tau^\perp \cap M$ for $\tau \preccurlyeq \sigma$ (see Exercise 12.3). Then, $\gamma \in U_\tau$ and using the minimality of σ we get $\tau = 0$. Therefore $\gamma \in \mathcal{O}(\sigma)$ and we get the equality by transitivity of the action.

(ii) Immediate from Lemma 12.6, indeed we have that $\mathcal{O}(\sigma) \cong T_{N(\sigma)}$ and the latter have dimension $n - \dim(\sigma)$ as $N(\sigma) = N/N_\sigma$.

(iii) The affine variety U_σ is T_N-invariant, so it is a union of orbits. Suppose that $\tau \preccurlyeq \sigma$, then $\mathcal{O}(\tau) \subseteq U_\tau \subseteq U_\sigma$. Hence τ must be a face of σ, it yields that each orbit composing U_σ corresponds to an $\mathcal{O}(\tau)$ for $\tau \preccurlyeq \sigma$.

(iv) First, we are doing it with the classicle topology.

From Exercise 12.3, we know that $\overline{\mathcal{O}(\tau)}$ is T_N-invariant. Suppose that $\mathcal{O}(\sigma) \subseteq \overline{\mathcal{O}(\tau)}$. Then $\mathcal{O}(\tau) \subseteq U_\sigma$, otherwise their intersection must be empty since we work with the classical topology and this is impossible since both contain $\mathcal{O}(\sigma)$. Thus $\tau \preccurlyeq \sigma$ using the previous part.

Suppose that $\tau \preccurlyeq \sigma$, it is enough to show that $\sigma \cap \overline{\mathcal{O}(\tau)} \neq \emptyset$. Consider γ_τ the distinguished point of τ. Let $u \in \text{Relint}(\sigma)$ and $t \in \mathbb{C}^*$, we define $\gamma(t) = \lambda^u(t) \cdot \gamma_\tau \in U_\tau$ by T_N-invariance of U_τ. Now for any $m \in M$ we have

$$\gamma(t)(m) = \chi^m(\lambda^u(t)) \cdot \gamma_\tau(m) = t^{\langle m,u\rangle} \cdot \gamma_\tau(m).$$

But $\langle m,u\rangle > 0$ if $m \in \sigma^\vee \setminus \sigma^\perp$ and $\langle m,u\rangle = 0$ if $m \in \sigma^\perp$. Thus, by taking the limit in the sense of the classical topology, we have $\gamma(0) = \lim_{t\to 0} \gamma(t)$ it exists and lies in $\mathcal{O}(\sigma)$ by Proposition 12.3, hence $\gamma(0) \in \mathcal{O}(\sigma) \cap \overline{\mathcal{O}(\tau)} \neq \emptyset$. Furthermore we have $\overline{\mathcal{O}(\tau)} = \bigcup_{\sigma' \succcurlyeq \tau} \mathcal{O}(\sigma')$ coming from the classical topology.

Now, for the Zariski topology. Let $\tau' \in \Sigma$, then we have

$$\overline{\mathcal{O}(\tau)} \cap U_{\tau'} = \bigcup_{\tau \preccurlyeq \sigma' \preccurlyeq \tau'} \mathcal{O}(\tau') = \mathbb{V}(I) \subseteq U_{\tau'}$$

where $I = \langle \chi^m \mid m \in \tau^\perp \cap (\tau')^\vee \cap M\rangle \subseteq \mathbb{C}[S_{\tau'}]$. Then $\overline{\mathcal{O}(\tau)}$ is also the closure of $\mathcal{O}(\tau)$ in the sense of the Zariski topology.

□

12.4 Closure of a T_N-orbit

Let Σ be a fan with associated toric variety X_Σ. For a given $\tau \in \Sigma$, we denote $V(\tau) := \overline{\mathcal{O}(\tau)}$ which is a toric variety with torus $T_{N(\tau)}$. Consider also the set

$$\text{Star}(\tau) := \{\overline{\sigma} \in N(\tau)_\mathbb{R} \mid \tau \preccurlyeq \sigma \in \Sigma\}$$

where $\overline{\sigma}$ corresponds to the image of σ via the quotient map $N \to N(\tau)$.

Proposition 12.9. *For any $\tau \in \Sigma$, $V(\tau) \cong X_{\text{Star}(\tau)}$.*

Proof. Omitted. □

Remark 12.10. If P is a full-dimensional lattice polytope, we have a toric variety $X_P \cong X_{\Sigma_P}$ where $\Sigma_P = \{\sigma_Q \mid Q \preccurlyeq P\}$. Thus $V(\sigma_Q) \cong X_Q \cong X_{\Sigma_Q}$ (Q is full-dimensional in its fan).

Proposition 12.11. $V(\sigma_Q) \cong X_Q$.

Proof. Here is a sketch. Take a facet presentation of P as P is full-dimensional

$$P = \{m \in M_\mathbb{R} \mid \langle m, u_F\rangle \geq -a_F,\ \forall F \preccurlyeq P \text{ facet}\}.$$

By doing a translation of the polytope, we may assume that the origin is in Q. If $Q \preccurlyeq F \preccurlyeq P$, we get that $a_F = 0$. Thus $\sigma_Q^\perp = \text{Span}(Q)$. And then, $N(\sigma_Q)$ is the dual to $\text{Span}(Q) \cap M$.
Now, take $V(\sigma_Q) = V(\sigma_{Q,P})$ as before, we have

$$\begin{aligned}
\text{Star}(\sigma_{Q,P}) &= \{\overline{\sigma} \in N(\sigma_{Q,P})_\mathbb{R} \mid \sigma_{Q,P} \preccurlyeq \sigma \in \Sigma_P\} \\
&= \{\overline{\sigma_{Q',P}} \in N(\sigma_{Q,P})_\mathbb{R} \mid \sigma_{Q,P} \preccurlyeq \sigma_{Q',P} \in \Sigma_P\} \\
&= \{\overline{\sigma_{Q',P}} \in N(\sigma_{Q,P})_\mathbb{R} \mid Q' \preccurlyeq Q\} \\
&= \Sigma_Q.
\end{aligned}$$

□

12.5 Exercises

Exercise 12.1. See the first example given in class or equivalently Example 3.2.1 of [CLS]:

(i) Compute the remaining limits of one parameter subgroups of $\mathbb{P}^2$

(ii) Compute the remaining $(\mathbb{C}^*)^2-$orbits in $\mathbb{P}^2$

(iii) Show that the limit point equals the distinguished point γ_σ of the corresponding cone in each case.

Exercise 12.2. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone . This exercise will consider $\lim_{t\to 0} f(t)$ where $f : \mathbb{C}^* \to T_N$ is an arbitrary function.

(i) Prove that $\lim_{t\to 0} f(t)$ exists in U_σ if and only if $\lim_{t\to 0} \chi^m(f(t))$ exists in $\mathbb{C}$ $\forall m \in S_\sigma$.
Hint: Consider a finite set of characters $\mathcal{A}$ such that $S_\sigma = \mathbb{N}\mathcal{A}$.

(ii) When $\lim_{t\to 0} f(t)$ exists in U_σ, prove that the limit is given by the following semigroup homomorphism $S_\sigma \to \mathbb{C}, m \mapsto \lim_{t\to 0} \chi^m(f(t))$

Exercise 12.3. This exercise is concerned with the proof of the theorem of Orbit-Cone correspondence.

(i) Let $\gamma : S_\sigma \to \mathbb{C}$ be a semigroup homomorphism giving a point in U_σ using the bijection seen many times. Prove that $\{m \in S_\sigma \mid \gamma(m) \neq 0\} = \tau \cap M$ for some face $\tau \preccurlyeq \sigma^\vee$.

(ii) Show that $\overline{O(\tau)}$ is invariant under the action of T_N.

(iii) Prove that $\overline{O(\tau)} \cap U_{\sigma'}$ is the variety of the ideal
$I = \langle \chi^m \mid m \in \tau^\perp \cap (\sigma')^\vee \cap M\rangle \subseteq S_{\sigma'}$.

Exercise 12.4. The objective of this exercise is to show that any normal separated toric variety can be obtained from a fan.

(i) Use Theorem 3.1.7 from the book to show that any normal separated toric variety X has an open cover consisting of affine toric varieties $U_i = U_{\sigma_i}$ for some collection of cones σ_i. Show that for all i, j, $U_i \cap U_j$ is also affine. *Hint: Use that X is separated.*

(ii) Show that $U_i \cap U_j$ is the affine toric variety corresponding to the cone $\tau = \sigma_i \cap \sigma_j$.

(iii) If $\tau = \sigma_i \cap \sigma_j$ show that τ is a face of both σ_i and σ_j. *Hint: You may use Exercise 3.2.10 [CLS].*

(iv) Deduce that $X = X_\Sigma$ where Σ is the fan consisting of all the σ_i and all their faces.

Chapter 13. Singular (co)homology

Clotilde Freydt after the talk of Joel Hakavuori, Isak Sundelius

This section is based on the book Algebraic Topology by Allen Hatcher.

Let X be a topological space. A singular n-simplex in X is a continuous map σ from the standard n-simplex Δ^n to X. The singular n-chain group $\Delta_n(X)$ is the free abelian group generated by the singular n- simplices in X. Its elements are called singular n-chains in X. The inclusion $\Delta^{n-1} \hookrightarrow \Delta^n$ induces the following sequence called chain complex:

$$... \to \Delta_{n+1}(X) \xrightarrow{d_{n+1}} \Delta_n(X) \xrightarrow{d_n} \Delta_{n-1}(X) \to ...$$

such that $d_{n+1} \circ d_n = 0$
And the n-th homology group is defined as

$$H_n(X;\mathbb{Z}) := ker(d_n)/Im(d_{n+1})$$

The dualization of the chain complex induces the sequence

$$... \leftarrow \Delta^{n+1}(X) \xleftarrow{d^{n+1}} \Delta^n(X) \xleftarrow{d^n} \Delta^{n-1}(X) \leftarrow ...$$

the n-th cohomology group is defined as

$$H^n(X;\mathbb{Z}) := ker(d^n)/Im(d^{n-1})$$

We say that $\varphi \in \Delta^n(X)$ has compact support if $supp(\varphi)$ is compact in X.
We define the subcomplex

$$\Delta^n_c(X) := \{\varphi \in \Delta^n(X) : supp(\varphi) \text{is compact in } X\}$$

And the associated cohomology group

$$H^n_c(X;\mathbb{Z}) := ker(d^n_c)/Im(d^{n-1}_c)$$

We recall the three following results:

- (Künneth formula) Let X,Y topological space, the following sequence

  $$0 \to \bigoplus_{p+q=n} H^p(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H^q(Y;\mathbb{Z}) \to H^n(X \times Y;\mathbb{Z}) \to \bigoplus_{p+q=n-1} Tor^1_{\mathbb{Z}}(H^p(X;\mathbb{Z}), H^q(Y;\mathbb{Z})) \to 0$$

  is a short exact sequence.

- (Poincaré duality) Let R be a ring and M be a closed orientable n-manifold, the following holds:

  $$H^{n-k}(M;R) \cong H_k(M;R)$$

  for all $0 \le k \le n$.

- (Relative cohomology groups) Let $A \subseteq X$ be a subspace. The n-th relative cohomology group is defined as follows:

  $$H^n(X, A) := H^n(X)/H^n(A)$$

13.1 Spectral sequences

Definition 13.1. A (cohomology) spectral sequence is a collection of abelian groups $E^{p,q}$ and homomorphisms $d_r^{p,q}$ with the following structure and properties:

(i) The groups are $E_r^{p,q}$ are indexed by integers p,q,r. Fixing r, we obtain one sheet of the spectral sequence, which is visualized as a diagram of groups indexed by integer lattice point in the plane.

(ii) In the r*th* sheet, there are homomorphisms

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all p,q,r: In other words, the r*th* sheet splits up into a collection of cochain complexes in which the differentials are all mappings of bidegree $(r, 1-r)$ for the indexing by p, q.

(iii) The (r+1)*st* sheet is the cohomology of $(E_r^{p,q}, d_r^{p,q})$ *i.e.*

$$E_r^{p,q} = ker(d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1})/im(d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q})$$

Remarks: - We will only work with *first quadrant* spectral sequences, for which $E_r^{p,q} = 0$ when $p < 0$ or $q < 0$. Thus, in each sheet, the nonvanishing terms lie in the quadrant where $p, q \geq 0$.

- For a first quadrant spectral sequence, the differentials mapping to $E_r^{p,q}$ and from $E_r^{p,q}$ for fixed p, q vanish when r is sufficiently large. It follows that for each p, q, there exists some r such that

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = ...$$

This common value is defined to be $E_\infty^{p,q}$.

Definition 13.2. A first quadrant spectral sequence $(E_r^{p,q}, d_r^{p,q})$ converges to a sequence of abelian groups $H^k, k \geq 0$ if there is a filtration

$$0 = F^{k+1}H^k \subseteq F^kH^k \subseteq F^{k-1}H^k \subseteq ... \subseteq F^1H^k \subseteq F^0H^k = H^k$$

of H^k by subgroups such that

$$E_\infty^{p,q} \simeq F^pH^{p+q}/F^{p+1}H^{p+q}$$

For an E_1 or E_2 spectral sequence we write this as

$$E_1^{p,q} \Rightarrow H^{p,q} \text{ or } E_2^{p,q} \Rightarrow H^{p,q}$$

respectively.

Definition 13.3. We say that a spectral sequence degenerates at the E_r sheet if the differential $d_s^{p,q} = 0$ for all p, q and all $s \geq r$

Note that degeneration at E_r implies that $E_\infty^{p,q} \simeq E_r^{p,q}$ for all p, q so we have a strong form of convergence in this case.

13.2 Singular Cohomology of Toric Varieties

In this section we focus on the singular cohomology groups of a toric variety X_Σ. We will describe them using, firstly, the singular cohomology of the toric varieties U_σ for a cone $\sigma \in \Sigma_{max}$. Secondly, using the singular cohomology of the torus orbits $O(\sigma)$ for $\sigma \in \Sigma$. The spectral sequences will establish the connection between these two approaches.

Proposition 13.4. *Let $\sigma \subseteq N_\mathbb{R}$ be a cone. Then*

$$H^\bullet(U_\sigma; \mathbb{Z}) \overset{(i)}{\simeq} H^\bullet(T_{N(\sigma)}; \mathbb{Z}) \overset{(ii)}{\simeq} \wedge^\bullet \mathbb{Z}^n$$

Proof. First, note that $O(\sigma) \simeq T_{N(\sigma)}$ is a deformation retract of U_σ (see Proposition 12.1.9 in Toric Varieties by Cox, Little and Schenck), so the first isomorphism (i) follows by excision.
We compute the cohomology group of the torus. The torus $(\mathbb{C}^*)^n$ contains the real torus $(\mathbb{S}^1)^n$ as a deformation retract via $(t_1, ..., t_n) \mapsto (t_1/|t_1|, ..., t_n/|t_n|)$. Hence

$$H^\bullet((\mathbb{C}^*)^n; \mathbb{Z}) \;\simeq\; H^\bullet((\mathbb{S}^1)^n; \mathbb{Z}) \;\simeq\; \wedge^\bullet \mathbb{Z}^n$$

and (ii) follows from the duality of $N \to N(\sigma)$ and $M(\sigma) \subseteq M$ □

Example 13.5. Let $\sigma = Cone(e_1)$ then the dual $\sigma^\vee = Cone(e_1, -e_1, e_2)$. $U_\sigma = Spec(\mathbb{C}[x, x^{-1}, y]) \cong \mathbb{C}^* \times \mathbb{C}$ that deformation retracts to $\mathbb{C}^*$. We have that $N(\sigma) = \mathbb{Z}$, so that

$$H^\bullet(U_\sigma; \mathbb{Z}) \simeq H^\bullet(T_{N(\sigma)}; \mathbb{Z})$$

13.2.1 The spectral sequence of a filtered topological space

Let X be a topological space and consider the filtration

$$\varnothing := X_{-1} \subseteq X_0 \subseteq X_1 ... \subseteq X_n := X$$

Theorem 13.6. *Let X as above and R a ring. Then there is a spectral sequence $(E_r^{p,q}, d_r^{p,q})$ with $E_1^{p,q} = H^{p+q}(X_p, X_{p-1}; R) \Rightarrow H^{p+q}(X; R)$ where the filtration is given by*

$$F^p H^{p+q}(X; R) = ker(H^{p+q}(X; R) \to H^{p+q}(X_p; R)),$$

the kernel of the map induced by the inclusion $X_p \hookrightarrow X$.

Remark: Note that this theorem still holds for cohomology with compact support.

13.3 A family of complexes

The aim is now to compute the cohomology groups of a toric variety.
We consider a fan Σ and the associated toric variety X_Σ. We begin by discussing a notion of orientation for a pair of cones $\sigma \prec \tau$ with $dim\tau = dim\sigma + 1$. First for each cone σ we may pick an orientation of the linear subspace $(N_\sigma)_\mathbb{R}$ by choosing a basis. Now let $\nu \in \tau$ be any vector not contained in σ. Then ν together with a basis of $(N_\sigma)_\mathbb{R}$ together with ν form a basis of $(N_\tau)_\mathbb{R}$ and defines an orientation.

Definition 13.7. The **orientation coefficient** related to the cones σ, τ as above is defined as follows:

$$c_{\sigma,\tau} = \begin{cases} 1 \text{ if the orientation of } \tau \text{ determined by } \sigma \text{ agrees with the chosen one} \\ -1 \text{ if not} \\ 0 \text{ if } \sigma \text{ is not a face of } \tau \end{cases}$$

Fix an integer q, $0 \leq q \leq n$ and consider the abelian groups and maps:

$$C^\bullet(\Sigma, \wedge^q) = \{C^p(\Sigma, \wedge^q), \delta^p) | p \in \mathbb{Z}\}$$

defined as follows: first we take

$$C^p(\Sigma, \wedge^q) = \bigoplus_{\tau \in \Sigma(n-p)} \wedge^q M(\tau)$$

where $M(\tau) = \tau^\perp \cap M$ as usual. This is a free abelian group with

$$rank C^p(\Sigma, \wedge^q) = \binom{p}{q} |\Sigma(n-p)|$$

Then $\delta^p : C^p(\Sigma, \wedge^q) \to C^{p+1}(\Sigma, \wedge^q)$ is the map defined on the components corresponding to the cones (σ, τ) in the two direct sums as

$$c_{\sigma,\tau} i^q_{\sigma,\tau}$$

where $c_{\sigma,\tau}$ are the orientation coefficient and $i^q_{\sigma,\tau} : \wedge^q M(\tau) \to \wedge^q M(\sigma)$ is induced by the inclusion $\tau^\perp \subseteq \sigma^\perp$: In other words the component of δ^p in the summand for the cone σ in $C^{p+1}(\Sigma, \wedge^q)$ is given by

$$\sum_{\sigma \prec \tau} c_{\sigma,\tau} i^q_{\sigma,\tau}$$

Lemma 13.8. *$C^\bullet(\Sigma, \wedge^q)$ is a cochain complex i.e. $\delta^{p+1} \circ \delta^p = 0$ for all p.*

Proof. Exercise □

Example 13.9. Consider the fan defining $\mathbb{P}^2$: Denote ρ_i =Cone(u_i) where $u_0 = -e_1 - e_2$ and $u_i = e_i$ for $i = 1, 2$.
The following diagram represents the complex $C^\bullet(\Sigma, \wedge^q)$ for $q = 0, 1, 2$:

$$q = 2 : \quad 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow ...$$
$$q = 1 : \quad 0 \longrightarrow 0 \longrightarrow \mathbb{Z}^3 \xrightarrow{C} \mathbb{Z}^2 \longrightarrow 0 \longrightarrow ...$$
$$q = 0 : \quad 0 \longrightarrow \mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^3 \xrightarrow{B} \mathbb{Z} \longrightarrow 0 \longrightarrow ...$$

Recall that

$$U_\sigma = \bigcup_{\tau \prec \sigma} O(\sigma), \quad X_\Sigma = \bigcup_{\sigma \subseteq \Sigma} U_\sigma$$

And the closure

$$\overline{O(\tau)} = \bigcup_{\tau \prec \sigma} O(\sigma)$$

We define a filtration of X_Σ by $X_p = \bigcup_{\sigma \subseteq \Sigma(n-p)} V(\sigma) = \bigsqcup_{\tau \subseteq \Sigma(l), l \geq n-p} O(\tau)$.

When working with general Σ, where X_Σ may not be compact we will consider cohomology with compact support (cf first section): We have that $E_1^{p,q} = H_c^{p+q}(X_p, X_{p-1}; \mathbb{Z}) \Rightarrow H_c^{p+q}(X_\Sigma; \mathbb{Z})$.

Proposition 13.10. *For $p, q \geq 0$, we have*

$$E_1^{p,q} \simeq \bigoplus_{\tau \in \Sigma(n-p)} \wedge^q M(\tau) = C^p(\Sigma, \wedge^q)$$

Moreover the differentials $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ agree with the coboundary maps in the complex $C^\bullet(\Sigma, \wedge^q)$ so that

$$E_2^{p,q} = H^p(\Sigma, \wedge^q)$$

Proof. By the excision property of cohomology with compact supports, we have

$$E_1^{p,q} \simeq \bigoplus_{\tau \in \Sigma(n-p)} H^{p+q}(O(\tau), \mathbb{Z}))$$

Furthermore the homeomorphism $O(\tau) \cong \mathbb{R}^p_{>0} \times S_{N(\tau)}$ and the Künneth formula imply that

$$H_c^{p+q}(O(\tau), \mathbb{Z})) \simeq \bigoplus_{k+l=p+q} H_c^k(\mathbb{R}^p_{>0}, \mathbb{Z}) \otimes_{\mathbb{Z}} H_c^l(S_{N(\tau)}, \mathbb{Z})$$

By the Poincaré duality $H_c^k(\mathbb{R}^p_{>0}, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } k = p \\ 0 \text{ otherwise} \end{cases}$
By Proposition 3.1, for each cone τ of dimension $n - p$,

$$H_c^{p+q}(O(\tau), \mathbb{Z})) \simeq H^q(S_{N(\tau)}, \mathbb{Z})) \simeq \wedge^q M(\tau)$$

Hence $E_1^{p,q} = C^p(\Sigma, \wedge^q)$ as desired.
The second part of the proof is left as an exercise. □

Example 13.11. We compute the E_2 sheet of the spectral sequence arising from the fan for $\mathbb{P}^2$. By a direct computation the $q = 0$ row is $E_2^{0,0} = \mathbb{Z}$, $E_2^{1,0} = 0$, $E_2^{2,0} = 0$. On the second row the kernel of C is 1−dimensional and the image of C is $\mathbb{Z}^2$. Hence $E_2^{1,1} \simeq \mathbb{Z}$, and $E_2^{2,1} = 0$. Finally $E_2^{2,2} = \mathbb{Z}$. Hence the E_2 sheet of the spectral sequence is just:

0	0	$E^{2,2} \simeq \mathbb{Z}$
0	$E^{1,1} \simeq \mathbb{Z}$	0
$E^{0,0} \simeq \mathbb{Z}$	0	0

13.4 Rational coefficients

To avoid torsion in cohomology, we look at coefficient in $\mathbb{Q}$ instead of $\mathbb{Z}$.
By the same argument as in Proposition 3.8

$$E_1^{p,q} \simeq \bigoplus_{\tau \in \Sigma(n-p)} \wedge^q M(\tau)_{\mathbb{Q}} \tag{6}$$

Proposition 13.12. *The spectral sequence $E_1^{p,q} = H_c^{p+q}(X_p, X_{p-1}; \mathbb{Q}) \Rightarrow H_c^{p+q}(X_\Sigma; \mathbb{Q})$ degenerates at E_2*

Proof. We show that $d_r^{p,q} = 0$ for all $r \geq 2$ and all (p, q), so that $E_2^{p,q} = E_\infty^{p,q}$. For any positive integer l the multiplication map

$$\overline{\varphi}_l : N \to N$$
$$a \mapsto l \cdot a$$

is compatible with Σ so there is a corresponding toric morphism $\varphi_l : X_\Sigma \to X_\Sigma$ whose restriction to $T_N \subseteq X_\Sigma$ is the group homomorphism

$$\varphi_l|_{T_N}(t_1, ..., t_n) \mapsto (t_1^l, ..., t_n^l)$$

and similarly on each torus orbit. Because φ_l respects the orbit decomposition of X_Σ, it respects the filtration of section 3.1 and induces homomorphisms

$$\varphi_l^* : E_r^{p,q} \to E_r^{p,q}$$

for each r. These commute with the differentials since the spectral sequence is functorial with respect to maps that preseves the filtration. One can use that $E_1^{p,q} \simeq \bigoplus_{\tau \in \Sigma(n-p)} H_c^{p+q}(O(\tau), \mathbb{Z}))$ to show that φ_l^* acts on $E_1^{p,q}$ by multiplication by l^q: Then the same holds for all r since $E_{r+1}^{p,q}$ is a quotient of subspaces of $E_r^{p,q}$. Let $\beta \in E_r^{p,q}$, for $r \geq 2$. Since $d_r^{p,q}(\beta) \in E_r^{p+r,q-r+1}$, we have

$$\begin{aligned} l^{q-r+1} d_r^{p,q}(\beta) \\ &= \varphi_l^*(d_r^{p,q}(\beta)) \\ &= d_r^{p,q}(\varphi_l^*(\beta)) \\ &= d_r^{p,q}(l^q \beta) \\ &= l^q d_r^{p,q}(\beta) \end{aligned}$$

Since we use coefficients in $\mathbb{Q}$, this implies that $d_r^{p,q}(\beta) = 0$ for all β. □

One has the following result for complete simplicial toric varieties:

Theorem 13.13. *If X_Σ is complete and simplicial, then $E^2_{p,q} = 0$ when $p \neq q$ in the spectral sequence 13.12. In particular,*

(i) $H^{2k+1}(X_\Sigma; \mathbb{Q}) = 0$ for all k.

(ii) $H^{2k}(X_\Sigma; \mathbb{Q}) \simeq E^2_{k,k}$ for all k.

Proof. See Theorem 12.3.11 in Toric Varieties by Cox, Little and Schenck □

There are interesting combinatorial consequences of this theorem on relations between the numbers of cones of various dimensions in simplicial fans and the Betti numbers of the corresponding toric varieties.

Theorem 13.14. *Let Σ be a complete simplicial fan in $N_\mathbb{R} \simeq \mathbb{R}^n$. Then the Betti numbers of X_Σ are given by*

$$b_{2k}(X_\Sigma) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} |\Sigma(n-i)|$$

and satisfy

$$b_{2k}(X_\Sigma) = b_{2n-2k}(X_\Sigma)$$

Proof. The $E_2^{p,q}$ terms of the spectral sequence $E_1^{p,q} = H_c^{p+q}(X_p, X_{p-1}; \mathbb{Q}) \Rightarrow H_c^{p+q}(X_\Sigma; \mathbb{Q})$ are the cohomology of the $E_1^{p,q}$ terms, and we also have $E_1^{p,q} = 0$ for $p < q$ by.
Since $E_2^{p,q} = 0$ unless $p = q$ by theorem 3.10, it follows that

$$0 \to E_2^{k,k} \to E_1^{k,k} \to E_1^{k+1,k} \to ...$$

is an exact sequence. Hence

$$b_{2k}(X_\Sigma) = dim E_2^{k,k} = \sum_{i=k}^{n} (-1)^{i-k} dim E_1^{i,k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} |\Sigma(n-i)|$$

where the last equality holds by 6.
The second assertion follows from Poincaré duality. □

13.5 Exercises

Exercise 13.1. Show that the family of complexes $(C^\bullet(\Sigma, \Lambda), \delta)$ constructed during the lectures forms a chain complex, i.e., $\delta^p \circ \delta^{p-1} = 0$. See Lemma 12.3.3 [CLS] for hints.

Exercise 13.2.

(i) Construct the family of complexes $(C^\bullet(\Sigma, \Lambda), \delta)$ for the fan of $\mathbb{P}^2$ (generated by e_1, e_2 and $-e_1 - e_2$ in $\mathbb{Z}^2$).

(ii) Use this to compute the cohomology of $\mathbb{P}^2$.

Exercise 13.3. Let $\Sigma \subseteq N \cong \mathbb{Z}^n$ be a fan. Consider the multiplication map $\bar{\varphi}_\ell : N \to N$, $\bar{\varphi}_\ell : a \mapsto \ell \cdot a$ for $\ell > 0$. The map $\bar{\varphi}_\ell$ is compatible with the fan Σ, so there is a corresponding toric morphism $\varphi_\ell : X_\Sigma \to X_\Sigma$ (Theorem 3.3.4). Show that φ_ℓ restricted to the torus of X_Σ acts by $\varphi_\ell\,|_{T_{X_\Sigma}} : (t_1, \dots, t_n) \mapsto (t_1^\ell, \dots, t_n^\ell)$. Deduce that for any $\tau \in \Sigma(n-p)$ the induced map φ_ℓ^* on $H_c^q(\mathcal{O}(\tau), \mathbb{Q})$ is multiplication by ℓ^q.

Exercise 13.4.

(i) Suppose $E_1^{p,q} \Rightarrow H^{p+q}$ is a first-quadrant spectral sequence with the property that $E_2^{p,q} = 0$ for $p \neq q$. Show that $E_2^{k,k} \cong H^{2k}$.

(ii) Consider the spectral sequence (with $\mathbb{Z}$-coefficients) $E_1^{p,q} \Rightarrow H_c^{p+q}(X_\Sigma, \mathbb{Z})$ associated to the orbit filtration of X_Σ. Define

$$\chi(E_r) := \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \text{rank } E_r^{p,q}.$$

Show that $\chi(E_r) = \chi(E_{r+1})$ for $r \geq 1$.

(iii) The Euler characteristic χ of X_Σ is given by

$$\chi(X_\Sigma) := \sum_{i=0}^{2n} \dim H_c^k(X_\Sigma, \mathbb{Z}).$$

Show that $\chi(X_\Sigma) = \chi(E_\infty)$ and that $\chi(X_\Sigma) = |\Sigma(n)|$, where n is the rank of $N \supseteq \Sigma$. Here Σ is not necessarily complete.

Note: this shows that even though we necessarily cannot find all the individual Betti numbers from the combinatorics of Σ when Σ is not a complete simplicial fan, we can still compute the Euler characteristic of X_Σ from the structure of Σ.

Chapter 14. The McMullen conjecture

Elsa Maneval after the talk of Zichen Gao and Matthew Dupraz

14.1 The statement of McMullen's condition

We fix the following set of data :

- $\mathcal{P}$ is a convex polytope of dimension d
- $f_i = \#\{\text{faces of } \mathcal{P} \text{ of dimension i}\}$
- $\vec{f} = (f_0, \ldots, f_d)$ is called the f**-vector** of $\mathcal{P}$.
- We define
$$h_i = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1}$$
where $f_{-1} = 1$.
- $\vec{h} = (h_0, \ldots, h_d)$ is called the h**-vector** of $\mathcal{P}$.

Proposition 14.1 (Dehn-Sommerville equations)**.** *For any simplicial polytope $\mathcal{P}$ of dimension d,*

$$\forall\ 0 \leq i \leq d, \qquad h_i = h_{d-i}$$

Remark 14.2. When $i = 0$, $h_0 = h_d$ computes the Euler-Poincaré characteristic of the boundary of the simplicial polytope :

$$\chi(\partial\mathcal{P}) = 1 - (-1)^d$$

We now turn to the definition of M-vectors. For k, i natural numbers, there exists a unique decomposition of the following type
$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}$$
where $1 \leq j \leq n_j < \cdots < n_{i-1} < n_i$.
We can define
$$k^{\langle i \rangle} := \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}$$
and $0^{\langle i \rangle} := 0$.

Definition 14.3 (M-vector)**.** Let $\vec{k} = (k_0, \ldots, k_d)$. $\vec{k}$ is an M-vector if $k_0 = 1$ and for all $1 \leq i < d$,
$$k_{i+1} \leq k_i^{\langle i \rangle}$$

Theorem 14.4 (McMullen's condition)**.** *The following are equivalent :*

(i) There exists a simplicial polytope $\mathcal{P}$ with f-vector $\vec{f}$

(ii) $(h_0, h_1 - h_0, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ is an M-vector and for all $0 \leq i \leq d$, $h_i = h_{d-i}$.

McMullen stated the conjecture in 1971. The necessity was proved by Stanley in 1979 and the suffisance was proved by Lee and Billera in 1981. The proof of suffisance was done by constructing a polytope. In this lecture we focus on the proof of necessity which can be done using toric geometry.
Fix a simplicial polytope $\mathcal{P}$. Dehn-Sommerville equations implies $h_i = h_{d-i}$. It suffices to prove that $(h_0, h_1 - h_0, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ is a M-vector to prove the necessity of McMullen's condition. We will sketch the argument, which uses Macaulay criterion.

14.2 Macaulay's criterion for M-vectors

The fact that $(h_0, h_1 - h_0, \dots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ is a M-vector can be deduced from the existence of a certain algebra. We do not prove this result here.

Theorem 14.5 (Macaulay, 1926)**.** *The following are equivalent :*

(i) $(k_0, \dots, k_d)$ *is a M-vector*

(ii) There exists a graded commutative algebra $R = R_0 \oplus \dots R_d$ *over the field* $K = R_0$ *which is generated by* R_1 *and such that* $H(R, n) = dim_K R_n = k_n$*.*

Remark 14.6. $H(R, n)$ is called the Hilbert function.

We fix a simplicial polytope $\mathcal{P}$ of dimension d, f-vector $\vec{f}$ and h-vector $\vec{h}$. Without lost of generality we can embed $\mathcal{P}$ in $\mathbb{R}^d$ so that it is full dimensional. We can also assume that $\mathcal{P}$ is rational and $0 \in \text{int}(\mathcal{P})$. Let $\alpha \preceq \mathcal{P}$. $\sigma_\alpha := \text{cone}(\alpha)$. Define the complete simplicial fan $\Sigma := \{\sigma_\alpha \ : \ \alpha \preceq \mathcal{P}\}$.

Recall from last lecture that the Betti numbers of X_Σ verifies :

$$b_{2k+1}(X_\Sigma) = 0 \tag{7}$$

$$b_{2k}(X_\Sigma) = \sum_{i=k}^{d} \binom{i}{k} (-1)^{i-k} |\Sigma(d-i)| \tag{8}$$

$$b_{2d-2k}(X_\Sigma) = b_{2k}(X_\Sigma) \tag{9}$$

Remark 14.7. The following fact indicate that $H^*(X_\Sigma; \mathbb{Q})$ could be useful to build an algebra R satisfying Macaulay criterion :

$$b_{2k}(X_\Sigma) = h_k$$

Indeed, $\dim(\sigma_\alpha) = \dim(\alpha) + 1$ so that $|\Sigma(d-i)| = f_{d-i-1}$, and then it suffices to change by $i \to d - i$ and $k \to d - k$ in equation (8).

14.3 Cup-product

We use the cup-product to give $H^*(X_\Sigma; \mathbb{Q})$ an algebra structure.

Definition 14.8 (cup-product)**.** Let R be a ring, X a topological space. Let $\varphi \in C^k(X; R)$, $\psi \in C^l(X; R)$. Let $\sigma \ : \ \Delta^{k+l} \to X$,

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma_{|[v_0, \dots v_k]}) \cdot \psi(\sigma_{|[v_k, \dots v_{k+l}]})$$

defines a $(k+l)$-cochain $\varphi \smile \psi \in C^{k+l}(X; R)$.
We are implicitly using inclusions of faces $\Delta^k \hookrightarrow \Delta^{k+l}$, $\Delta^l \hookrightarrow \Delta^{k+l}$ and the map of cochain complexes induced by the diagonal morphism $X \to X \times X$

Lemma 14.9. *The cup-product has the following compatibility with the boundary map :* $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$*.*

Thus the cup-product induces a map on cohomology. We still call it cup-product.

$$H^k(X; R) \times H^l(X; R) \xrightarrow{\smile} H^{k+l}(X; R)$$

For $A, B \subset X$ it also induces a map in relative cohomology :

$$H^k(X, A; R) \times H^l(X, B; R) \xrightarrow{\smile} H^{k+l}(X, A \cup B; R)$$

Proposition 14.10. *Let* X*,* Y *be topological spaces,* R *a ring.*

(i) If $f : X \to Y$ continuous map, the induced map

$$f^* : H^*(Y;R) \to H^*(X;R)$$

is a ring morphism, i.e. $f^*(\varphi \smile \psi) = f^*(\varphi) \smile f^*(\psi)$.

(ii) If R is commutative, the cup-product is skew-commutative :

$$\varphi \smile \psi = (-1)^{k+l} \psi \smile \varphi$$

(iii) If R has a unit 1_R, the 0-cochain

$$\begin{aligned} \epsilon : C_0(X) &\longrightarrow R \\ \sum_i a_i x_i &\longmapsto \sum_i a_i \end{aligned}$$

is the identity for the cup-product operation.

Thus, if R is commutative, the cup-product makes $H^*(X;R)$ a graded ring.

Example 14.11. The cohomology of a complex projective space is the following graded ring :

$$\begin{aligned} H^*(\mathbb{CP}^n;\mathbb{Q}) &\cong \mathbb{Q}[u]/u^{n+1} \\ H^*(\mathbb{CP}^\infty;\mathbb{Q}) &\cong \mathbb{Q}[u] \end{aligned}$$

where $u \in H^2(\mathbb{CP}^n;\mathbb{Q})$.

Remark 14.12. In the case of $H^*(X_\Sigma;\mathbb{Q})$, there is no odd-degree cohomology so the ring is commutative. Define

$$A_i := H^{2i}(X_\Sigma;\mathbb{Q})$$

then

$$A := \oplus_{i=0}^d A_i$$

is a graded commutative ring and $H(A,n) = h_n$.

14.4 The Hard Lefschetz theorem

We use Hard Lefschetz theorem and the algebra A to define an algebra R satisfying Macaulay criterion for $(h_0, h_1 - h_0, \dots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$. The Hard Lefschetz theorem in intersection cohomology is a difficult result that goes beyond the scope of this course.

Theorem 14.13 (Hard Lefschetz)**.** *Let X be a normal projective variety of dimension d. There exists a canonical Chern class $\omega \in H^2(X;\mathbb{Q})$ inducing a map in intersection cohomology :*

$$IH^j(X;\mathbb{Q}) \xrightarrow{- \smile \omega} IH^{j+2}(X;\mathbb{Q})$$

such that for all $0 \le i \le d$, its composition i times is an isomorphism

$$IH^{d-i}(X;\mathbb{Q}) \xrightarrow{\sim} IH^{d+i}(X;\mathbb{Q})$$

In our case Σ is complete and simplicial so X_Σ is a projective orbifold. It implies that singular cohomology coincides with intersection cohomology :

$$H^i(X_\Sigma;\mathbb{Q}) \cong IH^i(X;\mathbb{Q})$$

Now, for $0 \le i \le \lfloor \frac{d}{2} \rfloor$ we have an isomorphism $\omega^{d-2i} : A^i \xrightarrow{\sim} A^{d-i}$, so that $\omega : A^i \to A^{i+1}$ is injective. We define I to be the ideal generated by ω and $A_{\lfloor \frac{d}{2} \rfloor + 1}$. We finally define the graded algebra

$$R := A/I$$

such that

$$R_k = A_k/\omega(A_{k-1}) \quad \text{if } k \leq \lfloor \frac{d}{2} \rfloor$$

and $R_k = \{0\}$ otherwise. Using $h_{-1} = 0$, for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$, we observe that

$$\dim R_k = h_k - h_{k-1}$$

Now it suffices to prove that R is generated by R_1 to use Macaulay criterion and deduce that $(h_0, h_1 - h_0, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ is a M-vector. It is a corollary of the following.

Proposition 14.14. *$H^*(X_\Sigma;\mathbb{Q})$ is generated by $H^2(X_\Sigma;\mathbb{Q})$*

The rest of lecture will be dedicated to sketch the proof of this proposition.

14.5 A presentation of $H^*(X_\Sigma;\mathbb{Q})$

Definition 14.15 (Prime divisors and linear equivalence)**.** Let X be an algebraic variety. A **prime divisor** is a codimension-1 subvariety. **Weil divisors** are formal sums of prime divisors. Let D, E be Weil divisors. If there exists $f \in \mathbb{C}(X)^*$ such that $\operatorname{div}(f) = D - E$, we say that D and E are **linearly equivalent** and we write $D \sim E$.

Proposition 14.16 (Some principal divisors)**.** *Let X_Σ be the toric variety of a fan Σ. For $\rho \in \Sigma(1)$ a ray denote the minimal generator u_ρ. Let $m \in M$, $\chi^m \in \mathbb{C}(X_\Sigma)^*$ its associated character.*

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle \nu_\rho$$

where ν_ρ is the closure of T-orbits of ρ, $\nu_\rho = \overline{\mathcal{O}(\rho)}$. It is a prime divisor.

We want to build a surjective ring morphism

$$F \ : \ \mathbb{Q}[x_1, \ldots, x_r] \longrightarrow H^*(X_\Sigma;\mathbb{Q})$$

such that the algebra generators x_i are sent to $H^2(X_\Sigma;\mathbb{Q})$.

Refined cohomology. We use refined cohomology to define F. Let X be a complete rational smooth variety. Let $W \subset X$ be an irreducible subvariety of dimension k. It has a refined cohomology class

$$[W]_r \in H^{2n-2k}(X, X \setminus W;\mathbb{Q})$$

The image of $[W]_r$ in $H^{2n-2k}(X;\mathbb{Q})$ is a cohomology class denoted $[W]$.

Proposition 14.17. *Let D, E be Weil divisors. If $D \sim E$ then $[D] = [E] \in H^2(X;\mathbb{Q})$.*

Let Σ be a complete simplicial fan. We write $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$ and u_i minimal generator of ρ_i. We can define F :

$$\begin{aligned} F \ : \ \mathbb{Q}[x_1, \ldots, x_r] &\longrightarrow H^*(X_\Sigma;\mathbb{Q}) \\ x_i &\longmapsto [\nu_{\rho_i}] \end{aligned}$$

We denote $D_i := [\nu_{\rho_i}] = [\overline{\mathcal{O}(\rho_i)}]$ the image of x_i. We now want to show that F is surjective. We will first explicit the kernel of F.
If $\rho_{i_1} + \cdots + \rho_{i_s}$ is not a cone in Σ then $D_{i_1} \cap \cdots \cap D_{i_s} = \emptyset$ so

$$[D_{i_1}]_r \cap \cdots \cap [D_{i_s}]_r \in H^*(X, X \setminus \cap_{j=1}^s D_{i_j};\mathbb{Q}) = H^*(X, X \setminus X;\mathbb{Q})$$

which implies that

$$[D_{i_1}] \cap \cdots \cap [D_{i_s}] = 0 \in H^*(X;\mathbb{Q})$$

It is now clear that the following ideals I and J are in the kernel of F.

$$I := \langle x_{i_1} \dots x_{i_s} \mid i_1, \dots, i_s \text{ distincts and } \rho_{i_1} + \dots + \rho_{i_s} \text{is not a cone in } \Sigma \rangle$$
$$J := \Big\langle \sum_{i=1}^{r} \langle m, u_i \rangle x_i \mid m \in M \Big\rangle$$

Indeed, note that the generators of J are sent to $[\mathrm{div}(\chi^m)] = 0$ by Proposition 14.16 and 14.17. We can now consider :

$$\bar{F} \ : \ \mathbb{Q}[x_1, \dots, x_r]/I + J \longrightarrow H^*(X_\Sigma; \mathbb{Q})$$

Theorem 14.18. *$\bar{F}$ is an isomorphism.*

This theorem implies Proposition 14.14. Its proof relies on the description of equivariant cohomology of toric varieties. After introducing equivariant cohomology, we sketch the proof of Theorem 14.18 in the last section.

14.6 Equivariant cohomology

Proposition 14.19. *Let G be a Lie group.*

- *There exists a contractible space EG with a free right G-action.*
  If G acts on X, then G acts on $EG \times X$ by $g \cdot (e, x) = (e \cdot g^{-1}, g \cdot x)$. We define $EG \times_G X = E \times X/G$
- *The homotopy type of $EG \times_G X$ does not depend of the choice of EG.*

Definition 14.20. Let R be a ring. The G-equivariant cohomology group of X is the usual cohomology of $EG \times_G X$. We denote it

$$H^*_G(X; R) := H^*(EG \times_G X; R)$$

Example 14.21. For $X = \{*\}$, G Lie group. $EG \times_G \{*\} \cong EG/G$ is the classifying space of G denoted BG. We denote its cohomology ring by

$$\Lambda_G := H^*_G = H^*(BG; R)$$

Remark 14.22. There is always a map $X \longrightarrow \{*\}$. It induces a map in cohomology

$$\Lambda_G \longrightarrow H^*_G(X; R)$$

which makes $H^*_G(X; R)$ a Λ_G-module.

Example 14.23. For $G = T_N$ a torus, with $M = \mathrm{Hom}(N; \mathbb{Z})$ the equivariant cohomology of the point is

$$\Lambda_{T_N} \cong \mathrm{Sym}_{\mathbb{Q}}(M)$$

The equivariant cohomology of toric varieties is of interest because of the following facts. Let X_Σ be a complete simplicial toric variety and $T = T_N$ its torus.

$$H^*_T(X_\Sigma; \mathbb{Q}) \xrightarrow{\sim} \Lambda_T \otimes_{\mathbb{Q}} H^*(X_\Sigma; \mathbb{Q})$$

is an isomorphism of Λ_T-modules.
The inclusion map $i_{X_\Sigma} \ : \ X_\Sigma \longrightarrow EG \times_G X_\Sigma$ induces

$$i^*_{X_\Sigma} \ : \ H^*_T(X_\Sigma; \mathbb{Q}) \to H^*(X_\Sigma; \mathbb{Q})$$

- $i^*_{X_\Sigma}$ is surjective
- its kernel is $I_T H^*_T(X_\Sigma; \mathbb{Q})$ where $I_T := \{\text{positive degree elements in } \Lambda_T\}$

Now (equivariant) intersection theory provides a map

$$[-]_T \ : \ \{\text{T-invariant divisors}\} \subset \mathrm{Div}(X_\Sigma) \longrightarrow H^2_T(X_\Sigma;\mathbb{Q})$$

Proposition 14.24. *Let $m \in M$. Recall that $m \in \Lambda_T$. Denote $1 \in H^0_T(X_\Sigma;\mathbb{Q})$ a generator. Then*

(i) $[div(\chi^m)]_T = -m \cdot 1$.

(ii) Moreover, the following diagram commutes :

$$\begin{array}{ccc} \{T\text{-invariant divisors}\} & \xrightarrow{[-]_T} & H^2_T(X_\Sigma;\mathbb{Q}) \\ & \searrow^{[-]} & \downarrow i^*_{X_\Sigma} \\ & & H^2(X_\Sigma;\mathbb{Q}) \end{array}$$

14.7 Proof of Theorem 14.18

Recall that we want to prove that

$$\bar{F} \ : \ \mathbb{Q}[x_1,\dots,x_r]/I+J \longrightarrow H^*(X_\Sigma;\mathbb{Q})$$

is an isomorphism. We can reduce it to the following theorem in equivariant cohomology.

Theorem 14.25. *There is a ring isomorphism*

$$G \ : \ \mathbb{Q}[x_1,\dots,x_r]/I \xrightarrow{\sim} H^*_T(X_\Sigma;\mathbb{Q})$$

given by $x_i \longmapsto [D_i]_T \in H^2_T(X_\Sigma;\mathbb{Q})$.
Moreover

$$G(J) = I_T H^*_T(X_\Sigma;\mathbb{Q})$$

We denote $R_\mathbb{Q}(\Sigma) := \mathbb{Q}[x_1,\dots,x_r]/I+J$ and $SR_\mathbb{Q}(\Sigma) := \mathbb{Q}[x_1,\dots,x_r]/I$. The reduction to Theorem 14.25 of the Theorem 14.18 relies on the following commutative diagram. The vertical maps are surjective and Theorem 14.25 implies that G restrict to an isomorphism on their kernels.

$$\begin{array}{ccc} SR_\mathbb{Q} & \xrightarrow{G} & H^*_T(X_\Sigma;\mathbb{Q}) \\ \downarrow & & \downarrow i^*_{X_\Sigma} \\ R_\mathbb{Q} & \xrightarrow{F} & H^*(X_\Sigma;\mathbb{Q}) \end{array}$$

Idea of the proof. The second part about $G(J)$ comes from Proposition 14.24. Now for the first claim, recall that $SR_\mathbb{Q} = \mathbb{Q}[x_1,\dots,x_r]/I$ where x_i corresponds to a ray ρ_i in Σ. For $\sigma \in \Sigma$, we define $\mathbb{Q}[\sigma] := \mathbb{Q}[x_{i_1},\dots,x_{i_l}]$ with $x_{i_1},\dots,x_{i_l}$ corresponding to the rays $\rho_{i_1},\dots,\rho_{i_l}$ in σ. The map G fits into the following diagram :

$$\begin{array}{ccccc} SR_\mathbb{Q} & \xrightarrow{\alpha} & \bigoplus_{\sigma\in\Sigma(d)} \mathbb{Q}[\sigma] & \xrightarrow{\beta} & \bigoplus_{\tau\in\Sigma(d-1)} \mathbb{Q}[\tau] \\ \downarrow G & & \downarrow A & & \downarrow B \\ H^*_T(X_\Sigma;\mathbb{Q}) & \xrightarrow{\alpha'} & \bigoplus_{\sigma\in\Sigma(d)} H^*_T(U_\sigma;\mathbb{Q}) & \xrightarrow{\beta'} & \bigoplus_{\tau\in\Sigma(d-1)} H^*_T(U_\tau;\mathbb{Q}) \end{array}$$

- α is defined as follows : $\forall f \in SR_\mathbb{Q}$, $\alpha(f) = (\alpha_{\sigma_1}(f),\dots,\alpha_{\sigma_k}(f))$ where

$$\alpha_\sigma(x_{i_1}\dots x_{i_l}) = \begin{cases} 0 \text{ if there is } i_j \text{ such that } \rho_{i_j} \notin \sigma(1) \\ x_{i_1}\dots x_{i_l} \end{cases}$$

- β is defined as follows :
  $\forall g = (g_1, \ldots, g_k) \in \bigoplus_{\sigma \in \Sigma(d)} \mathbb{Q}[\sigma]$, $\beta(g) = (\beta_{\tau_1}(g), \ldots, \beta_{\tau_{k'}}(g))$ where

$$\beta_\tau(g) = g_{i|x_i=0} - g_{j|x_j=0}$$

  where $i < j$ is such that $\sigma_i = \tau + \rho_i$, $\sigma_j = \tau + \rho_j$ are the cones in $\Sigma(d)$ containing τ (recall that Σ is simplicial).

Lemma 14.26. *The first row of the diagram is exact.*

We must finally prove that

- A and B are isomorphisms
- α' is injective

For the first claim,

$$H_T^*(U_\sigma; \mathbb{Q}) \cong H_T^*(\{x_\sigma\}; \mathbb{Q}) \cong \Lambda_T$$

as U_σ deformation retracts to the torus fixed point x_σ.

Injectivity of α follows from the localisation theorem.

Theorem 14.27 (Localisation)**.** *Let X be a toric variety with torus T. Then the inclusion $i : X^T \hookrightarrow X$ induces an isomorphism*

$$I_T^{-1} H_T^*(X; \mathbb{Q}) \xrightarrow[I_T^{-1} i^*]{\sim} I_T^{-1} H_T^*(X^T; \mathbb{Q})$$

We have the following diagram :

$$\begin{array}{ccc}
H_T^*(X_\Sigma; \mathbb{Q}) & \xrightarrow{\alpha'} & \bigoplus_{\sigma \in \Sigma(d)} H_T^*(U_\sigma; \mathbb{Q}) \\
 & \searrow & \downarrow \wr \\
 & \varphi & \bigoplus_{\sigma \in \Sigma(d)} H_T^*(\{x_\sigma\}; \mathbb{Q}) \\
 & & \downarrow \wr \\
 & & H_T^*(X_\Sigma^T; \mathbb{Q})
\end{array}$$

The Localisation theorem states that φ become an isomorphism after tensorisation with I_T^{-1}. It means that the kernel of φ is torsion. But $H_T^*(X_\Sigma; \mathbb{Q})$ is a free finitely generated Λ_T-module so it does not have torsion. Thus, φ is injective and so α' has to be.

□

Chapter 15. Solutions to exercises

15.1 Solutions to Chapter 1

Solutions written by Sergej Monavari

Solution 1.1. Define the map

$$V_f \to \mathbf{V}(I(V), x_{n+1}g - 1)$$

by sending $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \frac{1}{g(x_1,\ldots,x_n)})$. This is well-defined since g is a lift of f and therefore does not vanish on the tuple of points $(x_1, \ldots, x_n) \in V_f$ and is easily seen to be a bijection. Therefore we have that

$$\mathbb{C}[V_f] \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{I(V)}[f^{-1}] \cong \mathbb{C}[V]_f.$$

Solution 1.2. Let R be a UFD, and consider the inclusion $R \hookrightarrow K$ into its fraction field. Let $x \in K$ be any non-zero element, satisfying an equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

with $a_i \in R$. Since R is an UFD, we can write $x = \frac{p}{q}$, where $p, q \in R$ are elements with no common divisors. Then the above equation implies that

$$p^n = -a_{n-1}p^{n-1}q - \cdots - a_0q^n = q(-a_{n-1}p^{n-1} - \cdots - a_0q^n),$$

which implies that $q|p$ and therefore $p = 1$ or q, and therefore $x \in R$.

Solution 1.3. Consider the maps

$$\mathbb{C}[V] \hookrightarrow R \hookrightarrow \mathbb{C}(V),$$

where R is the normalisation of $\mathbb{C}[V]$. Surely, it has to contain the $\overline{y}/\overline{x}$, since this element is integral over $\mathbb{C}[V]$. We claim now that $\mathbb{C}[V][\overline{y}/\overline{x}]$ is normal. This will imply that then it is already the normalisation, therefore $R \cong \mathbb{C}[V][\overline{y}/\overline{x}]$. To show this, consider the map

$$\mathbb{C}[t] \to \mathbb{C}[V][\overline{y}/\overline{x}]$$

sending $t \mapsto \overline{y}/\overline{x}$. This is easily seen to be an isomorphism (by showing it is both surjective and injective!). Therefore, $\mathbb{C}[V][\overline{y}/\overline{x}] \cong \mathbb{C}[t]$, but the latter is a UFD, and therefore normal, by exercise 2.

Solution 1.4.

(i) Let $S^{-1}R \hookrightarrow K$ be the inclusion and take $x = \frac{p}{q} \in K$. Take an integral expression

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

with $a_i \in S^{-1}R$. Take $s \in S$ such that $a_i|s$ for all i. Then we have

$$(sx)^n + sa_{n-1}(sx)^{n-1} + \cdots + s^na_0 = 0,$$

which implies that $s^{n-i}a_i \in R$ and therefore $sx \in R$, since R is normal.

(ii) Let $R = \bigcap_i R_i \hookrightarrow K$ and take $a \in K$ integral over R, i.e.

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

with $a_j \in R$. Then $a_j \in R_i$ for all $i \in I$, therefore $x \in R_i$ for all i and $x \in R$.

15.2 Solutions to Chapter 2

Solutions written by Clotilde Freydt

Solution 2.1. Let V be an affine algebraic variety. Let $p \in V$ such that p is smooth. We know that the minimal primes ideal of the local ring $\mathcal{O}_{V,p}$ are in bijection with the irreducible components of V passing through p. Furthermore by definition of p being smooth the local ring $\mathcal{O}_{V,p}$ is regular, therefore a domain by the indication.
Now as irreducible components are maximal irreducible closed subsets, they correspond to minimal prime ideals. As a domain has a unique minimal prime, (0), if $\mathcal{O}_{V,p}$ is a domain, p is on a unique irreducible component of V.
Now suppose that V is a connected, smooth variety. By definition each $p \in V$ is smooth. Therefore each point of the variety belongs to a unique irreducible component and as the variety is connected, each point belongs to the same irreducible component that thus equals V, showing its irreducibility.

Solution 2.2.

(i) First by definition

$$\mathbb{C}[V \times W] = \frac{\mathbb{C}[x_1, ...x_n, y_1, ..., y_m]}{I(V \times W)}$$

where $I(V \times W)$ is the ideal corresponding to the product of the varieties V and W respectively embedded in $\mathbb{A}^n$ and $\mathbb{A}^m$.
We have the following:

$$I(V \times W) = I(V \times \mathbb{A}^m \cap \mathbb{A}^n \times W) = I(V \times \mathbb{A}^m) + I(\mathbb{A}^n \times W) = I(V) + I(W)$$

Therefore we rewrite,

$$\frac{\mathbb{C}[x_1, ...x_n, y_1, ..., y_m]}{I(V \times W)} = \frac{\mathbb{C}[x_1, ...x_n, y_1, ..., y_m]}{I(V) + I(W)}$$

Now, on the other hand, using commutative algebra we have:

$$\begin{aligned} &\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[W] \\ &= \frac{\mathbb{C}[x_1, ...x_n]}{I(V)} \otimes_{\mathbb{C}} \frac{\mathbb{C}[y_1, ..., y_m]}{I(W)} \\ &\cong \frac{\mathbb{C}[x_1, ...x_n] \otimes_{\mathbb{C}} [y_1, ..., y_m]}{I(V) + I(W)} \\ &\cong \frac{\mathbb{C}[x_1, ...x_n, y_1, ..., y_m]}{I(V) + I(W)} \end{aligned}$$

We can conclude $\mathbb{C}[V \times W] \cong \mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[W]$

(ii) We start by writing

$$\bar{S} \times W = \left(\bigcap_{\substack{S \subseteq B \\ B \text{ closed}}} B \right) \times W = \left(\bigcap_{\substack{S \subseteq B \\ B \text{ closed}}} V(I_B) \right) \times W$$

and

$$\overline{S \times W} = \bigcap_{\substack{S \times W \subseteq A \\ A \text{ closed}}} A = \bigcap_{\substack{S \times W \subseteq A \\ A \text{ closed}}} V(I_A)$$

Where I_B, I_A denote the ideal corresponding to the closed sets B and A, by defintion of closed sets in the Zariski topology. Now we have

$$\left(\bigcap_{\substack{S\subseteq B\\ B \text{ closed}}} V(I_B)\right) \times W = \bigcap_{\substack{S\subseteq B\\ B \text{ closed}}} (V(I_B)\times W) = \bigcap_{\substack{S\subseteq B\\ B \text{ closed}}} (V(I_B)\times V((0))$$

This writing makes clear that $\bar{S}\times W \supseteq \overline{S\times W}$ since $V(I_B)\times V((0)$ is closed in $V\times W$.

On the other hand we consider the closed set $V(I_A)$ and we claim that it has the same form as in the the above expression, namely $V(I_A) = V(I_B)\times V((0))$ where B is a closed set containing S. This fact will derive from the fact that $I_A = I_B + (0)$ with B closed containing S.

We have

$$V(I_A) = \{(a,b) \in V\times W \mid f(a,b) = 0 \;\forall\; f\in I_A\}$$

By (a), any polynomial $f \in I_A \subseteq \mathbb{C}[V\times W] = \mathbb{C}[V]\otimes_{\mathbb{C}} \mathbb{C}[W]$ can be written as $\sum_{i\in I} g_i\otimes h_i$ for some $g_i\in\mathbb{C}[V]$ and $h_i\in\mathbb{C}[W]$. So that,

$$V(I_A) = \{(a,b)\in V\times W \mid f(a,b) = \sum_{i\in I} g_i(a)\otimes h_i(b) = 0 \;\forall\; f\in I_A\}$$

As A contains $S\times W$, in particular the condition $\sum_{i\in I} g_i(a)\otimes h_i(b) = 0$ for all $(a,b)\in S\times W$ is verified. Now fix $a\in S$. One has $\mathbb{C}[W]\ni f(a,-) = \sum_{i\in I} g_i(a)\otimes h_i(-) = 0$ on the whole W and the only polynomial of $\mathbb{C}[W]$ that satisfies it is trivial. It implies that the generators of I_A are all of the form $g_i\otimes \mathbf{1}$ where g_i's are generators of B.

Therefore, $I_A = I_B + (0)$ so that $V(I_A) = V(I_B)\times V((0))$ the claimed form, showing the reverse inclusion.

(iii) We will use that any variety is irreducible if and only if its coordinate ring is a domain. As V and W are irreducible $\mathbb{C}[V]$ and $\mathbb{C}[W]$ are both domains. Now as $\mathbb{C}$ is algebraically closed the tensor product $\mathbb{C}[V]\otimes_{\mathbb{C}}\mathbb{C}[W]$ is again a domain and therefore by (b), $\mathbb{C}[V\times W]$ is a domain, thus $V\times W$ is irreducible.

Solution 2.3.

(i) We will show that $\Phi(\mathbb{C}^2) = \hat{\mathbb{C}}_d = V(I)$

The first inclusion $\hat{\mathbb{C}}_d \subseteq V(I)$ is straightforward as $x = (s^d, s^{d-1}t, ..., t^d)$ is a point of the vanishing locus of $x_i x_{j+1} - x_j x_{i+1}$ for any $s,t\in\mathbb{C}^2$ and $0\le i<j\le d-1$.

For the second inclusion $V(I)\subseteq\hat{\mathbb{C}}_d$, we first observe that the generators of I are exactly the maximal minors by i-th and j-th columns of the matrix

$$A(x) := \begin{pmatrix} x_0 & x_1 & ... & x_{d-1}\\ x_1 & x_2 & ... & x_d\end{pmatrix}$$

Therefore we have, $x = (x_0, ..., x_d)\in V(I)$ if and only if all minors of $A(x)$ vanish, if and only if $rank(A)\le 1$.

So now if $x\in V(I)$ there exists some p, $q\in\mathbb{C}$ $(p,q)\neq(0,0)$ such that $pA(x)_1 - qA(x)_2 = 0$ (where $A(x)_i$ denotes the i-th row of $A(x)$).
-When $p=0$, $A(x)_1 = 0$ and x is the image of $(0, x^{1/d})\in\mathbb{C}^2$ Note that here we take the complex d-th root of x, as this is not uniquely defined we choose the principal root. -When $p\neq 0$, we have $x = (x_0, q/px_0..., (q/p)^d x_0)$.

By setting $s = {x_0}^{1/d}$ (by taking again the principal d-th root) and $t = \frac{p}{q}s$ we obtain $x = (s^d, s^{d-1}t, ..., t^d)$. therefore $V(I)\subseteq\hat{\mathbb{C}}_d$.

(ii) We have $\Phi((\mathbb{C})^2) = \hat{C}_d$. The space $\mathbb{C}^2$ is the product of irreducible sets and is therefore irreducible. The set $\hat{C}_d$ is irreducible as the image of an irreducible set under the continuous map Φ.

15.3 Solutions to Chapter 3

Solutions written by Matthew Dupraz

Solution 3.1. Let I, J denote the left and right ideal in the above equality. It is clear that $I \subseteq J$ as its set of generators of I is contained in the set of generators of J.
Let us show that $I \supseteq J$. Let $x^\alpha - x^\beta$ be an element in the set of generators of J, i.e. $\alpha, \beta \in \mathbb{N}^s$ are such that $\alpha - \beta \in L$. Let $l = \alpha - \beta$. Then we have

$$l_+ - l_- = \alpha - \beta.$$

Since l_+ and l_- have disjoint supports, $\alpha \geq l_+$. Indeed, for any i s.t. $(l_+)_i$ is non-zero, we have that $(l_-)_i = 0$ and so

$$(l_+)_i = \alpha_i - \beta_i.$$

So consider

$$\alpha - l_+ = \beta - l_- = \gamma \in \mathbb{N}^s.$$

We get that

$$x^\alpha - x^\beta = (x^{l_+} - x^{l_-})x^\gamma \in I.$$

This implies that $I \supseteq J$ and so we get the desired equality.

Solution 3.2.

(i) We have that for $g \in \mathbb{C}[x_1, \ldots, x_s]$, $\Phi^*(g) = g \circ \Phi \in \mathbb{C}[V]$, so we have that

$$\begin{aligned} \ker(\Phi^*) &= \{g \in \mathbb{C}[x_1, \ldots, x_s] : g \circ \Phi \equiv 0\} \\ &= \{g \in \mathbb{C}[x_1, \ldots, x_s] : \forall y \in \operatorname{im}(\Phi), g(y) = 0\} \end{aligned}$$

But any polynomial $g \in \mathbb{C}[x_1, \ldots, x_s]$ which is zero on $\operatorname{im}(\Phi)$ is also zero on the Zariski closure $\overline{\operatorname{im}(\Phi)} = Y$. So in fact we have that

$$\ker(\Phi^*) = \{g \in \mathbb{C}[x_1, \ldots, x_s] : \forall y \in Y : g(y) = 0\} = \mathbf{I}(Y)$$

(ii) In the proof of the proposition which tells us that semigroup algebras give rise to affine toric varieties, we start with a finite generating subset $\mathcal{A} = \{m_1, \ldots, m_s\} \subset S$ of a semigroup $S \subset M$. We apply the exercise to the variety $V = T_N$, $f_i = \chi^{m_i}$, so that $\operatorname{im}\Phi = Y_\mathcal{A}$. So we get by part 1 that $\mathbf{I}(Y_\mathcal{A}) = \ker(\Phi^*)$, which yields

$$\mathbb{C}[Y_\mathcal{A}] = \mathbb{C}[x_1, \ldots, x_s]/\mathbf{I}(Y_\mathcal{A}) = \mathbb{C}[x_1, \ldots, x_s]/\ker(\Phi^*) \cong \mathbb{C}[S],$$

and so shows that $\operatorname{Spec}\mathbb{C}[S] = Y_\mathcal{A}$, i.e. an affine toric variety.

Solution 3.3. Denote J the lattice ideal of L. Clearly, $I \subseteq J$ as the generators correspond to $(2,0,0), (1,1,0), (0,1,1)$, which are all in L. So we just have to show $I \supseteq J$.
First, notice that in $R = \mathbb{C}[x, y, z]/I$,

$$\bar{x} - \bar{y} = \bar{x} - \bar{x}^2\bar{y} = \bar{x}(\bar{1} - \bar{x}\bar{y}) = \bar{0},$$

so $x - y \in I$. Similarly,

$$\bar{x} - \bar{z} = \bar{x} - \bar{x}^2\bar{z} = \bar{x}(\bar{1} - \bar{x}\bar{z}) = \bar{x}(\bar{1} - \bar{y}\bar{z}) = \bar{0},$$

so $x - z \in I$.
Now let $f = x^a y^b z^c - x^d y^e z^f$ a generator of J, i.e. $(a-d, b-e, c-f) \in L$. We have that in R,

$$\bar{f} = \bar{x}^a\bar{y}^b\bar{z}^c - \bar{x}^d\bar{y}^e\bar{z}^f = \bar{x}^{a+b+c} - \bar{x}^{d+e+f}.$$

Since $\bar{x}^2 = 1$, the exponent in R may be seen modulo 2, and since by definition of L, we get that $a + b + c \equiv d + e + f \mod 2$, we conclude that in fact $\bar{f} = \bar{0}$, i.e. $f \in I$.
Hence we conclude that $I \supseteq J$ and so we get the desired equality.

Solution 3.4. Identify $M \cong \mathbb{Z}^n$ via a basis $e_1, \dots, e_n$. In fact φ is a $\mathbb{Z}$-module isomorphism, so we may represent it by an invertible matrix $A = (a_{ij}) \in \mathbb{Z}^{n\times n}$ (whose inverse has also coefficients in $\mathbb{Z}$). Let $\mathcal{A} = \{m_1, \dots, m_s\}$, so that $\mathcal{B} = \{\varphi(m_1), \dots, \varphi(m_s)\}$. Consider

$$\begin{aligned}\Phi_{\mathcal{A}} : T_n &\to (\mathbb{C}^*)^s \\ t &\mapsto (\chi^{m_1}(t), \dots, \chi^{m_s}(t))\end{aligned}$$

and also

$$\begin{aligned}\Phi_{\mathcal{B}} : T_n &\to (\mathbb{C}^*)^s \\ t &\mapsto (\chi^{\varphi(m_1)}(t), \dots, \chi^{\varphi(m_s)}(t)).\end{aligned}$$

We have that $Y_{\mathcal{A}}$ is just the closure in $\mathbb{C}^s$ of the image of $\Phi_{\mathcal{A}}$ and $Y_{\mathcal{B}}$ is $\overline{\operatorname{im}\Phi_{\mathcal{B}}}$. Notice that the image of a map does not change if it's precomposed with a surjective map, so let's consider the map

$$\begin{aligned}\psi : T_n &\to T_n \\ t &\to (t^{\varphi(e_1)}, \dots t^{\varphi(e_n)}) = (t^{\sum_{i=1}^n a_{i1}e_i}, \dots, t^{\sum_{i=1}^n a_{in}e_i})\end{aligned}$$

This is actually a bijection of inverse

$$\begin{aligned}\psi^{-1} : T_n &\to T_n \\ t &\to (t^{\varphi^{-1}(e_1)}, \dots t^{\varphi^{-1}(e_n)})\end{aligned}$$

Indeed,

$$\begin{aligned}\psi \circ \psi^{-1}(t) &= (\psi^{-1}(t)^{\varphi(e_1)}, \dots, \psi^{-1}(t)^{\varphi(e_n)}) \\ &= \left(\prod_{i=1}^n t^{a_{i1}\varphi^{-1}(e_i)}, \dots, \prod_{i=1}^n t^{a_{in}\varphi^{-1}(e_i)}\right) \\ &= \left(t^{\sum_{i=1}^n \varphi^{-1}(a_{i1}e_i)}, \dots, t^{\sum_{i=1}^n \varphi^{-1}(a_{in}e_i)}\right) \\ &= (t^{\varphi^{-1}\varphi(e_1)}, \dots, t^{\varphi^{-1}\varphi(e_n)}) \\ &= t.\end{aligned}$$

This shows surjectivity of ψ, but the other direction is symmetric.
If we precompose $\Psi_{\mathcal{A}}$ with ψ, we get:

$$\begin{aligned}\Phi_{\mathcal{A}} \circ \psi(t) &= (\chi^{m_1}(\psi(t)), \dots, \chi^{m_s}(\psi(t))) \\ &= \left(\prod_{i=1}^n \chi^{m_{1i}}(t^{\varphi(e_i)}), \dots, \prod_{i=1}^n \chi^{m_{ni}}(t^{\varphi(e_i)})\right) \\ &= \left(\prod_{i=1}^n \chi^{m_{1i}\varphi(e_i)}(t), \dots, \prod_{i=1}^n \chi^{m_{ni}\varphi(e_i)}(t)\right) \\ &= \left(\chi^{\sum_{i=1}^n \varphi(m_{1i}e_i)}(t), \dots, \chi^{\sum_{i=1}^n \varphi(m_{ni}e_i)}(t)\right) \\ &= \left(\chi^{\varphi(m_1)}(t), \dots, \chi^{\varphi(m_n)}(t)\right) \\ &= \Phi_{\mathcal{B}}(t).\end{aligned}$$

This shows that $\operatorname{im}\Phi_{\mathcal{B}} = \operatorname{im}\Phi_{\mathcal{A}}$ and so in particular, this induces an isomorphism $Y_{\mathcal{A}} \cong Y_{\mathcal{B}}$ (as affine varieties).
Since the tori associated to the affine toric varieties are just the tori $\operatorname{im}\Phi_{\mathcal{A}}$ and $\operatorname{im}\Phi_{\mathcal{B}}$, which are equal as we have just shown and their action is induced from the multiplication in $(\mathbb{C}^*)^s$, we deduce that these actions coincide. We conclude that the isomorphism $Y_{\mathcal{A}} \cong Y_{\mathcal{B}}$ is actually an isomorphism of of toric varieties.

15.4 Solutions to Chapter 4

Solutions written by Louis Gogniat

Solution 4.1. We denote respectively $i)$, $ii)$, $iii)$, and $iv)$ as the four equivalent properties in the exercise (from top to bottom). We then prove that $i) \implies iv) \implies ii) \implies iii) \implies i)$.

- $i) \implies iv)$: Suppose $\dim \sigma^\vee < n$. Then there exists a hyperplane $H \subseteq M_\mathbb{R}$ containing $\sigma^\vee$, i.e., there exists $u \in N_\mathbb{R} \setminus 0$ such that
$$\sigma^\vee \subseteq H_u = \{m \in M_\mathbb{R} \mid \langle m, u \rangle = 0\}.$$
In fact, the condition $\langle m, u \rangle = 0$ for every $w \in \sigma^\vee$ implies that $u \in (\sigma^\vee)^\vee = \sigma$. Therefore, for any $w \in \sigma^\vee$, we have
$$u \in \sigma \cap H_w = \{v \in \sigma \mid \langle w, v \rangle = 0\}.$$
Given that u is non-zero, it leads us to the conclusion that $\{0\}$ cannot be a face of σ.

- $iv) \implies ii)$: Let us assume there exists a non-zero element $v \in \sigma \cap -\sigma$. Notice that if $w \in \sigma^\vee$, then $\langle w, v \rangle \geq 0$ (by definition of $\sigma^\vee$). Moreover, since $-v \in \sigma$, we also have $\langle w, -v \rangle \geq 0$. By bilinearity of the inner product, we conclude that $\langle w, v \rangle = 0$ for every $w \in \sigma^\vee$, or equivalently
$$\sigma^\vee \subseteq H_v = \{m \in M_\mathbb{R} \mid \langle m, v \rangle = 0\}.$$
Given that v is non-zero, H_v is indeed a hyperplane, and we consequently deduce that $\dim \sigma^\vee \leq n-1$.

- $ii) \implies iii)$: Suppose that σ contains a positive-dimensional subspace of $N_\mathbb{R}$. Then, it is clear that $-\sigma$ also contains the same subspace, and thus $\sigma \cap -\sigma \neq \{0\}$.

- $iii) \implies i)$: For this implication, we show that $\sigma \cap -\sigma$ is a face of σ, and as a result, if $\sigma \cap -\sigma = \{0\}$, we conclude that $\{0\}$ is a face of σ. Recall that if τ is a face of a cone, then $\tau^* = \sigma^\vee \cap \tau^\perp$ is a face of the dual $\sigma^\vee$ (see Proposition 1.2.10 in [CLS]). In particular, if we apply this proposition to the face $\tau = \sigma^\vee$ of the dual, we find that $(\sigma^\vee)^\vee \cap (\sigma^\vee)^\perp$ is a face of $(\sigma^\vee)^\vee$. Since $\sigma = (\sigma^\vee)^\vee$, we then see that
$$(\sigma^\vee)^\vee \cap (\sigma^\vee)^\perp = \{u \in \sigma \mid \langle w, u \rangle = 0 \ \forall w \in \sigma^\vee\}$$
is a face of σ. Let us show that this latter set is nothing other than $\sigma \cap -\sigma$. On one hand, if $u \in \sigma \cap -\sigma$, then for all $w \in \sigma^\vee$ we have $\langle w, \pm u \rangle \geq 0$, and thus $\langle w, u \rangle = 0$. Conversely, if $u \in \sigma$ satisfies $\langle w, u \rangle = 0$ for any $w \in \sigma^\vee$, then $-u$ also satisfies this property. Thus, we have $\pm u \in (\sigma^\vee)^\vee = \sigma$, from which we conclude that $u \in \sigma \cap -\sigma$.
In conclusion, we indeed have $(\sigma^\vee)^\vee \cap (\sigma^\vee)^\perp = \sigma \cap -\sigma$, demonstrating that $\sigma \cap -\sigma$ is a face of σ.

Solution 4.2.

(i) First, let us note that $\sigma^\vee$ is strongly convex since it satisfies property $iv)$ of the first exercise. Indeed, the fact that σ has maximal dimension implies that $\dim \sigma = \dim(\sigma^\vee)^\vee = n$. This, in particular, means that $\{0\}$ is a face of $\sigma^\vee$, so that we can find some $u \in \sigma \setminus 0$ with $\sigma^\vee \cap H_u = 0$. Since σ is a rational cone, $\sigma^\vee$ is also rational. Therefore, we may assume that $u \in \sigma \cap N \setminus 0$. We then have $\langle m, u \rangle \in \mathbb{N}$, with $\langle m, u \rangle = 0$ if and only if $m = 0$ (since $\sigma^\vee \cap H_u = 0$).

We now prove that $\mathcal{H}$ generates S_σ. Consider $m \in S_\sigma$. If m is irreducible, then $m \in \mathcal{H}$. So we may assume that m is not irreducible, which means that there exist $m', m'' \in S_\sigma \setminus 0$ such that $m = m' + m''$. We then observe that
$$\langle m, u \rangle = \langle m', u \rangle + \langle m'', u \rangle.$$
Since m', m'' are not zero, we have $\langle m', u \rangle$ and $\langle m'', u \rangle > 0$, hence
$$\langle m', u \rangle < \langle m, u \rangle \text{ and } \langle m'', u \rangle < \langle m', u \rangle.$$
By induction on $\langle m, u \rangle$, we conclude that every element $m \in S_\sigma$ is a finite sum of elements from $\mathcal{H}$, meaning that $\mathcal{H}$ is a generating set for S_σ.

Additionally, according to Gordan's lemma (Proposition 1.2.17 in [CLS]), S_σ is finitely generated. Consequently, S_σ contains a finite number of irreducible elements, from which we conclude that $\mathcal{H}$ is finite.

(ii) We first establish the following lemma (see Lemma 1.2.7 in [CLS]).

Lemma 4.1. *Let τ be a face of a polyhedral cone σ. If $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$.*

Proof. Since τ is a face of σ, there exists an element $m \in \sigma^\vee$ such that $\tau = \sigma \cap H_m$. Now, observe that if $u + v \in \tau$, then

$$\langle m, u + v\rangle = \langle m, u\rangle + \langle m, v\rangle = 0.$$

Furthermore, as $m \in \sigma^\vee$, we have $\langle m, u\rangle \geq 0$ and $\langle m, v\rangle \geq 0$. Consequently, we obtain that

$$\langle m, u\rangle = \langle m, v\rangle = 0,$$

meaning that both u and v are in τ. □

Now let ρ be an edge of $\sigma^\vee$ and denote by m_ρ the ray generator of $\sigma^\vee \cap M$. Let us assume for the sake of contradiction that m_ρ is not irreducible, meaning that there exist m' and m'' in $S_\sigma \setminus 0$ such that $m_\rho = m' + m''$. Using the above lemma, we then deduce that both m' and m'' are in ρ. Since m_ρ generates $\sigma^\vee \cap M$, there exist k' and k'' in $\mathbb{N} \setminus 0$ such that $m' = k' m_\rho$ and $m'' = k'' m_\rho$. Therefore we obtain

$$m_\rho = m' + m'' = k' m_\rho + k'' m_\rho = (k' + k'') m_\rho.$$

This implies that $k' + k'' = 1$, which contradicts our initial choice of k' and k'' as positive integers. Thus, we conclude that m_ρ is irreducible, i.e. $m_\rho \in \mathcal{H}$.

(iii) Let $\mathcal{G}$ be a generating set of S_σ. By contradiction, suppose that there exists $h \in \mathcal{H} \setminus \mathcal{G}$. Since $\mathcal{G}$ generates S_σ, there exist $g_1, \ldots, g_k \in \mathcal{G}$ and $n_1, \ldots, n_k \in \mathbb{N} \setminus 0$ such that $h = \sum n_i g_i$. However, this contradicts the irredicibility of h. Therefore, $\mathcal{H} \subseteq \mathcal{G}$, and consequently, $\mathcal{H}$ is the minimal generating set of S_σ with respect to the inclusion.

Solution 4.3.

(i) First, note that if $\sigma = Cone(s_1, \ldots, s_k)$, then $w \in \sigma^\vee$ if and only if $\langle w, s_i\rangle \geq 0$ for all $i = 1, \ldots, k$. In fact, as any $u \in \sigma$ can be expressed as a conical combination of the s_i, i.e. $u = \sum \lambda_i s_i$ for some $\lambda_i \in \mathbb{R}_{\geq 0}$, we have

$$\langle w, u\rangle = \langle w, \sum \lambda_i s_i\rangle = \sum \lambda_i \langle w, s_i\rangle \geq 0,$$

as long as $\langle w, s_i\rangle \geq 0$ for all i.

For $\sigma = Cone(3e_1 - 2e_2, e_1)$, we obtain that $w = xe_1 + ye_2 \in \sigma^\vee$ if and only if

$$x \geq 0 \quad \text{and} \quad 3x - 2y \geq 0.$$

We illustrate the cone σ and its dual in Figure 18a.

According to Figure 18a, it is easily seen that

$$m_1 = 2e_1 + 3e_2, \; m_2 = e_1 + e_2, \; \text{and } m_3 = -e_2$$

are the only irreducible elements in $S_\sigma = \sigma^\vee \cap \mathbb{Z}^2$. Hence, Exercise 2 implies that $\mathcal{H} = \{m_1, m_2, m_3\}$ is the minimal generating set of S_σ.

(ii) Since $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$, and because $S_\sigma = \mathbb{N}\mathcal{H}$, we find that U_σ is precisely the affine toric variety $Y_\mathcal{H}$. Thus, the toric ideal associated with U_σ is obtained by considering the kernel L of the morphism $\hat{\Phi}_\mathcal{H} : \mathbb{Z}^3 \to \mathbb{Z}^2$ given by the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}.$$

This kernel is precisely $L = \ker \hat{\Phi}_\mathcal{H} = \langle (1, -2, 1) \rangle_\mathbb{Z}$. Therefore, the toric ideal of U_σ is

$$\mathbb{I}(Y_\mathcal{H}) = \langle \mathbf{x}^{l_+} - \mathbf{x}^{l_-} \mid l \in L\rangle = \langle x^k z^k - y^{2k} \mid k \in \mathbb{N}\rangle = \langle xz - y^2\rangle,$$

where the last equality come from the fact that

$$(xz)^k - y^{2k} = (xz - y^2)(\sum_{i=0}^{k-1} y^{2i} (xz)^{k-1-i}) \quad \text{for any} \quad k \in \mathbb{N} \setminus 0.$$

Solution 4.4.

(i) Using the same argument as in exercise 3 part (a), we have that $w = xe_1 + ye_2 + ze_3 \in \sigma^\vee = \text{Cone}(e_1, e_2, e_1 + e_2 + 2e_3)^\vee$ if and only if

$$x \geq 0,\ y \geq 0, \text{ and } x + y + 2z \geq 0.$$

We graphically represent the dual $\sigma^\vee$ in Figure 18b. To find generators of S_σ, we first decompose

$$S_\sigma = S_\sigma^- \cup S_\sigma^+,$$

where $S_\sigma^\pm := \{(x, y, z) \in S_\sigma \mid \pm z \geq 0\}$. Note that, since $\mathbb{R}^3_{\geq 0} \subset \sigma^\vee$, the three elements

$$m_1 = e_1,\ m_2 = e_2, \text{ and } m_3 = e_3$$

belong to S_σ and clearly generate the subset S_σ^+. Let us turn our attention to S_σ^-. We claim that the set

$$\mathcal{H}^- = \{m_1,\ m_2,\ m_4 = 2e_1 - e_3,\ m_5 = 2e_2 - e_3,\ m_6 = e_1 + e_2 - e_3\}$$

generates S_σ^-. For $z \in \mathbb{Z}_{\leq 0}$, we denote A_z the set defined by the intersection of the line with equation $x + y + 2z = 0$ and S_σ^- (see Figure 19), explicitly, we have

$$A_z = \{(m, -2z - m, z) \mid m = 0, \dots, -2z\} \text{ for any } z \in \mathbb{Z}_{\leq 0}.$$

With these notations, it is not difficult to see that if $\mathcal{H}^-$ generates A_z for all $z \in \mathbb{Z}_{\leq 0}$, then $\mathcal{H}^-$ generates S_σ^-. By induction on $z \in \mathbb{Z}_{\leq 0}$, let us verify that this is indeed the case.

If $z = 0$, then $A_z = \{(0, 0, 0)\}$, and the claim is satisfied. Similarly, if $z = -1$, then $A_z = \{m_4, m_5, m_6\}$, and the claim holds true as well. Now, assume the result is true for some $z \leq -1$, and let us show that it is also true for $z - 1$. Note that for $m = 0, \dots, -2z$, we have

$$(m, -2(z - 1) - m, z - 1) = (m, -2z - m, z) + m_5,$$

for $m = -2z + 1$,

$$(m, -2(z - 1) - m, z - 1) = (-2z + 1, 1, z - 1) = (-2z, 0, z) + m_6,$$

and for $m = -2z + 2$,

$$(m, -2(z - 1) - m, z - 1) = (-2z + 2, 0, z - 1) = (-2z, 0, z) + m_4.$$

Hence, we see that each element within A_{z-1} can be expressed as the sum of an element from A_z along with one element from $\mathcal{H}^-$. With this established, we obtain by applying the induction hypothesis, that $\mathcal{H}^-$ generates A_{z-1}.

Since on one hand, $\{m_1, m_2, m_3\}$ generates S_σ^+, and on the other hand, $\mathcal{H}^-$ generates S_σ^-, we then deduce that

$$\mathcal{H} = \{m_1, m_2, m_3, m_4, m_5, m_6\}$$

is a generating set for S_σ (in fact, it is a minimal generating set, as all the m_i are irreducible).

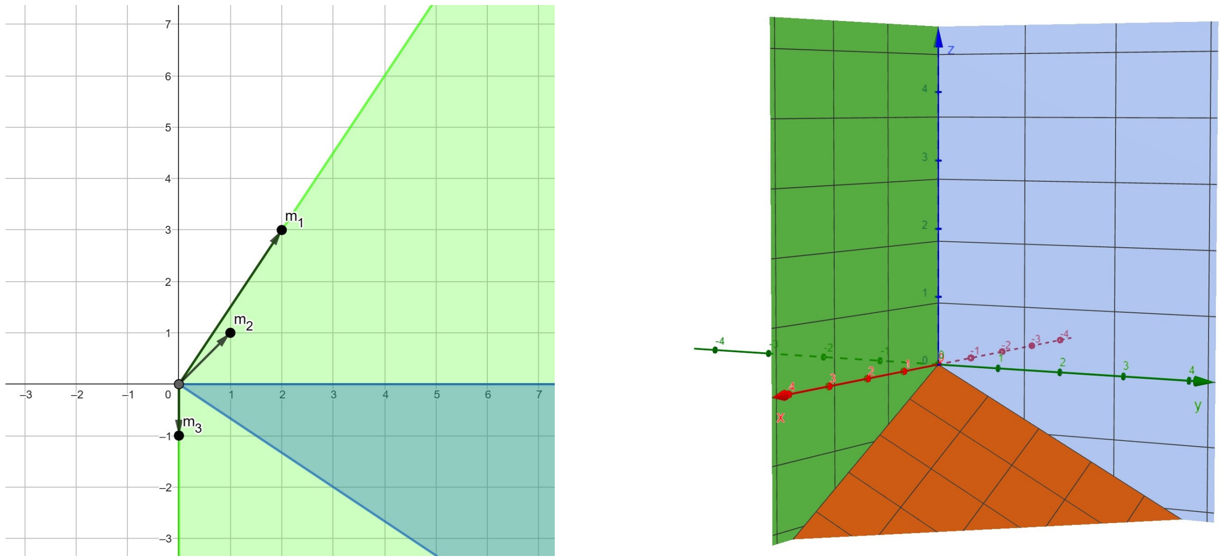
(ii) We propose here a different method than the one presented in exercise 3 part (b) to compute the toric ideal associated with U_σ. According to section 1.1 of [CLS], the affine toric variety U_σ is the Zariski closure of the image of the map $\Phi : (\mathbb{C}^*)^3 \to \mathbb{C}^6$ defined by

$$\Phi(r, s, t) = (r, s, t, r^2 t^{-1}, s^2 t^{-1}, rst^{-1}).$$

We claim that $\overline{\text{im}\Phi} = \mathbb{V}(I)$ where I is the ideal

$$I = \langle zu - x^2,\ zv - y^2,\ zw - xy \rangle \subseteq \mathbb{C}[x, y, z, u, v, w].$$

Figure 18



(a) The blue surface represents the cone σ, the green surface its dual $\sigma^\vee$. The vectors m_1, m_2, m_3 are the irreducible elements that generate S_σ.

(b) The three green, blue, and red faces represent the three faces of the dual convex cone $\sigma^\vee$.

In what follows, we will assume without providing a proof that I is indeed a prime ideal. So in particular, $\mathbb{V}(I)$ is irreducible. First, it is easily verified that $\mathrm{im}\Phi \subset \mathbb{V}(I)$. Conversely, suppose that $\mathbf{x} = (x, y, z, u, v, w) \in \mathbb{V}(I)$. We then have that

$$zu = x^2, \tag{10}$$

$$zv = y^2, \tag{11}$$

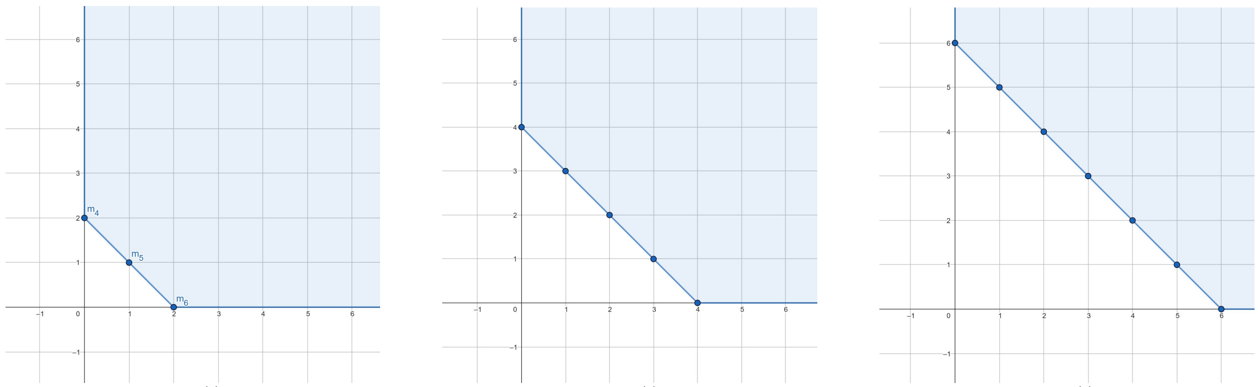
$$zw = xy. \tag{12}$$

If $z \neq 0$, let us set $r := x$, $s := y$, and $t := z$. In this case, we observe that by (1), we have $u = r^2 t^{-1}$, by (2), $v = s^2 t^{-1}$, and by (3), $w = rst^{-1}$. Therefore, we conclude that if $x, y, z \neq 0$, then $\mathbf{x} \in \mathrm{im}\Phi$, and thus we have $\mathbb{V}(I) \cap (\mathbb{C}^*)^3 = \mathrm{im}\Phi$. Note that since $U := \mathbb{V}(I) \cap (\mathbb{C}^*)^3$ is an open (non-empty) subset of $\mathbb{V}(I)$ and $\mathbb{V}(I)$ is irreducible, then U is also irreducible and dense in $\mathbb{V}(I)$. Taking the Zariski closure, one obtains

$$U_\sigma = \overline{\mathrm{im}\Phi} = \overline{U} = \mathbb{V}(I),$$

so that I is the toric ideal of U_σ.

Figure 19



(a) Projection of $\sigma^\vee$ onto the plane with equation $z = -1$. The dark blue points represent the set $A_{-1} = \{m_4, m_5, m_6\}$.

(b) Projection of $\sigma^\vee$ onto the plane with equation $z = -2$. The 5 five dark blue points represent the set A_{-2}.

(c) Projection of $\sigma^\vee$ onto the plane with equation $z = -3$. The seven dark blue points represent the set A_{-3}.

15.5 Solutions to Chapter 5

Solutions written by Isak Gustaf Salomon Sundelius

Solution 5.1.

(a) The $\mathbb{C}$-algebra homomorphism induced by γ is given by

$$\mathbb{C}[S] \to \mathbb{C}$$
$$f(x_1, \ldots, x_s) \mapsto f(\gamma(m_1), \ldots, \gamma(m_s))$$

where the elements x_i correspond to the characters χ^{m_i} and the multiplication is given by the semigroup structure of S. With this it is clear that the kernel of this map must be $f \in \mathbb{C}[S]$ such that

$$f(\gamma(m_1), \ldots, \gamma(m_s)) = 0$$

and since we in the exercise description get that $p := (\gamma(m_1), \ldots, \gamma(m_s))$, this condition on f means that $f(p) = 0$, so we are done.

(b) We want to show that the affine semigroup homomorphism $m \mapsto \chi^m(t)\gamma(m)$ has induced semigroup algebra homomorphism $\mathbb{C}[S] \to \mathbb{C}$ has kernel the maximal ideal corresponding to the point

$$(\chi^{m_1}(t), \ldots, \chi^{m_s}(t)) \cdot (\gamma(m_1), \ldots, \gamma(m_s)).$$

The corresponding homomorphism of semigroup algebras is given in the obvious way, so the kernel will be precisely

$$\{f \in \mathbb{C}[S] : f(\chi^{m_1}(t)\gamma(m_1), \ldots, \chi^{m_s}(t)\gamma(m_s)) = 0\}.$$

It is clear that we have

$$(\chi^{m_1}(t)\gamma(m_1), \ldots, \chi^{m_s}(t)\gamma(m_s)) = (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)) \cdot (\gamma(m_1), \ldots, \gamma(m_s))$$

The action by $t \in T_N$ on $p \in Y_{\mathcal{A}}$ is given by multiplication, and the affine semigroup homomorphism corresponding to a given point in $Y_{\mathcal{A}}$, for instance $t \in T_N \subseteq Y_{\mathcal{A}}$, is given by

$$m \mapsto \chi^m(t).$$

Then the semigroup homomorphism corresponding to the point $t \cdot p$ is exactly

$$m \mapsto \chi^m(t \cdot p) = \chi^m(t) \cdot \chi^m(p) = \chi^m(t) \cdot \gamma(m).$$

by definition of p, and so we are done.

Solution 5.2. We use the proposition preceding this corollary, from Chapter 5. Part (a) of this proposition states that the torus action on an affine toric variety $V = \mathrm{Spec}(\mathbb{C}[S])$ has a fixed point if and only if S is pointed.
By definition, S_σ is pointed if and only if $S_\sigma \cap (-S_\sigma) = \{0\}$. By definition of S_σ this amounts to

$$\{0\} = (\sigma^\vee \cap M) \cap (-(\sigma^\vee \cap M)) = \sigma^\vee \cap (-\sigma^\vee) \cap M,$$

since $M = -M$. We proceed by proving the following lemma:

Lemma. *Under the assumptions of the exercise description,*

$$\sigma^\vee \cap (-\sigma^\vee) \cap M = \{0\} \iff \sigma^\vee \cap (-\sigma^\vee) = \{0\}.$$

Proof. The implication " $\Longleftarrow$ " is trivial.
For the other direction, consider the generators of S_σ, given by a subset $\{e_1, \dots, e_\ell\} \subseteq M$. In particular, the $\mathbb{Z}$-linear combinations of these elements constitute S_σ, while the $\mathbb{R}$-linear combinations of these elements constitute $\sigma^\vee$. An element of $\sigma^\vee$ is given by

$$m' = \sum_{i=1}^{\ell} \alpha_i e_i, \quad \alpha_i \in \mathbb{R}.$$

By definition of the dual, $m' \in \sigma^\vee \cap (-\sigma^\vee)$ if and only if

$$\langle m', u \rangle = 0 \quad \forall u \in \sigma.$$

By choosing a set of generators $f_1, \dots, f_s \in N$ of σ, this may be rephrased as

$$\langle m', f_j \rangle \quad \forall j$$

and so

$$\sum_{i=1}^{\ell} \langle e_i, f_j \rangle \quad \forall j.$$

Since every $\langle e_i, f_j \rangle \in \mathbb{Z}$ we obtain a matrix $\{\langle e_i, f_j \rangle\}_{i,j}$, which has full rank since the e_i and f_j are linearly independent by assumption. This gives us that the product of this matrix by the vector $\{\alpha_i\}_i$ equals zero, which in turn implies that all α_i are integers, up to some common multiple of some nonzero scalar $r \in \mathbb{R}$. With this we get that

$$\sum_{i=1}^{\ell} \frac{\alpha_i}{r} e_i \in M.$$

This, together with the fact that

$$\langle m', u \rangle = 0 \implies \left\langle \frac{m'}{r}, u \right\rangle = \frac{1}{r} \langle m', u \rangle = 0,$$

gives us that $\frac{m'}{r} \in S_\sigma \cap (-S_\sigma)$ which implies that $m'/r = 0$, so $m' = 0$ and with this we are done. ☐

We have from Chapter 4, in particular by using it for the dual $\sigma^\vee$ in place of σ and vice versa (since $(\sigma^\vee)^\vee = \sigma$), that

$$\sigma^\vee \cap (-\sigma^\vee) = \{0\} \iff \dim \sigma = \dim M_\mathbb{R} = \dim N_\mathbb{R}$$

so with this we have proven the first statement of the exercise description.
Furthermore, part (a) of the proposition referenced above states that in this case, where S_σ is pointed, the unique fixed point of U_σ under the action of its torus is given by the affine semigroup homomorphism

$$\gamma : S_\sigma \to \mathbb{C}$$

$$m \mapsto \begin{cases} 1, \ m = 0 \\ 0, \ m \neq 0 \end{cases} .$$

We then want to calculate the kernel of the corresponding semigroup algebra homomorphism. To do this we use exercise 1:

$$\mathbb{C}[S] \to \mathbb{C}$$

$$x \mapsto \begin{cases} 1, \text{ if } x \text{ equals } \chi^0 \\ 0, \text{ otherwise} \end{cases}$$

With this description of the induced semigroup algebra homomorphism it becomes clear that the kernel is given by

$$\langle \chi^m \mid m \in S_\sigma \text{ and } m \neq 0 \rangle = \langle \chi^m \mid m \in S_\sigma \setminus \{0\} \rangle,$$

so we are done.

Solution 5.3. The Hilbert basis is defined as

$$\mathscr{H} = \{m \in S_\sigma : \ m \text{ irreducible}\}.$$

where m is irreducible if there exist no $m_1, m_2 \in S_\sigma \setminus \{0\}$ such that $m_1 + m_2 = m$.
We begin by calculating the dual $\sigma^\vee$:

$$\begin{bmatrix} d \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff dx - y = 0 \iff dx = y,$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff y = 0$$

so

$$\sigma^\vee = \mathrm{Span}_{\mathbb{R}_{\geq 0}} \left\{ \begin{bmatrix} 1 \\ d \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \mathrm{Span}_{\mathbb{R}_{\geq 0}} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ d \end{bmatrix} \right\}.$$

With this we get that when intersecting with M we get

$$S_\sigma = \mathrm{Span}_{\mathbb{Z}_{\geq 0}} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ d \end{bmatrix} \right\}.$$

so it is clear that the smallest choice of irreducible elements spanning M are given by

$$\mathscr{H} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \dots, \begin{bmatrix} 1 \\ d \end{bmatrix} \right\}.$$

For the next part, we have stated in Chapter 5 that $\dim T_{p_\sigma}\mathcal{U}_\sigma \leq \ell$ if $\mathcal{U}_\sigma \to \mathbb{C}^\ell$ is any embedding, where p_σ is the unique fixed point of the torus action on $\mathcal{U}_\sigma$. A more general version of this statement is given in Lemma 1.0.6 of .
The last lemma of lecture states that $\dim T_{p_\sigma}\mathcal{U}_\sigma = |\mathscr{H}|$. Since we already know that we can embed $\hat{\mathbb{C}}_d$ into $\mathbb{C}^{d+1}$, we then get that the minimal dimension ℓ of affine space in which we can embed $\hat{\mathbb{C}}_d$ is $|\mathscr{H}| = d + 1$.

Solution 5.4. An affine semigroup $S \subseteq M$ is saturated by definition if for any $k \in \mathbb{N} \setminus \{0\}$ and any $m \in M$, $km \in S$ implies $m \in S$.
For $S = \mathbb{N}\mathcal{A}$, and by the semigroup structure of any affine semigroup S, we see that S is saturated if and only if for every point $m \in \mathbb{N}\mathcal{A}$, the $\mathbb{R}_{\geq 0}$-span of m inside $M_\mathbb{R}$ intersected with M occurs in $\mathbb{N}\mathcal{A}$. Since M is a lattice and by definition of the cone $\mathrm{Cone}(\mathcal{A})$, we see that this is equivalent to

$$\mathbb{N}\mathcal{A} = \mathrm{Cone}(\mathcal{A}) \cap M,$$

since $\mathrm{Cone}(\mathcal{A})$ collects $\mathbb{R}_{\geq 0}$-linear combinations of the elements of $\mathcal{A}$ and M only contains points with integer coordinates (and additionally, since $\mathbb{N}\mathcal{A} \subseteq M$).

15.6 Solutions to Chapter 6

Solutions written by Juan Felipe Celis Rojas

Solution 6.1. Let V_1, V_2 be two affine toric varieties. Recall that we proved that an affine variety morphism is toric iff the corresponding $\mathbb{C}$-algebra homomorphism is induced by a semi-group homomorphism iff the morphism can be restricted to the tori inside the varieties and this restriction is a group homomorphism.
First assume that the morphism $\varphi : T_{N_1} \to T_{N_2}$ extends to a toric morphism $\varphi : U_{\sigma_1} \to U_{\sigma_2}$. We have the following diagram.

$$\begin{array}{ccc} T_{N_1} & \xrightarrow{\varphi} & T_{N_2} \\ \downarrow & & \downarrow \\ U_{\sigma_1} & \dashrightarrow & U_{\sigma_2} \end{array}$$

Now since $Spec(-)$ is a contravariant functor the diagram above is equivalent to the diagram

$$\begin{array}{ccc} \mathbb{C}[M_1] & \longleftarrow & \mathbb{C}[M_2] \\ \uparrow & & \uparrow \\ \mathbb{C}[S_{\sigma_1}] & \dashleftarrow & \mathbb{C}[S_{\sigma_2}] \end{array}$$

where the map in the bottom $\mathbb{C}[S_{\sigma_2}] \to \mathbb{C}[S_{\sigma_1}]$ is induced by an affine semi-group homomorphism, because $U_{\sigma_1} \to U_{\sigma_2}$ is a toric morphism.
Then all the morphisms in this diagram are entirely determined by semi-group homomorphisms. Observe that this is true only if $U_{\sigma_1} \to U_{\sigma_2}$ is toric. So this diagram is equivalent to

$$\begin{array}{ccc} M_1 & \longleftarrow & M_2 \\ \uparrow & & \uparrow \\ S_{\sigma_1} & \dashleftarrow & S_{\sigma_2} \end{array}$$

Observe that applying functor $- \otimes_{\mathbb{Z}} \mathbb{R}$ gives us an equivalence of diagrams because S_{σ_1} and S_{σ_2} are always saturated. Recall σ_1 and σ_2 are strongly convex rational polyhedral cones, and we have seen that this is equivalent to $U_{\sigma_1}, U_{\sigma_2}$ being normal, and equivalently $S_{\sigma_1}, S_{\sigma_2}$ being saturated.

$$\begin{array}{ccc} (M_1)_{\mathbb{R}} & \longleftarrow & (M_2)_{\mathbb{R}} \\ \uparrow & & \uparrow \\ \sigma_1^\vee & \dashleftarrow & \sigma_2^\vee \end{array}$$

Then we can dualize to get another equivalent diagram

$$\begin{array}{ccc} (N_1)_{\mathbb{R}} & \xrightarrow{\overline{\varphi}_{\mathbb{R}}} & (N_2)_{\mathbb{R}} \\ \downarrow & & \downarrow \\ \sigma_1 & \dashrightarrow & \sigma_2 \end{array}$$

yielding $\overline{\varphi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$.
First we denote the map

$$\Phi : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1} : \underline{a} \mapsto (f_0(\underline{a}), \ldots, f_m(\underline{a}))$$

we want to show that it induces a map from V to $\mathbb{P}^m$. Let $[\underline{a}] = [a_0 : \cdots : a_n] \in V$. Since

$$V \cap \mathbb{V}_p(f_0, \ldots, f_m) = \emptyset$$

it follows that for any representative $\underline{a} \in \mathbb{C}^{n+1}$ of $[\underline{a}] \in V$ we get

$$\Phi(\underline{a}) \neq \underline{0}$$

because
$$\Phi(\underline{a}) = \underline{0} \iff \underline{a} \in \mathbb{V}_p(f_0, \ldots, f_m).$$
Therefore $\Phi(\underline{a})$ defines an element $[\Phi(\underline{a})] \in \mathbb{P}^m$.
As the polynomials $f_0, \ldots, f_m$ are homogeneous the choice of representative $\underline{a} \in \mathbb{C}^{n+1}$ of $[\underline{a}] \in V$ does not change the class of $\Phi(\underline{a})$ in $\mathbb{P}^m$. Whence the map
$$\Phi : V \to \mathbb{P}^m : [\underline{a}] \mapsto [f_0(\underline{a}) : \cdots : f_m(\underline{a})]$$
is well defined.

Solution 6.2.

(i) To show that the Segre embedding is indeed an embedding we must show that it is injective, continuous and closed. Observe that it is continuous because it is defined by polynomials. Moreover, *exercise 4* implies that the Segre embedding is closed. It remains to show that it is injective.
Let $([\underline{a}], [\underline{b}]), ([\underline{a}'], [\underline{b}']) \in \mathbb{P}^n \times \mathbb{P}^m$ be such that
$$\sigma_{n,m}([\underline{a}], [\underline{b}]) = \sigma_{n,m}([\underline{a}'], [\underline{b}'])$$
then there is $\lambda \in \mathbb{C}^*$ such that
$$\lambda a_i b_j = a_i' b_j' \quad \forall\ 0 \leq i \leq n, 0 \leq j \leq m.$$
Observe that there must exist $0 \leq k \leq n, 0 \leq l \leq m$ such that $a_k' \neq 0$ and $b_l' \neq 0$. Then we get
$$a_i' = \frac{\lambda b_l}{b_l'} a_i \quad \forall\ 0 \leq i \leq n$$
$$b_j' = \frac{\lambda a_k}{a_k'} b_j \quad \forall\ 0 \leq j \leq m.$$
Therefore $[\underline{a}] = [\underline{a}']$ and $[\underline{b}] = [\underline{b}']$. In other words $\sigma_{n,m}$ is injective.

(ii) Now let I be the ideal generated by
$$\{z_{ij}z_{kl} - z_{il}z_{kj} \quad | \quad 0 \leq i \leq n, 0 \leq j \leq m\}.$$
We denote
$$U_i = \mathbb{P}^n \backslash \mathbb{V}_p(x_i) \subset \mathbb{P}^n$$
$$V_j = \mathbb{P}^m \backslash \mathbb{V}_p(y_j) \subset \mathbb{P}^m$$
$$W_{ij} = \mathbb{P}^{nm+n+m} \backslash \mathbb{V}_p(z_{ij}) \subset \mathbb{P}^{nm+n+m}$$
for all $0 \leq i \leq n, 0 \leq j \leq m$. Notice that $\{U_i\}, \{V_j\}, \{W_{ij}\}$ form open covers for $\mathbb{P}^n, \mathbb{P}^m$ and $\mathbb{P}^{nm+n+m}$ respectively. We claim that
$$\sigma_{n,m}(U_i \times V_j) = \mathbb{V}_p(I) \cap W_{ij}.$$
Let $([\underline{a}], [\underline{b}]) \in U_i \times V_j$, then
$$\sigma_{n,m}([\underline{a}], [\underline{b}])_{ij = a_i b_j \neq 0}$$
and
$$(z_{rs}z_{kl} - z_{rl}z_{ks})([\underline{a}], [\underline{b}]) = a_r b_s a_k b_l - a_r b_l a_k b_s = 0.$$
It follows that
$$\sigma_{n,m}(U_i \times V_j) \subseteq \mathbb{V}_p(I) \cap W_{ij}.$$
Now let $[\underline{z}] \in \mathbb{V}_p(I) \cap W_{ij}$. Then define
$$a_k = z_{kj} \quad \forall\ 0 \leq k \leq n$$
$$b_l = \frac{z_{il}}{z_{ij}} \quad \forall\ 0 \leq l \leq m.$$

Observe that this is well defined because $z_{ij} \neq 0$. Moreover $[\underline{a}] \in U_i$ and $[\underline{b}] \in V_j$. It remains to see that it is a pre-image of $[\underline{z}]$ under the Segre embedding. Indeed

$$\sigma_{n,m}([\underline{a}],[\underline{b}])_{kl} = a_k b_l = z_{kj}\frac{z_{il}}{z_{ij}} = z_{kl}$$

because $[\underline{z}] \in \mathbb{V}_p(I)$ implies

$$z_{ij}z_{kl} - z_{il}z_{kj} = 0.$$

Hence we have proven that $\sigma_{n,m}(U_i \times V_j) = \mathbb{V}_p(I) \cap W_{ij}$. Now it is enough to see the following:

$$\begin{aligned}
\sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) &= \sigma_{n,m}\left(\bigcup_{i,j} U_i \times V_j\right) \\
&= \bigcup_{i,j} \sigma_{n,m}(U_i \times V_j) \\
&= \bigcup_{i,j} \mathbb{V}_p(I) \cap W_{ij} \\
&= \mathbb{V}_p(I) \cap \bigcup_{i,j} W_{ij} \\
&= \mathbb{V}_p(I) \cap \mathbb{P}^{nm+n+m} \\
&= \mathbb{V}_p(I).
\end{aligned}$$

Solution 6.3. Let $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ defined by $f_l(x,y) = 0$ where f_l is bihomogeneous of bidegree (a_l, b_l) for $l = 0, ..., s$. The goal of this exercise is to show that V can be viewed as a projective variety of $\mathbb{P}^{nm+n+m}$ via the Segre embedding.

(i) For each l, consider $d_l \geq \max\{a_l, b_l\}$ and $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^n$, $\beta = (\beta_0, \ldots, \beta_n) \in \mathbb{N}^m$ be such that

$$\sum_{i=0}^{n} \alpha_i = d_l - a_l$$
$$\sum_{j=0}^{m} \beta_j = d_l - b_l.$$

Define the bihomogeneous polynomial

$$g_{l,\alpha,\beta}(x,y) = x^\alpha y^\beta f_l(x,y).$$

Note that this polynomial is bihomogeneaous of degree (d_l, d_l). We will show that

$$V = \mathbb{V}_p\left(g_{l,\alpha,\beta} \,|\, l \in \{0,\ldots,s\}, \alpha, \beta\right) =: V'.$$

It is clear that V is included in the vanishing locus of the $g_{l,\alpha,\beta}$'s by definition of these polynomials. It remains to show the other inclusion. Let $(p,q) \in V'$. Since $p \in \mathbb{P}^n$ and $q \in \mathbb{P}^m$ there exist $0 \leq i \leq n, 0 \leq j \leq m$ such that $p_i \neq 0$ and $q_j \neq 0$. Moreover

$$g_{l,\alpha,\beta}(p.q) = 0$$

for all choices of l, α, β. In particular it is true for α and β with $\alpha_i = d_l - a_l$ and $\beta_j = d_l - b_l$. Thus we have

$$p_i^{d_l - a_l} q_j^{d_l - b_l} f_l(p,q) = 0$$

which implies that

$$f_l(p,q) = 0$$

for all $l \in \{0, \ldots, s\}$. Whence

$$V = V'.$$

(ii) We want to show that $\sigma_{n,m}(V)$ is a projective sub-variety of $\mathbb{P}^{nm+n+m}$.

Let us use part (a). Observe that the polynomials $g_{l,\alpha,\beta}$ are bihomogeneous of bidegree (d_l, d_l). So we can view them as follows:

$$g_{l,\alpha,\beta} \in (\mathbb{C}[z_0, \dots, z_{nm+n+m}])_{2d_l}.$$

Then we have

$$\sigma_{n,m}(V) = \mathbb{V}_p(g_{l,\alpha,\beta} \,|\, l \in \{0, \dots, s\}, \alpha, \beta).$$

It follows that $\sigma_{n,m}(V)$ is indeed a closed sub-variety of $\mathbb{P}^{nm+n+m}$. In particular the Segre embedding is a closed map.

15.7 Solutions to Chapter 7

Solutions written by Zichen Gao

Solution 7.1.

(a) First of all, in any abelian category $\mathscr{A}$, and an object M in $\mathscr{A}$, the functor $\mathrm{Hom}(\cdot, M)$ is a left exact functor. In other words, if

$$N_1 \to N_2 \to N_3 \to 0$$

is exact, then

$$0 \to \mathrm{Hom}(N_3, M) \to \mathrm{Hom}(N_2, M) \to \mathrm{Hom}(N_1, M)$$

is exact. The category of abelian affine group schemes over $\mathbb{C}$ is abelian, so here we can apply the above result to the exact sequence $T \to T' \to T'' \to 0$, and $M = \mathbb{C}^*$. Now we only need to prove that for an injective morphism $\alpha : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$, it induces a surjection $\alpha^* : \mathrm{Hom}((\mathbb{C}^*)^m, \mathbb{C}^*) \to \mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*)$.

Recall that $(\mathbb{C}^*)^n \simeq \mathrm{Spec}(\mathbb{C}[\mathbb{Z}^n])$ and $(\mathbb{C}^*)^m \simeq \mathrm{Spec}(\mathbb{C}[\mathbb{Z}^m])$. So the morphism $\alpha : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ induces a map $\alpha^\vee : \mathbb{C}[\mathbb{Z}^m] \to \mathbb{C}[\mathbb{Z}^n]$. And since α is toric, $\alpha^\vee$ restricts to a morphism of lattices $\mathbb{Z}^m \to \mathbb{Z}^n$, which corresponds to the homomorphism $\alpha^* : \mathrm{Hom}((\mathbb{C}^*)^m, \mathbb{C}^*) \to \mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*)$. So we only need to show the surjectivity of $\mathbb{C}[\mathbb{Z}^m] \to \mathbb{C}[\mathbb{Z}^n]$. In fact, since α is a morphism of tori, its image is a close subgroup of the targeting torus, and α induces an isomorphism onto its image. In particular, topologically, α is a homeomorphism onto a closed subset of the targeting space. Combining with the fact that all the involved schemes are affine schemes, we know that α is a closed immersion of affine schemes. Hence $\mathbb{C}[\mathbb{Z}^m] \to \mathbb{C}[\mathbb{Z}^n]$ is surjective.

(b) For the first part, tensoring with $\mathbb{Q}$ is the same as localizing at $\mathbb{Z} \setminus \{0\}$, and localization is an exact functor. For the second part, taking dual is an exact functor in the category of $\mathbb{Q}$-vector spaces.

Solution 7.2.

(a) $\mathbb{Z}'\mathcal{A}$ is a subgroup of the lattice M, hence is still a lattice.

(b) The smallest affine subspace containing $\mathcal{A}$ is the affine space $H = m_1 + \sum \mathbb{R}(m_i - m_1)$. It is also easy to verify that $\mathbb{Z}'\mathcal{A} = \sum \mathbb{Z}(m_i - m_1)$, since the sum of the coefficients of the m_i's is zero. So $\dim H = \dim_{\mathbb{R}} \sum \mathbb{R}(m_i - m_1) = \dim_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}'\mathcal{A} = \mathrm{rank}\ \mathbb{Z}'\mathcal{A}$.

(c) If $\exists u \in N$, $k \in \mathbb{N} \setminus \{0\}$, s.t. $\langle m_i, u \rangle = k$ for each i, then $\langle \sum_{i=1}^s a_i m_i, u \rangle = k(\sum_{i=1}^s a_i)$, which gives us the exact sequence

$$0 \longrightarrow \mathbb{Z}'\mathcal{A} \longrightarrow \mathbb{Z}\mathcal{A} \xrightarrow{\langle\ ,u\rangle} k\mathbb{Z} \longrightarrow 0$$

Then $k > 0$ implies that $\mathrm{rank}\mathbb{Z}\mathcal{A} - 1 = \mathrm{rank}\,\mathbb{Z}'\mathcal{A}$.

If there isn't an $u \in N$ and $k \in \mathbb{N} \setminus \{0\}$, s.t. $\langle m_i, u \rangle = k$ for each i, then by Proposition 2.1.4 of [CLS], I_L is not homogeneous, where L is the kernel in the exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \longrightarrow \mathbb{Z}\mathcal{A}$$

where the last map sends e_i to m_i, and $I_L = \langle x^\alpha - x^\beta | \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L\rangle$. Since I_L is not homogeneous, some generator $x^\alpha - x^\beta$ is not homogeneous, so that $(\alpha-\beta)\cdot(1,\ldots,1) \neq 0$, but $\alpha - \beta \in L$. This implies that in the exact sequence

$$0 \longrightarrow \mathcal{M}_{s-1} \longrightarrow \mathbb{Z}^s \xrightarrow{\cdot(1,\ldots,1)} \mathbb{Z} \longrightarrow 0$$

the image of $L \subseteq \mathbb{Z}^s$ is $l\mathbb{Z} \subseteq \mathbb{Z}$ for some $l > 0$. This gives a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L \cap \mathcal{M}_{s-1} & \longrightarrow & L & \longrightarrow & l\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_{s-1} & \longrightarrow & \mathbb{Z}^s & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}'\mathcal{A} & \longrightarrow & \mathbb{Z}\mathcal{A} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. The exactness of the columns and the first two rows is clear. The exactness of the third row can be shown by five-lemma. Hence $\operatorname{rank} \mathbb{Z}\mathcal{A} = \operatorname{rank} \mathbb{Z}'\mathcal{A}$.

Solution 7.3.

(a) Suppose $\operatorname{rank} M = n$, and $\mathcal{A} = \{m_1, \ldots, m_s\}$. Then the projective variety induced by $\mathcal{A} + m$ is the closure in $\mathbb{P}^{s-1}$ of the set

$$\{[\chi^{m_1+m}(t_1,\ldots,t_n) : \cdots : \chi^{m_s+m}(t_1,\ldots,t_n)] | t_i \in \mathbb{C}^*\}$$

but

$$\begin{aligned}
&[\chi^{m_1+m}(t_1,\ldots,t_n) : \cdots : \chi^{m_s+m}(t_1,\ldots,t_n)] \\
=&[\chi^{m_1}(t_1,\ldots,t_n)\cdot\chi^{m}(t_1,\ldots,t_n) : \cdots : \chi^{m_s}(t_1,\ldots,t_n)\cdot\chi^{m}(t_1,\ldots,t_n)] \\
=&[\chi^{m_1}(t_1,\ldots,t_n) : \cdots : \chi^{m_s}(t_1,\ldots,t_n)]
\end{aligned}$$

The closure of all these latter points is the projective toric variety induced by $\mathcal{A}$, so $\mathcal{A}$ and $\mathcal{A} + m$ induce the same projective toric variety.

(b) Let $\mathcal{A} = \{0, 1\} \subseteq \mathbb{Z}$. Then the affine toric variety induced by $\mathcal{A}$ is the Zariski closure in $\mathbb{C}^2$ of the set $\{(1, t) | t \in \mathbb{C}^*\}$, hence is a copy of $\mathbb{C}$. Let $m = 2$, then $m + \mathcal{A} = \{2, 3\}$, and the affine toric variety induced by $m + \mathcal{A}$ is the curve $y^2 - x^3 = 0$, which is not isomorphic to $\mathbb{C}$, since it is not smooth, for example.

Solution 7.4.

(a) For an element $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$, let's denote the assigned permutation matrix by M_{ijk}. We explain here why $M_{123} + M_{231} + M_{312} = M_{132} + M_{321} + M_{213}$. For the three matrices M_{123}, M_{231} and M_{312}, since the corresponding permutations send i to the three different elements , the place of the 1 in the i-th rows of these matrices should be different. Hence the i-th row of their sum should be $(1, 1, 1)$. More concretely,

$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, M_{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

so

$$M_{123} + M_{231} + M_{312} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The same is true for the other three matrices:

$$M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, M_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, M_{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so

$$M_{132} + M_{321} + M_{213} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

In summary, $M_{123} + M_{231} + M_{312} = M_{132} + M_{321} + M_{213}$ Therefore,

$$\chi^{M_{123}} \cdot \chi^{M_{231}} \cdot \chi^{M_{312}} = \chi^{M_{123}+M_{231}+M_{312}} = \chi^{M_{132}+M_{321}+M_{213}} = \chi^{M_{132}} \cdot \chi^{M_{321}} \cdot \chi^{M_{213}}$$

This implies that $x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213} \in \mathbf{I}(X_{\mathcal{P}_3})$.

(b) It is straightforward to compute that $\mathbb{Z}'\mathcal{P}_3$ is generated over $\mathbb{Z}$ by

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

And it's clear that these generators are linearly independent, so $\operatorname{rank} \mathbb{Z}'\mathcal{P}_3 = 4 = \dim X_{\mathcal{P}_3}$

(c) By (a), $X_{\mathcal{P}_3} \subseteq \mathbf{V}(x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213})$. The latter is an irreducible hypersurface, hence of dimension 4. But $X_{\mathcal{P}_3}$ is also an irreducible variety of dimension 4, so $X_{\mathcal{P}_3} = \mathbf{V}(x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213})$. In particular, $\mathbf{I}(X_{\mathcal{P}_3}) = \langle x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213}\rangle$.

15.8 Solutions to Chapter 8

Solutions written by Maxence Alexandre Coppin.

Solution 8.1. Let P be a polytope.
"$\Rightarrow$" Assume that P is not full dimensional. Let F be a facet of P with a supporting hyperplane $H_{u,a}$. Since P is not full dimensional, there exists $n \in (\operatorname{Span}(P)^\vee)^\perp$. Consider the hyperplane $H_{n+u,a}$, we want to prove that it is a suupporting hyperplane of F different from $H_{u,a}$. We have

$$H_{n+u,a} \cap P = \{m \in P \mid \langle n+u, m\rangle = a\} = \{m \in P \mid \langle u, m\rangle = a\}$$
$$= H_{u,a} \cap P = F,$$

where the second equality holds because $\langle n, m\rangle = 0$ since $n \in (\operatorname{Span}(P)^\vee)^\perp$. Now suppose that these define the same supporting hyperplane, then for every $m \in M \setminus \operatorname{Span}(P)$, we have $\langle n+u, m\rangle = a = \langle u, m\rangle$, hence $\langle n, m\rangle = 0$ for any m. Since this paring is non-degenerate we must have $n = 0$, we get a contradiction.
"$\Leftarrow$" Suppose that P is full dimensional. Let F be a facet of P and $H_{u,a}$, $H_{v,b}$ be two supporting hyperplane of F. Since P is full dimensional, the facet F is a polytope of dimension $\dim P - 1$, by definition it means that F is contained in an affine subspace of dimension $\dim P - 1$. But since $H_{u,a} \cap P = H_{v,b} \cap P = F$, the intersection $H_{u,a} \cap H_{v,b} \neq \emptyset$ and it must have dimension at least $\dim P - 1$, hence they must be equal.

Solution 8.2. Let P be a full dimensional polytope of dimension d with the origin as interior point.

(i) Notice that since the origin is an interior point of P, we have that for any facet F of P, $0 = \langle u_F, 0\rangle > -a_F$, hence $a_F > 0$.

" $\supseteq$ " Let $m \in P$, then for any F we have that

$$\langle u_F, m\rangle \geq -a_F \Leftrightarrow \langle \frac{1}{a_F} u_F, m\rangle \geq -1.$$

Thus $1/a_F u_F \in P^\circ$ for any facet F, then $\mathrm{Conv}(1/a_F u_F \mid F \text{ facet of } P) \subseteq P^\circ$.

" $\subseteq$ "

(ii) Since P is full-dimensional, each facet F of P has a unique supporting hyperplane $H_{u_F,-a_F}$. We have a bijection between the set of facets of P and vertices of P° given by

$$F \mapsto \frac{1}{a_F} u_F$$

This bijection induces a bijection between the set of faces of P and faces of P°. Indeed, for Q a facet of two facets of F, F' of P, the intersection of the two supporting hyperplanes associated to these facets is exactly $\mathrm{Span}(Q)$, thus Q correpond s uniquely to the edge between the points $1/a_F u_F$ and $1/a_{F'} u_{F'}$. And we can pursuit this construction inductively to obtain the wanted bijection.

It is immediate by construction that a face of P with dimension n is send to a face of P° with dimension $d - n - 1$ by this bijection and it is inclusion reversing.

(iii) Let $r > 0$, we want to prove that $(rP)^\circ = 1/rP^\circ$.

$$\begin{aligned}(rP)^\circ &= \{u \in N \mid \langle rm, u\rangle \geq -1 \text{ for all } m \in P\} \\ &= \{u \in N \mid \langle m, ru\rangle \geq -1 \text{ for all } m \in P\} \\ &= \{\frac{1}{r} u \in N \mid \langle m, u\rangle \geq -1 \text{ for all } m \in P\} = \frac{1}{r} P^\circ.\end{aligned}$$

Now if we take $2P$ where $P = \mathrm{Conv}(\pm e_1, \pm e_2 \subseteq \mathbb{R}^2$, we have that $(2P)^\circ = \frac{1}{2} P^\circ$ is well-defined, but it is not a lattice polytope.

Solution 8.3. Let P be a polytope.
"$\Rightarrow$" Assume that P is normal. Let $(m, k) \in C(P) \cap (M \times \mathbb{Z})$. Then $k \in \mathbb{Z}$ and the element m has height k, and by normality of P, we have that $m = \sum_{i=1}^k m_i$ for some $m_i \in P \cap M$. Thus we have that $(m, k) = \sum_{i=1}^k (m_i, 1)$, hence $(P \cap M) \times \{1\}$ generates the semigroup $C(P) \cap M \times \mathbb{Z}$.
"$\Leftarrow$" Assume that $(P \cap M) \times \{1\}$ generates $C(P) \cap (M \times \mathbb{Z})$. We always have the inclusion $P \cap M + \ldots + P \cap M \subseteq kP \cap M$. So let $m \in kP \cap M$, notice that $(m, k) \in C(P) \cap (M \times \mathbb{Z})$. By assumption we have that $(m, k) = \sum_{i=1}^k (m_i, 1)$ where $m_i \in P \cap M$, hence $m = \sum_{i=1}^k m_i \in P \cap M + \ldots + P \cap M$.

Solution 8.4. Consider $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$.

(i) Let $m \in P \cap \mathbb{Z}^3$, then $m = \alpha e_1 + \beta e_2 + \gamma(e_1 + e_2 + 3e_3) = (\alpha + \gamma)e_1 + (\beta + \gamma)e_2 + 3\gamma e_3$ where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$. If $\alpha = \beta = \gamma = 0$, we have that $m = 0 \in P$. Else, it yields three cases :

- If $\gamma = 0$, then $m = \alpha e_1 + \beta e_2$ whith $\alpha + \beta = 1$. If $\alpha = 0$, then $m = e_2$. If $\beta = 0$, then $m = e_1$.
- If $\gamma = 1$, then $\alpha = \beta = 0$, so $m = e_1 + e_2 + 3e_3$.
- If $\gamma = 1/3$, then $\alpha, \beta \geq 2/3$. Since $\alpha + \beta = 2/3$, we have that m cannot lies in $\mathbb{Z}^3$.

Thus the only lattice points of P are its vertices.

(ii) We have that $T_{P\cap\mathbb{Z}^3} \subseteq \mathbb{P}^3$ since $P \cap \mathbb{Z}^3$ has only four points. Furthermore $\dim T_{P\cap\mathbb{Z}^3} = \mathrm{rk}(\mathbb{Z}'\{0, e_1, e_2, e_1 + e_2 + 3e_3\}) = \mathrm{rk}(\mathbb{Z}\{0, e_1, e_2, e_1 + e_2 + 3e_3\}) = \mathrm{rk}(\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}(e_1 + e_2 + 3e_3)) = 3$ since $0 \in P \cap \mathbb{Z}^3$. Thus we have $T_{P\cap\mathbb{Z}^3} = \mathbb{P}^3$ because $\dim \mathbb{P}^3 = 3$.

(iii) It is just some computation to show that it is indeed the Hilbert basis of $C(P)$. Since the Hilbert basis of $C(P) \cap (\mathbb{Z}^3 \times \mathbb{Z})$ contains 6 elements, the set $(P \cap \mathbb{Z}^3) \times \{1\}$ cannot generates $C(P) \cap (\mathbb{Z}^3 \times \mathbb{Z})$ since it has less elements than 6 (it would contradicts the minimality of the Hilbert basis).

15.9 Solutions to Chapter 9

Solutions written by Emma Marie Billet

Solution 9.1. First $P \subseteq M_\mathbb{R}$ being very ample, it must be a lattice polytope so $P \cap M = \{m_1, ..., m_s\}$ are the vertices of P, and we may assume $m = m_s$ without loss of generality.

(i) Note that 0 is indeed in C because $m \in P \cap M$ by very ampleness. To show that $H_{u,0}$ is a supporting hyperplane of $0 \in C$ is equivalent to prove that $\{0\} = H_{u,0} \cap C$ and $u \in C^\vee = \sigma_m$.

"$\subseteq$" This inclusion is trivial because we just proved $0 \in C$ and clearly $\langle u, 0\rangle = 0$.

"$\supseteq$" Consider $m' \in C \cap H_{u,0}$, then $m' \in C$

$\implies m' = \sum_{i=1}^{s-1} \lambda_i(m_i - m)$ where $\lambda_i \geq 0, i \in \{1, ..., s-1\}$. And $m' \in H_{u,0} \implies \langle m', u\rangle = \sum_{i=1}^{s-1} \lambda_i \langle m_i - m, u\rangle = 0$, but this is equivalent to

$$\sum_{i=1}^{s-1} \lambda_i \langle m_i, u\rangle = \sum_{i=1}^{s-1} \lambda_i \langle m, u\rangle \tag{13}$$

Now we can use the assumption that $H_{u,a}$ is a supporting hyperplane of $m \in P$ which implies that $\{m\} = P \cap H_{u,a}$ and $P \subseteq H^+_{u,a}$. From these we obtain $\langle m, u\rangle = a$, $\langle m_i, u\rangle \neq a$, $\langle m_i, u\rangle \geq a$ so $\langle m_i, u\rangle > a$, $\forall i \in \{1, ...s-1\}$. Combining this with with (1) gives a contradiction, unless $\lambda_i = 0, \forall i \in \{1, ..., s-1\}$ or equivalently $m' = 0$ as desired.

Finally we have to prove, $u \in \sigma_m$ which is equivalent to $C \subseteq H^+_{u,0}$. Consider $\sum_{i=1}^{s} \lambda_i(m_i - m) \in C$ for some $\lambda_i \geq 0 \; \forall \; i \in \{1, ..., s\}$, denote $\sum_{i=1}^{s} \lambda_i =: \lambda > 0$ unless we consider 0. Then $x = \sum_{i=1}^{s} \frac{\lambda_i}{\lambda} m_i \in P$ which implies that we have $x \in H^+_{u,a} \implies \langle \sum_{i=1}^{s} \lambda_i m_i, u\rangle \geq a \implies \sum_{i=1}^{s} \lambda_i \langle m_i, u\rangle \geq \lambda a = \lambda\langle m, u\rangle$ by hypothesis on $m \in P$. This implies $\langle \sum_{i=1}^{s} \lambda_i(m_i - m), u\rangle \geq 0$ as desired. Remark that if we consider 0, clearly $0 \in H^+_{u,0}$. This finishes the proof of part (a).

(ii) Prove that $dim(C) = dim(P) = n$. Because P is full dimensional the only thing to prove is $dim(C) \geq dim(P)$, recall that the dimension of C, P is the dimension of the smallest subspace of $M_\mathbb{R}$ containing C and P respectively. Thus it suffices to prove that $C \subseteq H \implies P \subseteq H$ for any affine subspace $H \subseteq M_\mathbb{R}$. Assume $C \subseteq H$ for some subspace H. Then if we show $m_i \in H, \forall i \in \{1, ..., s\}$

$\implies P = Conv(m_i | i \in \{1, ..., s\}) \subseteq H$.

But using that $m_i - m \in C \subseteq H, i \in \{1, ..., s-1\}$, we are done if we prove $m \in H$.

Claim: Wlog, $m \in C = C_m \subseteq H$. This finishes the proof.
Proof of the claim: it always exists a translation of P under which $m \in C$. The dimension of a full dimension polytope is invariant under translation. □

Solution 9.2.

(i) P and $m + P$ have the same normal fan for $m \in M$ a lattice point. Note that $P + m$ is still a full dimensional lattice polytope as m is a lattice point and dimension is invariant under translation. We clearly have a bijective correspondence φ between faces of P and faces of $P + m$, given by $Q \preccurlyeq P \mapsto Q + m \preccurlyeq P + m$. Note that φ preserved face inclusion. Let's prove for $Q \preccurlyeq P$, that $\sigma_Q = \sigma_{\varphi(Q)}$.

Using $\sigma_Q = Cone(u_F | Q \preccurlyeq F \preccurlyeq P,\ F \text{ facet })$, it suffices to prove $u_F = u_{\varphi(F)}$ for any facet F. Indeed $u_F, u_{\varphi(F)}$ are uniquely defined (up to multiplication by a positive real number) because, P and $P + m$ are full dimensional.

Let $F \preccurlyeq P \implies P \subseteq H^+_{u_F, a_F} \implies \forall m \in P : \langle m, u_F\rangle \geq -a_F$.

Then consider some $\tilde{m} \in P + m \implies \tilde{m} = n + m \implies \langle \tilde{m}, u_F\rangle = \langle n, u_F\rangle + m_F \geq -(a_F - m_F) =: -a_{F+m}$ for $m_F := \langle m, u_F\rangle$ and using $n \in P$. This directly proves $P + m \subseteq H^+_{u_F, -a_{F+m}}$. Moreover, considering $\tilde{m} \in F + m$ makes all inequalities above being equalities, and this proves $F + m = H_{u_F, -a_{F+m}} \cap (P + m)$. Indeed for the other inclusion we have $\langle n + m, u_F\rangle = -(a_F - m_F) \implies \langle m, u_F\rangle = -a_F \implies m \in P \cap H_{u_f, a_F} = F$.

(ii) We basically use the same strategy, P and kP have the same normal fan for $k \geq 1$ an integer. Note that kP is still a full dimensional lattice polytope as k is an integer and dimension is invariant under homothety. Indeed the basis of the subspace containing P will still generate kP since a generating set allows multiplication by a scalar. We clearly have a bijective correspondence φ between faces of P and faces of kP, given by $Q \preccurlyeq P \mapsto kQ \preccurlyeq kP$. Also note that φ preserved face inclusion. Let's prove for $Q \preccurlyeq P$, that $\sigma_Q = \sigma_{\varphi(Q)}$.

Using $\sigma_Q = Cone(u_F | Q \preccurlyeq F \preccurlyeq P, F \text{ facet of } P)$, it suffices to prove $u_F = u_{\varphi(F)}$. Indeed, in this case they generate the same cone.

Let $F \preccurlyeq P \implies P \subseteq H^+_{u_F, a_F} \implies \forall m \in P : \langle m, u_F \rangle \geq -a_F$.

Then consider some $\tilde{m} \in kP \implies \tilde{m} = k.m \implies \langle \tilde{m}, u_F \rangle = \langle k.m, u_F \rangle = k\langle m, u_F \rangle \geq -k.a_F =: -a_{kF}$ using $m \in P$. This directly proves $kP \subseteq H^+_{u_F, -a_{kF}}$. Moreover, considering $\tilde{m} \in kF$ makes all inequalities above being equalities, and this proves $kF = H_{u_F, -a_{kF}} \cap kP$.

Indeed for the other inclusion we have $\langle k.m, u_F \rangle = -k.a_F \implies \langle m, u_F \rangle = -a_F \implies m \in P \cap H_{u_f, a_F} = F$.

This finishes the proof of exercise 2. □

Solution 9.3. The proof of this exercise is mostly based on exercise 8.2.b. This exercise tells us that it exists a bijection between faces of the polytope and faces of the dual polytope. Since 0 is an interior point of P, which is full dimensional, we have $P = \bigcap_{F\,facet} H^+_{u_F, -a_F}$ where the $a_F > 0$ for all facets. This allows us to characterize the dual polytope as $P^\circ = Conv(\frac{1}{a_F} u_F \,|\, F \text{ facet of P})$ so that the faces of P° are the convex hull of some vertices $\{\frac{1}{a_F} u_F\}$. Hence the bijection is given by $Conv(\frac{1}{a_F} u_F \,|\, F \text{ facet of P containing } Q) \preccurlyeq P^\circ$ is associated to the face $Q \preccurlyeq P$. Denote by φ this bijection.

Then, by definition the normal fan of the polytope $P \subseteq M_\mathbb{R}$ is $\Sigma_P = \{\sigma_Q \,|\, Q \preccurlyeq P\}$.

Claim: $\sigma_Q = Cone(Q') \subseteq N_\mathbb{R}$ where the latter means the cone generated by the vertices of Q' and $Q' = \varphi(Q)$. This claim directly proves the statement of the exercise since φ is bijective and $\varphi(Q) \preccurlyeq P^\circ$.

Proof of the claim: We have $\sigma_Q = Cone(u_F \,|\, F \text{ facet containing } Q)$; and $Cone(Q') = Cone(\frac{1}{a_F} u_F \,|\, F$ facet containing $Q)$, but we have an equality between both of them since more generally $Cone(m_1, m_2) = Cone(\lambda_1 m_1, \lambda_2 m_2)$ for any $\lambda_1, \lambda_2 \geq 0$. In fact this is a property of cones and you can clearly see this taking the definition. □

Solution 9.4.

(i) We set $\{e_1, ..., e_n\}$ canonical basis of $\mathbb{R}^n$ so that $\triangle_n = Conv(0, e_1, ..., e_n)$. We denote $A = \{0, e_1, ..., e_n\}$ and $e_0 = -\sum_{i=1}^n e_i$. We need to prove the following second equality (the first is the definition of the normal fan) :

$$\Sigma_{\triangle_n} = \{\sigma_Q \,|\, Q \preccurlyeq \triangle_n\} = \{Cone(S) \,|\, S \subsetneq \{e_0, ..., e_n\}\}$$

For the sake of simplicity, we denote $\{e_0, ..., e_n\} =: E$. Note that for any face $Q \preccurlyeq \triangle_n$, $Q = Conv(A \backslash \tilde{S})$ for $\tilde{S} \subsetneq \{0, e_1, ..., e_n\}$, indeed any proper face of the n-simplex is the convex hull of at most n vertices. Thus, any facet $F \preccurlyeq \triangle_n$, is of the form $F_i = Conv(A \backslash \{e_i\})$ for $i \in \{1, ..., n\}$ or $F_0 = Conv(A \backslash \{0\})$ since a facet is determine by n vertices.

Claim: $u_{F_i} = e_i$ for $i \in \{1, ..., n\}$ and $u_{F_0} = e_0$

Proof of the claim : First of all, u_F is well defined since $\triangle_n$ is full dimensional. In order to prove this we distinguish the cases $i \in \{1, ..., n\}$ or $i = 0$.

Case 1: We need to show $F_i = \{m \in \triangle_n \,|\, \langle m, e_i \rangle = 0\}$ and $\triangle_n \subseteq H^+_{e_i, 0}$. Let $i \in \{1, ..., n\}$.

"$\subseteq$" Consider $m = \sum_{j=1}^n \lambda_j e_j \in F_i = Conv(A\backslash\{e_i\}) \implies \lambda_i = 0$, and $\sum_{j=1}^n \lambda_j \leq 1,\ \lambda_j \geq 0,\ \forall j$. Then by orthonormality of the canonical basis $\langle m, e_i\rangle = \lambda_i = 0$ and $m \in \triangle_n$as desired.

"$\supseteq$" Consider $m \in \triangle_n \implies m = \sum_{j=1}^n \lambda_j e_j, \sum_{j=1}^n \lambda_j \leq 1,\ \lambda_j \geq 0\,\forall j$. Clearly $\langle m, e_i\rangle = 0 \implies \lambda_i = 0 \implies m \in Conv(A\backslash\{e_i\}) = F_i$.

Lastly, consider $m = \sum_{j=1}^n \lambda_j e_j \in \triangle_n, \sum_{j=1}^n \lambda_j \leq 1,\ \lambda_j \geq 0\ \forall j$. Then $\langle m, e_i\rangle = \lambda_i \geq 0 \implies m \in H^+_{e_i,0}$. This finishes the proof of case 1.

Case 2: Analogously we need to show $F_0 = \{m \in \triangle_n \,|\, \langle m, e_0\rangle = -1\}$ and $\triangle_n \subseteq H^+_{e_0,-1}$.

"$\subseteq$" Consider $m = \sum_{j=1}^n \lambda_j e_j \in F_0 = Conv(A\backslash\{0\}) \implies$ so that $\sum_{j=1}^n \lambda_j = 1$, and $\lambda_j \geq 0,\ \forall j$. Then by orthonormality of the canonical basis $\langle m, e_0\rangle = -\sum_{i=1}^n \sum_{j=1}^n \lambda_j \delta_{ij} = -\sum_{i=1}^n \lambda_i = -1$ and $m \in \triangle_n$ as desired.

"$\supseteq$" Consider $m \in \triangle_n \implies m = \sum_{j=1}^n \lambda_j e_j, \sum_{j=1}^n \lambda_j \leq 1,\ \lambda_j \geq 0,\ \forall j$. Clearly $\langle m, e_0\rangle = -1 \implies -\sum_{i=1}^n \lambda_i = -1 \implies m \in Conv(e_1, ..., e_n) = Conv(A\backslash\{0\}) = F_0$.

Lastly, consider $m = \sum_{j=1}^n \lambda_j e_j \in \triangle_n, \sum_{j=1}^n \lambda_j \leq 1,\ \lambda_j \geq 0\ \forall j$. Then $\langle m, e_0\rangle = -\sum_{i=1}^n \lambda_i \geq -1 \implies m \in H^+_{e_0,-1}$. This finishes the proof of case 2 and claim 1.

Finally, for any face $Q \preccurlyeq \triangle_n$ we have the following equality $\sigma_Q = Cone(u_F \,|\, \text{F facet containing Q})$. Thus, for $Q = Conv(A\backslash\tilde{S})$, F_i, $i \in \{0, 1, ..., n\}$ contains Q if and only if $e_i \in \tilde{S}$ (or $0 \in \tilde{S}$). Hence, using the claim 1, we obtain $\sigma_Q = Cone(e_i \,|\, e_i \in \tilde{S}$ or $0 \in \tilde{S})$. That is to say $\sigma_Q = Cone(S)$ for $S \subseteq E$.

Moreover we have a bijection given by : $\hat{\varphi} : Conv(A\backslash\tilde{S}) \leftrightarrow Cone(S)$ which is induced by $\varphi : A \to E : e_i \mapsto e_i$, for $i \in \{1, ..., n\}$; and $0 \mapsto e_0$. Note that φ is clearly a bijection. In fact $\hat{\varphi}(Conv(A\backslash\tilde{S})) = Cone(\varphi(\tilde{S}))$.

Combining everything, we obtain $\{\sigma_Q \,|\, Q \preccurlyeq \triangle_n\} = \{Cone(S) \,|\, S \subsetneq \{e_0, ..., e_n\}\}$ because $\sigma_Q = \hat{\varphi}(Q)$. This finishes the proof of part (a).

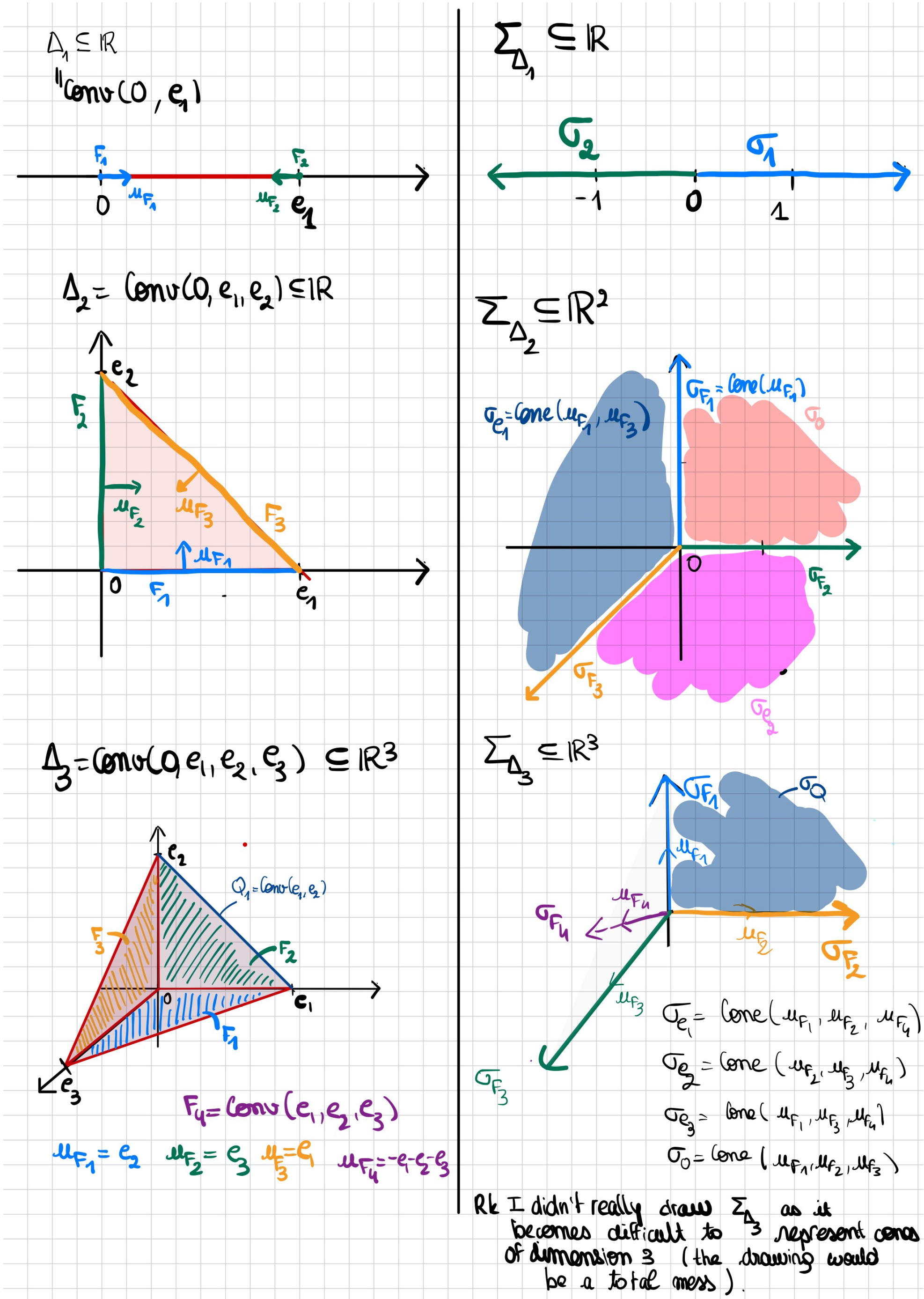


Figure 20: Pictures of normal fans of the n-simplex for n=1,2,3

(ii) Consider $k \in \mathbb{N}_{\geq 1}$, and $k\triangle_n \subseteq \mathbb{R}^n$ whose underlying lattice is $M = \mathbb{Z}^n$. Note that $k\triangle_n$ is full dimensional, and normal hence very ample for every such k. Suppose, $k\triangle_n \cap M = \{m_1, ..., m_{s_k}\}$, then the projective toric variety $X_{k\triangle_n \cap M}$ is given by the projective closure of the image of the map Φ:

$$T_n \cong (\mathbb{C}^*)^n \to \mathbb{C}^{s_k} \to \mathbb{P}^{s_k - 1}, \bar{t} = (t_1, ..., t_n) \mapsto [\chi^{m_1}(\bar{t}), ..., \chi^{m_s}(\bar{t})].$$

We try to compute $k\triangle_n \cap \mathbb{Z}^n$,

$m = (b_1, ..., b_n) \in k\triangle_n \cap M \iff b_i \in \mathbb{Z}^n_{\geq 0} \; \forall \, i \in \{1, ..., n\}$

and $m \in k\triangle_n = Conv(0, ke_1, ..., ke_n) \implies m = \sum_{i=0}^n a_i.ke_i$ (set $e_0 = 0$), and $\sum_{i=0}^n a_i = 1 \iff m = \sum_{i=1}^n b_i e_i$ where $b_1, ..., b_n$ are the coefficients in the canonical basis with $\sum_{i=1}^n b_i \leq k$, using $b_i = a_i.k \; \forall i \in \{1, ..., n\}$ and $b_0 = a_0.k \in \mathbb{Z}_{\geq 0}$

Hence

$$k\triangle_n \cap \mathbb{Z}^n = \{(b_1, ..., b_n) \in \mathbb{Z}^n_{\geq 0} \mid \sum_{i=1}^n b_i \leq k\}$$

Therefore, this set can be rewritten as

$$\{(b_0, b_1, ..., b_n) \in \mathbb{Z}^{n+1}_{\geq 0} \mid \sum_{i=0}^n b_i = k\}$$

(to see the bijection you can take $b_0 = n - \sum_{i=1}^n b_i \in \mathbb{Z}_{\geq 0}$). The cardinality of the latter set is known and equals $\binom{n+k}{n} =: s_k$ (number of ways to put n bars (space) among n+k bullets (sum of bullets between 2 spaces gives the value of some b_i).

Thus, $X_{k\triangle_n \cap M}$ is the projective closure of

$$\{[(\bar{t})^{m_1} : ... : (\bar{t})^{m_{s_k}}] \mid \bar{t} \in (\mathbb{C}^*)^n \mid m_i = (b_1, ..., b_n) \in \mathbb{Z}^n_{\geq 0} \mid \sum_{i=1}^n b_i \leq k\} \subseteq \mathbb{P}^{s_k - 1}$$

where for $m_i = (b_1, ..., b_n) \in \mathbb{Z}^n$ we have $(\bar{t})^{m_i} = t_1^{b_1} \cdot ... \cdot t_n^{b_n}$.

On another hand we have

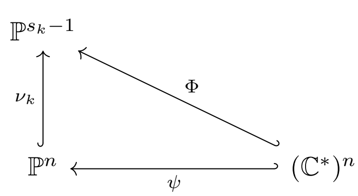
$$\nu_k(\mathbb{P}^n) = \{[P_1(\bar{x}) : ... : P_{s_k}(\bar{x})] \mid \bar{x} = [x_0 : ... : x_n] \in \mathbb{P}^n\} \subseteq \mathbb{P}^{s_k - 1}$$

where the $P_i(X_0, ..., X_n) \in \mathbb{C}[X_0, ..., X_n]$ are the $s_k := \binom{n+k}{n}$ monomials of total degree k. Such polynomials are of the form $P_i(X_0, ..., X_n) = X_0^{b_0} \cdot ... \cdot X_n^{b_n}$ where $(b_0, ..., b_n) \in \mathbb{Z}^{n+1}$ and $\sum_{i=0}^n b_i = k$.

Then, we prove that the two ways of seeing $X_{k\triangle_n}$ are equal or in other words we need to show

$$\overline{\Phi((\mathbb{C}^*)^n)} = \nu_k(\mathbb{P}^n)$$

- Firstly we have the following commutative diagram:



  Where Φ, ν_k have been defined previously, and ψ is the quotient map composed with the usual injection.

  Now consider some $(t_1, ..., t_n) \in (\mathbb{C}^*)^n$, $\nu_k \circ \psi(t1, ..., t_n) = \nu_k([1 : t_1 : ... : t_n])$ but this is clearly equal to $\Phi(t_1, ..., t_n)$ since each monomial $P_i(1, t_1, ..., t_n) = 1^{b_0} t_1^{b_1} ... t_n^{b_n}$ corresponds exactly to $\chi^{m_i}(t_1, ..., t_n)$ for $m_i = (b_1, ..., b_n) \in \mathbb{Z}^n ; \sum_{i=1}^n b_i \leq k$. This shows the diagram commutes. And in particular, it proves

$$\Phi((\mathbb{C}^*)^n) = \nu_k \circ \psi((\mathbb{C}^*)^n) \subseteq \nu_k(\mathbb{P}^n),$$

  we don't have an equality here since ψ isn't surjective.

- Moreover we have that $\nu_k(\mathbb{P}^n) \subseteq \mathbb{P}^{s_k-1}$ is closed since ν_k is proper. The details of this are left to the reader. This implies using the equality above that

$$\overline{\Phi((\mathbb{C}^*)^n)} \subseteq \nu_k(\mathbb{P}^n),$$

so we are left to prove $\nu_k(\mathbb{P}^n)$ is in fact the Zariski closure.

- Also, since both $\mathbb{P}^n$ and $(\mathbb{C}^*)^n$ are irreducible (cf. algebraic curves course). Their images under continuous maps are also irreducible, and $\overline{\Phi((\mathbb{C}^*)^n)}$ is also irreducible (being the closure of an irreducible).
- Now, we argue on dimensions, you can verify that both Φ and ν_k are injective continuous maps. Therefore $n = dim((\mathbb{C}^*)^n = dim(\mathbb{P}^n) \implies dim(\Phi((\mathbb{C}^*)^n)) = dim(\nu_k(\mathbb{P}^n)) = n$, and since $\Phi((\mathbb{C}^*)^n)$ is an open in its closure $dim(\overline{\Phi((\mathbb{C}^*)^n)}) = n$
- Finally the last argument is that $\overline{\Phi((\mathbb{C}^*)^n)} \subseteq \nu_k(\mathbb{P}^n)$ have the same dimension and if we we supposed that the inclusion is strict it would give a contradiction with irreducibility.

$$\implies \overline{\Phi((\mathbb{C}^*)^n)} = \nu_k(\mathbb{P}^n)$$

This finishes the proof.

15.10 Solutions to Chapter 10

Solutions written by Joel Jeremias Hakavuori

Solution 10.1.

(i) The facet presentation of P is given by

$$\begin{cases} x_i \geq -1 & \text{for all } 1 \leq i \leq n \\ -\sum_{i=1}^n x_i \geq -1. \end{cases}$$

The vertex corresponding to the origin of Δ_n is shifted to $(-1, \dots, -1)$, from which we see that we require $x_i \geq -1$, and the hyperplane connecting the vertices e_i is given by $-\sum_{i=1}^n x_i \geq -n$, which after shifting by $(-1, \dots, -1)$ gives $-\sum_{i=1}^n x_i \geq -1$.

To show that P is smooth, observe that P and Δ_n have the same normal fan, and as seen during Chapter 10, Δ_n is smooth as the corresponding variety X_{Δ_n} is $\mathbb{P}^n$, which is smooth. Recall that X_Q is a smooth projective variety for a full dimensional lattice polytope $Q \Leftrightarrow Q$ is a smooth polytope $\Leftrightarrow$ Σ_Q is a smooth fan. Hence we see that P is smooth.

Now that we have a facet presentation $P = \{m \in M_\mathbb{R} | \langle m, u_F \rangle \geq -a_F,\ F \text{ facet of } P\}$, we can use the result stating that $P^\circ = \mathrm{Conv}((\frac{1}{a_F})u_F \in N_\mathbb{R} \mid F \text{ facet of } P)$, which shows that $P^\circ = \mathrm{Conv}(e_0, e_1, \dots, e_n)$, where $e_0 = -\sum_{i=1}^n e_i$.

(ii) With the facet presentation of P, the fact that $P^\circ = \mathrm{Conv}((\frac{1}{a_F})u_F \in N_\mathbb{R} \mid F \text{ facet of } P)$, and that $(P^\circ)^\circ = P$, we can read off the facet presentation of P° from the presentation of P as the convex hull of the points $(-1, \dots, -1), p_1, \dots, p_n$, with p_i having i^{th} coordinate n and others -1. Thus P° has facet presentation $\{m \in M_\mathbb{R} | \langle m, u_F \rangle \geq -a_F,\ F \text{ facet of } P\}$ with all $a_F = 1$ and the collection of u_F consisting of the vector $u_{F_0}(-1, \dots, -1)$ and $u_{F_i} = (-1, \dots, n, \dots, -1)$ with n as i^{th} coordinate for $i = 1, \dots n$.

(iii) To show that P° is not smooth for $n \geq 2$, we again use the result stating that P is smooth if and only if X_{Σ_P} is smooth. By exercise 9.3, the normal fan Σ_{P° consists of the cones over the faces of $(P^\circ)^\circ = P$. To show that P° is not smooth, it suffices to show that one of the affine pieces of $X_{\Sigma_{P^\circ}}$ is not smooth, i.e., that one of the cones of Σ_{P° is not smooth. Consider the cone corresponding to $U_{\sigma_{e_0}} = \mathrm{Spec}(\mathbb{C}[\sigma_{e_0}^\vee \cap M])$. This cone $\sigma_{e_0}^\vee$ is the generated by the elements $(n+1, 1, \dots, 1), \dots, (1, \dots, n+1)$. Let a_i be the element with $n+1$ in the i^{th} position. A cone is smooth if the minimal generators (the ray generators of the edges) of $\sigma \subseteq N$ form part of a $\mathbb{Z}$-basis of N. The minimal generators in this case are exactly the elements $\{a_1, \ldots, a_n\}$. Observe that to generate the lattice points in $\sigma_{e_0}^\vee \cap \mathbb{Z}^n$, we require (at least) $(1, \ldots, 1) \in \sigma_{e_0}^\vee$ in addition to all $\{a_i\}_{i=1}^n$, as $(1, \ldots, 1)$ is in $\sigma_{e_0}^\vee \cap \mathbb{Z}^n$. If an integer linear combination $\sum_{i=1}^n k_i a_i$ of the elements $\{a_1, \ldots, a_n\}$ is on the diagonal, then $k_1 = k_2 = \ldots = k_n$. As $\sum_{i=1}^n k a_i = k(\sum_{i=1}^n a_i) = k(n+1)(1, \ldots, 1) \neq (1, \ldots, 1)$ for any nonzero integer k when $n \geq 2$, we see that $(1, \ldots, 1)$ is not generated as a $\mathbb{Z}$-linear combination of $\{a_1, \ldots, a_n\}$. Hence we require at least $n+1$ elements to form a minimal generating set of $\sigma_{e_0}^\vee$, which is never a subset of a basis of $\mathbb{Z}^n$. Thus P° is not smooth, as Σ_{P° has a cone which is not smooth.

Solution 10.2.

(i) In the context of this course R is a finitely generated k-algebra $R = k[x_1, \ldots, x_n]/I$, where k is a field and I is an ideal in $k[x_1, \ldots, x_n]$. We want to show that R is Noetherian, i.e., ideals $I \subseteq R$ are finitely generated. Observe that quotients of Noetherian rings are Noetherian: ideals of R/I are the ideals of R containing I, so every ascending chain of ideals in R/I will stabilize, which is an equivalent condition for a ring to be Noetherian. Hence it suffices to show that $k[x_1, \ldots, x_n]$ is Noetherian. Furthermore, if we show that $k[x]$ is Noetherian, then the result follows by induction, as $k[x_1, \ldots, x_n] = k[x_1, \ldots, x_{n-1}][x_n]$. A field k is obviously Noetherian, as (0) is the only proper ideal, and the fact that $k[x]$ is Noetherian follows from Hilberts basis theorem. Thus we get that every ideal $I \subseteq R$ is finitely generated.

(ii) Let $W \subset V$ be a subvariety defined by the ideal $I \subset R$. By part (a), $I = \langle f_1, \ldots, f_r\rangle$ for some $f_i \in R$, and hence $V \setminus W = V \setminus (\mathbb{V}(\langle f_1, \ldots, f_r\rangle)) = \cup_{i=1}^r D(f_i)$, where $D(f_i)$ are the principal opens of each f_i in $\mathrm{Spec}(R)$. As $D(f_i) \simeq V_{f_i}$, we get that $V \setminus W \simeq \cup_{i=1}^r V_{f_i}$.

(iii) Suppose we have an open cover $\cup_{j\in J} U_j = V$ of V, with $U_j = V \setminus X_j$ for some $X_j = \mathbf{V}(I_j)$. Then $V = \cup_J (V \setminus X_j) = (\cap_J \mathbf{V}(I_j))^c$, and hence we have that $1 \in \sum_J I_j$. The element 1 is generated by finitely many elements chosen from the collection $\{I_j\}_{j\in J}$, and choosing the finitely many open sets corresponding to these elements, we see that we get a finite subcover of V.

Solution 10.3.

(i) We need to show that $\sim$ is reflexive, symmetric and transitive. Reflexivity is clear, as $f|_U \sim f|_U$ for all $U \subset X$, and similarly if $f \sim g$ then $g \sim f$. To show transitivity, take $f \sim g$ and $g \sim h$, so $f|_U = g|_U$ and $g|_V = h|_V$ for some $U, V \subset X$, and hence $f|_{U\cap V} = h|_{U\cap V}$. As X is irreducible, all open subsets are dense, so $U \cap V$ is a nonempty open subset of X. Thus $f \sim h$.

(ii) Let $\langle U, f\rangle$ be a representative of an equivalence class, i.e. $\langle U, f\rangle \sim \langle V, g\rangle$ if there exists a nonempty open W where $f|_W = g|_W$. As X is irreducible, any nonempty opens have nonempty intersection, and we can turn the set of equivalence classes of part (a) into a ring by defining $\langle U, f\rangle + \langle V, g\rangle = \langle U \cap V, f + g\rangle$ and similarly for multiplication. If $f \neq 0$ is a rational function which is regular on some nonempty open $U \subseteq X$, then $\frac{1}{f}$ is regular on $D(f) \neq \varnothing$, so $\langle U, f\rangle$ has a multiplicative inverse, and we see that the set of equivalence classes form a field.

(iii) A rational function $f \in \mathbb{C}(X)$ is a regular function $f : U \to \mathbb{C}$ defined on some nonempty Zariski open subset $U \subseteq X$, with f and g equivalent if they agree on some nonempty open subset of X. Now, let $U \subseteq X$ be a nonempty open subset of X. With this definition it clear that $\mathbb{C}(U) \subseteq \mathbb{C}(X)$, as any $f \in \mathbb{C}(U)$ is regular on some nonempty $V \subseteq U \subseteq X$. Conversely, if $f \in \mathbb{C}(X)$, f is regular on some nonempty open $V \subseteq X$. As U is dense in X, $U \cap V$ is a nonempty open subset of U, and f is regular on $U \cap V$, so $f \in \mathbb{C}(U)$.

Solution 10.4.

(i) On W_{i-1} we have the relations $R := \{x_{i-1}y_j - x_{j-1}y_i\}$ for $1 \leq i < j \leq n$, so $y_j = \frac{x_{j-1}y_i}{x_{i-1}}$. Hence, we may simplify the coordinate ring of the affine open W_{i-1} to get

$$\mathrm{Spec}(\mathbb{C}[\frac{x_0}{x_{i-1}}, \ldots, \frac{x_n}{x_{i-1}}, y_1, \ldots, y_n] \ / \ R) = \mathrm{Spec}(\mathbb{C}[\frac{x_0}{x_{i-1}}, \ldots, \frac{x_n}{x_{i-1}}, y_i]),$$

as $y_j = \frac{x_{j-1}}{x_{i-1}} \cdot y_i$ for $j \neq i$ in the quotient.

(ii) We want to give the gluing data to identify $W_{i-1} \setminus \mathbf{V}(x_{j-1})$ and $W_{j-1} \setminus \mathbf{V}(x_{i-1})$. By the above, we have the coordinate rings $\mathbb{C}[\frac{x_0}{x_{i-1}}, \dots, \frac{x_n}{x_{i-1}}, y_i]_{\frac{x_{j-1}}{x_{i-1}}}$ and $\mathbb{C}[\frac{x_0}{x_{i-1}}, \dots, \frac{x_n}{x_{i-1}}, y_i]_{\frac{x_{i-1}}{x_{j-1}}}$ for $W_{i-1} \setminus \mathbf{V}(x_{j-1})$ and $W_{j-1} \setminus \mathbf{V}(x_{i-1})$ respectively. As in the case when gluing the affine pieces for projective space, the gluing map $\varphi_{ij} : W_{i-1} \setminus \mathbf{V}(x_{j-1}) \to W_{j-1} \setminus \mathbf{V}(x_{i-1})$ will send

$$\varphi_{ij} : \frac{x_k}{x_{i-1}} \mapsto (\frac{x_k}{x_{j-1}})/(\frac{x_{j-1}}{x_{i-1}}), (k \neq i-1),$$

and

$$(\frac{x_{j-1}}{x_{i-1}})^{-1} \mapsto \frac{x_{i-1}}{x_{j-1}},$$

which are well defined on the intersection. Due to the relations $x_{i-1}y_j - x_{j-1}y_i$, we have $y_i = \frac{x_{i-1}}{x_{j-1}} y_j$, and hence map $\varphi_{ij} : y_i \mapsto \frac{x_{i-1}}{x_{j-1}} y_j$, which is again well defined on the intersection. Observe that $\varphi_{ij} = \varphi_{ji}^{-1}$ and $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$ wherever all three maps are defined.

15.11 Solutions to Chapter 11

Solutions written by Matthias Georges A Schuller

Solution 11.1.

(i) Since U_α and U_β are open and both contain p, there is an open $W \subset U_\alpha \cap U_\beta$ containing p. The local ring is a local property thus $\mathcal{O}_{U_\alpha,p} \cong \mathcal{O}_{W,p}$ and similarly $\mathcal{O}_{U_\beta,p} \cong \mathcal{O}_{W,p}$. This yields $\mathcal{O}_{U_\alpha,p} \cong \mathcal{O}_{U_\beta,p}$. The Zariski tangent space is a property defined from the local ring and isn't changed by isomorphisms, so we deduce the isomorphism $T_{U_\alpha,p} \cong T_{U_\beta,p}$.

(ii) Suppose U_α, U_β are two affine open sets containing p. As previously there is an open $W \subset U_\alpha \cap U_\beta$ containing p. Lets show $\dim_p U_\alpha = \dim_p U_\beta$.
We do this by finding for each irreducible component of U_α containing p an irreducible component of W containing p of the same dimension, and similarly the other way around. Then doing the same between U_β and W, this will show $\dim_p U_\alpha = \dim_p W = \dim_p U_\beta$.
Let $p \in A \subset U_\alpha$ be an irreducible component, then $A \cap W$ is an irreducible component of W containing p. Furthermore A and $A \cap W$ have the same dimension. To see this, let $A = \mathrm{Spec}(R)$ where R is an integral domain. Since $A \cap W$ is a nonempty open subset of A, there is a nonzero $f \in R$ such that $D(f) \subset A \cap W$ where $D(f) = \{q \in A \mid f(q) \neq 0\}$. Thus we have $\dim D(f) \leq \dim(A \cap W) \leq \dim A$. Recall that for irreducible affine varieties the dimension is equal to the dimension of the coordinate ring. Note that $D(f) = \mathrm{Spec} R_f$. The integral $\mathbb{C}$-algebras R and R_f have the same fraction field and hence the same transcendence degree over $\mathbb{C}$. This implies $\dim D(f) = \dim A$, which gives the equality $\dim(A \cap W) = \dim A$.
On the other hand, let $p \in B \subset W$ be an irreducible component. Then $\bar{B} \subset U_\alpha$ is an irreducible component containing p and since $B = \bar{B} \cap W$ the above implies $\dim B = \dim \bar{B}$.
This proves the announced result and shows that the local dimension $\dim_p X$ is well defined.

(iii) Combining the two previous points shows that smoothness is well-defined for abstract varieties.

Solution 11.2.

(i) Let $V = \{y \in Y \mid f(y) = g(y)\}$ and

$$f \times g : \begin{cases} Y \to X \times X \\ y \mapsto (f(y), g(y)). \end{cases}$$

Also consider the diagonal map $\Delta : X \to X \times X$. Note that

$$V = (f \times g)^{-1}((f \times g)(Y) \cap \Delta(X)).$$

Since X is separated, $\Delta(X)$ is closed in $X \times X$, hence $(f \times g)(Y) \cap \Delta(X)$ is closed in $(f \times g)(Y)$. The map $f \times g : Y \to (f \times g)(Y)$ is a morphism so in particular Zariski continuous. Then V being the inverse image of a closed set is closed itself in Y.

(ii) Define

$$f : \begin{cases} U \cap V \to (U \times V) \cap \Delta(X) \\ p \mapsto (p,p) \end{cases}$$

and

$$g : \begin{cases} (U \times V) \cap \Delta(X) \to U \cap V \\ (p_1,p_2) \mapsto p_1. \end{cases}$$

It is easy to see that f and g are well defined and that they are polynomial maps when restricted between open subsets of affine open sets, thus f and g are morphisms of varieties. Furthermore, the two maps are inverses of each other, hence $(U \times V) \cap \Delta(X) \cong U \cap V$.
Note that U and V being affine implies that $U \times V$ is affine. Also $\Delta(X)$ is closed in $X \times X$ so $(U \times V) \cap \Delta(X)$ is closed in $U \times V$. Being closed in an affine variety implies being affine, therefore $(U \times V) \cap \Delta(X)$ is affine, and thus the same is true for the isomorphic $U \cap V$.

(iii) For a counterexample of point (a), take X to be the line with two origins obtained by gluing U, V, two copies of $\mathbb{C}$. Then the identity morphisms $f : \mathbb{C} \to U$ and $g : \mathbb{C} \to V$ give morphisms $f, g : \mathbb{C} \to X$. The set $\{y \in \mathbb{C} \mid f(y) = g(y)\}$ is $\mathbb{C}^*$, which is not closed in $\mathbb{C}$.
For a counterexample of point (b), imagine a construction of a plane with two origins. Take U, V two copies of $\mathbb{C}^2$ and glue them along their Zariski open subset $\mathbb{C}^2\backslash 0$ via the identity map, to form the abstract variety X. Then U and V are affine open subsets of X, however $U \cap V = \mathbb{C}^2\backslash 0$ is not affine.

Solution 11.3. Let $\sigma_0 = \text{Cone}(0)$, $\sigma_1 = \text{Cone}(e_1)$ and $\sigma_2 = \text{Cone}(-e_1)$. Let's compute U_{σ_i} for $i = 0, 1, 2$. First compute the dual cones : $\sigma_0^\vee = \text{Cone}(\pm e_1, \pm e_2)$, $\sigma_1^\vee = \text{Cone}(e_1, \pm e_2)$ and $\sigma_2^\vee = \text{Cone}(-e_1, \pm e_2)$. Then we get

$$\begin{aligned} U_{\sigma_0} &= \text{Spec}(\mathbb{C}[x^{\pm 1}, y^{\pm 1}]) \\ &= \text{Spec}(\mathbb{C}[x^{\pm 1}] \otimes \mathbb{C}[y^{\pm 1}]) \\ &= \text{Spec}(\mathbb{C}[x^{\pm 1}]) \times \text{Spec}(\mathbb{C}[y^{\pm 1}]), \end{aligned}$$

$$\begin{aligned} U_{\sigma_1} &= \text{Spec}(\mathbb{C}[x, y^{\pm 1}]) \\ &= \text{Spec}(\mathbb{C}[x] \otimes \mathbb{C}[y^{\pm 1}]) \\ &= \text{Spec}(\mathbb{C}[x]) \times \text{Spec}(\mathbb{C}[y^{\pm 1}]), \end{aligned}$$

$$\begin{aligned} U_{\sigma_2} &= \text{Spec}(\mathbb{C}[x^{-1}, y^{\pm 1}]) \\ &= \text{Spec}(\mathbb{C}[x^{-1}] \otimes \mathbb{C}[y^{\pm 1}]) \\ &= \text{Spec}(\mathbb{C}[x^{-1}]) \times \text{Spec}(\mathbb{C}[y^{\pm 1}]). \end{aligned}$$

U_{σ_1} and U_{σ_2} glue along U_{σ_0}. All the U_{σ_i} have the $\text{Spec}(\mathbb{C}[y^{\pm 1}])$ component in common, hence we have

$$X_\Sigma \cong P \times \text{Spec}(\mathbb{C}[y^{\pm 1}]) = P \times \mathbb{C}^*$$

where P is the variety obtained by gluing $\text{Spec}(\mathbb{C}[x])$ and $\text{Spec}(\mathbb{C}[x^{-1}])$ along $\text{Spec}(\mathbb{C}[x^{\pm 1}])$ via the maps $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x^{\pm 1}]$ and $\mathbb{C}[x^{-1}] \hookrightarrow \mathbb{C}[x^{\pm 1}]$. This glues to $\mathbb{P}^1$, therefore $X_\Sigma \cong P \times \mathbb{C}^* \cong \mathbb{P}^1 \times \mathbb{C}^*$.

Solution 11.4. Let's begin by showing that for $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$ we have $(\sigma_1 \times \sigma_2)^\vee = \sigma_1^\vee \times \sigma_2^\vee$. The point $(m, m') \in (M_1)_\mathbb{R} \times (M_2)_\mathbb{R}$ is in $(\sigma_1 \times \sigma_2)^\vee$ if and only if

$$0 \leq \langle (m, m'), (u, u') \rangle = \langle m, u \rangle + \langle m', u' \rangle$$

for all $(u, u') \in \sigma_1 \times \sigma_2$.
This implies directly the inclusion $\sigma_1^\vee \times \sigma_2^\vee \subset (\sigma_1 \times \sigma_2)^\vee$.
For the other inclusion, take $(m, m') \in (\sigma_1 \times \sigma_2)^\vee$. For all $u \in \sigma_1$, $(u, 0) \in \sigma_1 \times \sigma_2$ so we must have

$0 \leq \langle (m, m'), (u, 0) \rangle = \langle m, u \rangle$, which implies $m \in \sigma_1^\vee$. Similarly we find $m' \in \sigma_2^\vee$. This proves the equality. Then we have

$$S_{\sigma_1 \times \sigma_2} = (\sigma_1 \times \sigma_2)^\vee \cap (M_1 \times M_2) = (\sigma_1^\vee \cap M_1) \times (\sigma_2^\vee \cap M_2) = S_{\sigma_1} \times S_{\sigma_2}$$

and

$$\begin{aligned} U_{\sigma_1 \times \sigma_2} &= \operatorname{Spec}(\mathbb{C}[S_{\sigma_1 \times \sigma_2}]) \\ &= \operatorname{Spec}(\mathbb{C}[S_{\sigma_1} \times S_{\sigma_2}]) \\ &= \operatorname{Spec}(\mathbb{C}[S_{\sigma_1}] \otimes \mathbb{C}[S_{\sigma_2}]) \\ &= \operatorname{Spec}(\mathbb{C}[S_{\sigma_1}]) \times \operatorname{Spec}(\mathbb{C}[S_{\sigma_2}]) \\ &= U_{\sigma_1} \times U_{\sigma_2}. \end{aligned}$$

To conclude that $X_{\Sigma_1 \times \Sigma_2} \cong X_{\Sigma_1} \times X_{\Sigma_2}$ it remains to study how those affine pieces glue together. Take $\sigma_1 \times \sigma_2$, $\sigma_1' \times \sigma_2' \in \Sigma_1 \times \Sigma_2$. The pieces $U_{\sigma_1 \times \sigma_2}$ and $U_{\sigma_1' \times \sigma_2'}$ glue together along U_τ where $\tau = (\sigma_1 \times \sigma_2) \cap (\sigma_1' \times \sigma_2') = (\sigma_1 \cap \sigma_1') \times (\sigma_2 \cap \sigma_2') =: \tau_1 \times \tau_2$. The gluing maps are given by

$$\mathbb{C}[S_{\sigma_1 \times \sigma_2}] = \mathbb{C}[S_{\sigma_1}] \otimes \mathbb{C}[S_{\sigma_2}] \hookrightarrow \mathbb{C}[S_{\tau_1}] \otimes \mathbb{C}[S_{\tau_2}] = \mathbb{C}[S_{\tau_1 \times \tau_2}]$$

and similar for $\mathbb{C}[S_{\sigma_1' \times \sigma_2'}]$, which is exactly what we would expect for $X_{\Sigma_1} \times X_{\Sigma_2}$, therefore we have $X_{\Sigma_1 \times \Sigma_2} \cong X_{\Sigma_1} \times X_{\Sigma_2}$.

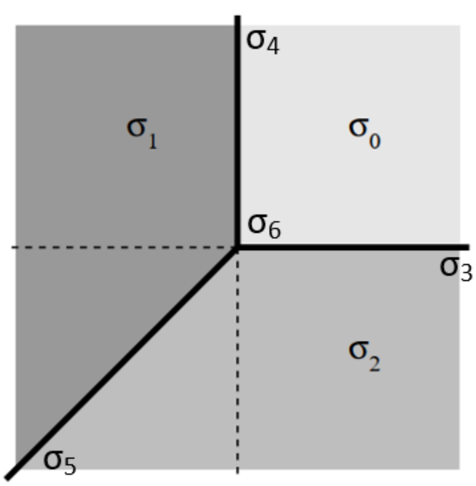
15.12 Solutions to Chapter 12

Solutions written by Julia Michèle Marie Morin

Solution 12.1. (i) Let $u = (a, b) \in \mathbb{Z}^2$ and $\lambda^u(t) : \mathbb{C}^* \to T_N$, $t \mapsto [1 : t^a : t^b]$ be a one-parameter subgroup. The book already deals with the limits $\lambda^u(t)$ as $t \to 0$ when $a, b > 0$ and $a = b < 0$.

- when $a = 0, b > 0$ we have $\lim_{t\to 0} \left[1 : t^a : t^b\right] = [1 : 1 : 0]$
- when $a > 0, b = 0$ we have $\lim_{t\to 0} \left[1 : t^a : t^b\right] = [1 : 0 : 1]$.
- when $a > b, b < 0$ we have $\lim_{t\to 0} \left[1 : t^a : t^b\right] = \lim_{t\to 0} \left[t^{-b} : t^{a-b} : 1\right] = [0 : 0 : 1]$ because $-b > 0$ and $a - b > 0$.
- when $a < 0, b > a$ we have $\lim_{t\to 0} \left[1 : t^a : t^b\right] = \lim_{t\to 0} \left[t^{-a} : 1 : t^{b-a}\right] = [0 : 1 : 0]$ because $-a > 0$ and $b - a > 0$
- when $a = b = 0$ we have $\lim_{t\to 0} \left(1, t^a, t^b\right) = [1 : 1 : 1]$

(ii) By the orbit-cone correspondence we know that there are 7 $(\mathbb{C}^*)^2$-orbits in $X_\Sigma \simeq \mathbb{P}^2$, each corresponding to a cone in the fan Σ, represented in the picture below :



The correspondence associates to each cone $\sigma \in \Sigma$ the orbit $O(\sigma) = T_N \cdot \gamma_\sigma = T_N \cdot \lim_{t\to 0} \lambda^u(t)$ when $u \in \text{Relint}(\sigma)$ by Proposition 3.2.2 of the book.

Recall that the action of $(\mathbb{C}^*)^2$ on $\mathbb{P}^2$ is given by

$$(\mathbb{C}^*)^2 \times \mathbb{P}^2$$

$$((t_1, t_2); [1 : x : y]) \to [1 : t_1 x : t_2 y]$$

because the description of $X_\Sigma \simeq \mathbb{P}^2$ arising from the polytope Δ_2 gives the inclusion $(\mathbb{C}^*)^2 \subseteq \mathbb{P}^2, (t_1, t_2) \mapsto [1 : t_1 : t_2]$

Therefore, let us compute :

- Let us take $u = (1, 1) \in \text{Relint}(\sigma_0)$, then $O(\sigma_0) = \{(t_1, t_2) \cdot \lim_{t\to 0} [1 : t : t] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = [1 : 0 : 0]$
- Let us take $u = (-1, 0) \in \text{Relint}(\sigma_1)$, then $O(\sigma_1) = \{(t_1, t_2) \cdot \lim_{t\to 0} \left[1 : t^{-1} : 1\right] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = \{(t_1, t_2) \cdot \lim_{t\to 0} [t : 1 : t] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = [0 : 1 : 0]$
- Let us take $u = (1, -1) \in \text{Relint}(\sigma_2)$, then $O(\sigma_2) = \{(t_1, t_2) \cdot \lim_{t\to 0} \left[1 : t : t^{-1}\right] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = \{(t_1, t_2) \cdot \lim_{t\to 0} \left[t : t^2 : 1\right] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = [0 : 0 : 1]$
- Let us take $u = (1, 0) \in \text{Relint}(\sigma_3)$, then $O(\sigma_3) = \{(t_1, t_2) \cdot \lim_{t\to 0} [1 : t : 1] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = \{[1 : 0 : t_2] \mid t_2 \neq 0\} = \{[x_1 : 0 : x_3] \mid x_1, x_3 \neq 0\}$
- Let us take $u = (0, 1) \in \text{Relint}(\sigma_4)$, then $O(\sigma_4) = \{(t_1, t_2) \cdot \lim_{t\to 0} [1 : 1 : t] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = \{[1 : t_1 : 0] \mid t_1 \neq 0\} = \{[x_1 : x_2 : 0] \mid x_1, x_2 \neq 0\}$
- Let us take $u = (-1, -1) \in \text{Relint}(\sigma_5)$, then $O(\sigma_5) = \{(t_1, t_2) \cdot \lim_{t\to 0} \left[1 : t^{-1} : t^{-1}\right] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = \{(t_1, t_2) \cdot \lim_{t\to 0} [t : 1 : 1] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = \{[0 : t_1 : t_2] \mid t_1, t_2 \neq 0\}$
- Let us take $u = (0, 0) \in \text{Relint}(\sigma_6)$, then $O(\sigma_6) = \{(t_1, t_2) \cdot \lim_{t\to 0} [1 : 1 : 1] \mid (t_1, t_2) \in (\mathbb{C}^*)^2\} = \{[x_1 : x_2 : x_3] \mid x_1, x_2, x_3 \neq 0\}$

(iii) For our toric variety $X_{\Delta_2} \simeq \mathbb{P}^2$ coming from the polytope $\Delta_2 = Conv(m_0, m_1, m_2)$ where $m_0 = (0, 0)$, $m_1 = (1, 0)$, $m_2 = (0, 1) \in \mathbb{R}^2$, let us denote the homogeneous coordinates of $\mathbb{P}^2$ as $[x_0 : x_1 : x_2]$. Then we have:

$$X_{\Delta_2} \cap U_i = U_{\sigma_{m_i}}$$

i.e $X_{\Delta_2} \cap U_i$ is the affine toric variety of the cone σ_{m_i} in the normal fan of Δ_2.

For simplicity we denote σ_{m_i} as σ_i. Let us describe the cases of one cone of each dimension, the other cases as similar :

- The limit point corresponding to the cone σ_0 is $[1 : 0 : 0]$. The distinguished point $\gamma_{\sigma_0} \in U_{\sigma_0}$ is given by the semi-group homomorphism:

$$\gamma_0 : S_{\sigma_0} \to \mathbb{C}$$

$$m \mapsto \begin{cases} 1 & \text{if } m \in \sigma_0^\perp \cap M, \\ 0 & \text{otherwise.} \end{cases}$$

  We know this means that γ_{σ_0} is the point $(\gamma_0(m_1), \gamma_0(m_2)) = (0, 0) \in (\mathbb{C}^*)^2$ where $m_1 = (1, 0)$ and $m_2 = (0, 1)$ are the generators of the semi-group S_{σ_0}. The isomorphism $(\mathbb{C}^*)^2 \cong U_0$ tells us that this is indeed the point $[1 : 0 : 0] \in U_{\sigma_0} \subseteq \mathbb{P}^2$.
- The limit point corresponding to the cone σ_3 is $[1 : 0 : 1]$. The distinguished point $\gamma_{\sigma_3} \in U_{\sigma_3} = X_{\Delta_2} \cap U_0 \cap U_2$ (see Proposition 0.13, week 5) is given by the semi-group homomorphism:

$$\gamma_3 : S_{\sigma_3} \to \mathbb{C}$$

$$m \mapsto \begin{cases} 1 & \text{if } m \in \sigma_3^\perp \cap M, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 1.1 of the notes of week 5, we know this means that γ_{σ_3} is the point $\gamma_3(m_1) = (0) \in \mathbb{C}^*$ where $m_1 = (1, 0)$ is the generator of the semi-group S_{σ_3}. The isomorphism $\mathbb{C}^* \cong U_0 \cap U_2$ tells us that this is indeed the point $[1 : 0 : 1] \in U_{\sigma_3} \subseteq \mathbb{P}^2$.

- The limit point corresponding to the cone σ_6 is $[1 : 1 : 1]$. The distinguished point $\gamma_{\sigma_6} \in U_{\sigma_6} = X_{\Delta_2} \cap U_0 \cap U_1 \cap U_2$ can only be the point $[1 : 1 : 1]$ because $U_0 \cap U_1 \cap U_2 = [1 : 1 : 1] \in \mathbb{P}^2$.

Solution 12.2. Suppose that $f : \mathbb{C}^* \to (\mathbb{C}^*)^m \simeq T_N$ is induced by the map $\mathbb{Z} \to \mathbb{Z}^m$, $1 \mapsto u = (e_1, ..., e_m)$ by tensoring with $- \otimes_{\mathbb{Z}} \mathbb{C}^*$, i.e f is given by

$$t \mapsto (t^{e_1}, ..., t^{e_m})$$

(i) $\Rightarrow$ Suppose that $lim_{t\to 0}(t^{e_1}, ..., t^{e_m})$ exists in U_σ. From proposition 3.2.2 of the book this means that $u \in \sigma$. Now let $m \in S_\sigma$ then

$$lim_{t\to 0}\chi^m(f(t)) = lim_{t\to 0}t^{\langle m,u\rangle}$$

But $m \in \sigma^\vee \cap M$, which means $\langle m, v\rangle \geq 0$ $\forall v \in \sigma$ by definition of $\sigma^\vee$. Therefore $\langle m, u\rangle \geq 0$ and $lim_{t\to 0}t^{\langle m,u\rangle}$ exists in $\mathbb{C}$.

$\Leftarrow$ Suppose that $lim_{t\to 0}\chi^m(f(t))$ exists in $\mathbb{C}$ for all $m \in S_\sigma$. This implies that $\langle m, u\rangle \geq 0$ for all $m \in S_\sigma$. Let $\mathcal{A} = \{m_1, ..., m_s\}$ be such that $S_\sigma = \mathbb{N}\mathcal{A}$. We have $\langle m_i, u\rangle \geq 0$ $\forall i \in \{1, ..., s\}$. Therefore $u \in H^+_{m_1} \cap ... \cap H^+_{m_s} = \sigma$ and by proposition 3.2.2 of the book again $lim_{t\to 0}f(t) \in U_\sigma$.

(ii) Suppose that $lim_{t\to 0}f(t) = lim_{t\to 0}(t^{e_1}, ..., t^{e_m})$ exists in U_σ. Then it corresponds to the semi-group homomorphism :

$$\gamma : S_\sigma \to \mathbb{C}$$
$$m \mapsto \chi^m(lim_{t\to 0}f(t))$$

Since the limit exists and χ^m is a continuous function $\forall m$, we have

$$\chi^m(lim_{t\to 0}(t^{e_1}, ..., t^{e_m})) = \chi^m(lim_{t\to 0}(t^{e_1}), ..., lim_{t\to 0}(t^{e_m})))$$
$$= lim_{t\to 0}\chi^m(t^{e_1}, ..., t^{e_m}) = lim_{t\to 0}\chi^m(f(t))$$

and we are done.

Solution 12.3. (i) Let σ be a cone in the fan Σ, σ is strongly convex rational polyhedral by definition of a fan. Therefore, $\sigma^\vee$ is rational too, let $\{m_1, ..., m_s\} \subseteq M$ be the finite set of its minimal generators. Therefore $S_\sigma = \mathbb{N}\mathscr{A}$ with $\mathscr{A} = \{m_1, ..., m_s\}$.
Let p be the point associated to the given semi-group homomorphism $\gamma : S_\sigma \to \mathbb{C}$. We know that $\gamma(m) = \chi^m(p)$ $\forall m \in S_\sigma$. Now let $m \in S_\sigma$, then there exists $a_1, ..., a_s$ in $\mathbb{N}$ such that $m = a_1m_1 + ... + a_sm_s$.

$$\gamma(m) = \chi^m(p) = \chi^{a_1m_1+...+a_sm_s}(p) \neq 0$$
$$\Leftrightarrow \chi^{a_1m_1}(p) \cdot ... \cdot \chi^{a_sm_s}(p) \neq 0$$
$$\Leftrightarrow (\chi^{m_1}(p))^{a_1} \cdot ... \cdot (\chi^{m_s}(p))^{a_s} \neq 0$$

Suppose that $(\chi^{m_i}(p)) \neq 0$ for all i. Then $\gamma(m) \neq 0$ $\forall m \in S_\sigma$ and the set $\{m \in S_\sigma \mid \gamma(m) \neq 0\} = \tau \cap M$ for $\tau = \sigma^\vee$.

Now suppose that there exists a non-empty indice set $I \subseteq \{1, ..., s\}$ such that $(\chi^{m_i}(p)) = 0$ $\forall i \in I$. Then

$$(\chi^{m_1}(p))^{a_1} \cdot ... \cdot (\chi^{m_s}(p))^{a_s} \neq 0$$
$$\Leftrightarrow a_i = 0 \quad \forall i \in I$$

Let us denote $J = \{j_1, ..., j_n\} = \{1, ..., s\} \setminus I$. Then the set $\{m \in S_\sigma \mid \gamma(m) \neq 0\} = \{m \in S_\sigma \mid m = \sum a_jm_j,$ for some $a_j \in \mathbb{N}, j \in J\} = \tau \cap M$ where $\tau = \text{Cone}(m_{j_1}, ..., m_{j_n}) \leq \text{Cone}(m_1, ..., m_s) = \sigma^\vee$.

(ii) Let $T_N \cong (\mathbb{C}^*)^s$ act on X_Σ and let $\tau \in \Sigma$. $O(\tau)$ is invariant under the action of T_N by definition. Now let $p' = \lim_{t\to 0} \lambda^u(t) \cdot p \in \overline{O(\tau)}$, for $p \in O(\tau)$ and a one-parameter subgroup λ^u with $u \in \mathbb{Z}^s$, $u \neq 0$. Then let us take $s \neq 1 \in T_N$ such that the action of s on p' is given by :

$$s \cdot \lim_{t\to 0} \lambda^u(t) \cdot p = s \lim_{t\to 0} \lambda^u(t) \cdot p = \lim_{t\to 0} \lambda^u(t) s \cdot p = \lim_{t\to 0} \lambda^u(t) \cdot s \cdot p$$

which is in $\overline{O(\tau)}$ since $s \cdot p \in O(\tau)$

(iii) Parts c) and d) of the Orbit-Cone correspondence theorem (Theorem 3.2.6 in the book) imply that

$$\overline{O(\tau)} \cap U_{\sigma'} = \bigcup_{\sigma \text{ is a face of } \sigma' \text{ containing } \tau} O(\sigma)$$

$$\simeq \bigcup_{\tau \subseteq \sigma \preceq \sigma'} \{\gamma : \mathrm{S}_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\}$$

.

First, notice that if $\tau \not\subseteq \sigma'$ then $\overline{O(\tau)} \cap U_{\sigma'} = \varnothing$ so let us assume that $\tau \subseteq \sigma'$.

We claim that

$$\bigcup_{\tau \subseteq \sigma \preceq \sigma'} \{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\}$$

$$\simeq \{\tilde{\gamma} : S_{\sigma'} \cap \tau^\perp \to \mathbb{C} \mid \tilde{\gamma} \text{ semi-group homomorphism } \}$$

where it is easy to verify that $S_{\sigma'} \cap \tau^\perp$ is an affine semi-group.

First let $\gamma : S_\sigma \to \mathbb{C}$ be a semi-group homomorphism such that $\gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M$ with $\tau \subseteq \sigma \preceq \sigma'$. Then we define

$$\tilde{\gamma} : S_{\sigma'} \cap \tau^\perp \to \mathbb{C}$$

$$m \mapsto \begin{cases} \gamma(m) & \text{if } m \in \sigma^\perp \cap S_{\sigma'}, \\ 0 & \text{otherwise.} \end{cases}$$

which is well defined because $\sigma^\perp \subseteq \tau^\perp$ and $S_{\sigma'} \subseteq S_\sigma$. It is a semi-group homomorphism because γ is.

Now let $\tilde{\gamma} : S_{\sigma'} \cap \tau^\perp \to \mathbb{C}$ be a semi-group homomorphism. Then σ' is a face of itself containing τ by assumption. We define

$$\gamma : S_{\sigma'} \to \mathbb{C}$$

$$m \mapsto \begin{cases} \tilde{\gamma}(m) & \text{if } m \in \sigma'^\perp \cap M, \\ 0 & \text{otherwise.} \end{cases}$$

But this set corresponds exactly to the affine toric variety $\mathrm{Spec}(\mathbb{C}[S_{\sigma'} \cap \tau^\perp]) = \mathrm{Spec}(\mathbb{C}[(\sigma')^\vee \cap M \cap \tau^\perp])$ which is the variety of the ideal generated by the χ^m with $m \in \tau^\perp \cap (\sigma')^\vee \cap M$.

Solution 12.4. (i) From Theorem 3.1.7 in the book, we know that every point $p \in X$ has a T_N-invariant affine open neighboorhood, let call it U_p. Then $\forall p \in X$, U_p is normal (because $O_{X,p} \simeq O_{U_p,p}$ and X is normal) and irreducible, as a non-empty open in an irreducible set. Therefore each U_p is a normal affine toric variety and from Theorem 1.3.5 of the book, there exists a strongly convex rational polyhedral cone σ_p such that $U_p = \mathrm{Spec}(\mathbb{C}[S_\sigma]) = U_{\sigma_p}$.

Therefore, $(U_{\sigma_p})_{p\in X}$ is an open cover of X with affine toric varieties and by quasi-compactness we can find a finite number of points $\{p_1, ..., p_n\}$ such that $X = \bigcup_{i=1}^n U_i$ where $U_i := U_{\sigma_{p_i}}$.

Moreover, since U_i and U_j are affine open subsets of the separated variety X, their intersection is also affine.

(ii) Let $U_{\sigma_i} = U_i \ \forall i$, then $U_i \cap U_j$ is T_N-invariant since both U_i and U_j are, it is non-empty open in the irreducible U_i so it is irreducible. Moreover, it is normal since $O_{U_i \cap U_j, p} = O_{U_i, p}$ for any $p \in U_i \cap U_j$ and U_i is normal. From point 1, we know that it is affine. Therefore, $U_i \cap U_j$ is an affine toric variety corresponding to a cone τ. Let us show that $U_i \cap U_j = U_{\sigma_i \cap \sigma_j}$.

Using question 3. of the same exercise, we have that $\sigma_i \cap \sigma_j$ is a face of both σ_i and σ_j. By exercise 3.2.9 of the book, we have that $U_{\sigma_i \cap \sigma_j}$ is an affine open subset of both U_{σ_i} and U_{σ_j}. In particular, $U_{\sigma_i \cap \sigma_j} \subseteq U_{\sigma_i} \cap U_{\sigma_j}$.

Moreover, both $U_{\sigma_i \cap \sigma_j}$ and $U_{\sigma_i} \cap U_{\sigma_j}$ are open and irreducible in U_i, so they are both of maximal dimension $\dim U_i$. As an affine toric variety, $U_{\sigma_i \cap \sigma_j}$ has a unique T_N-invariant subvariety of maximal dimension (which is $U_{\sigma_i \cap \sigma_j}$). It has to coincide with $U_{\sigma_i} \cap U_{\sigma_j}$ since it is T_N-invariant and of maximal dimension.

(iii) Let us show WLOG that $\sigma_i \cap \sigma_j$ is a face of σ_i. We have that $\sigma_i \cap \sigma_j \subseteq \sigma_i$ and $U_{\sigma_i \cap \sigma_j}$ is open in U_{σ_i}. From exercise 3.2.9 of the book, it follows that $\sigma_i \cap \sigma_j$ is a face of σ_i.

(iv) The collection of cones σ_i and their faces verifies the definition of a fan. Indeed, all cones are strongly convex rational polyhedral. Moreover, point 3. of the exercise exactly corresponds to the third condition in the definition of a fan (see Definition 3.1.2 in the book). Point 2. of the exercise corresponds to the gluing condition for X_Σ : X is constructed exactly as X_Σ from all its cones and faces so that $X \simeq X_\Sigma$.

15.13 Solutions to Chapter 13

Solutions written by Julie Estelle Marie Bannwart

Solution 13.1. If Σ is a fan in $N_\mathbb{R}$, fix an integer $0 \leq q \leq \operatorname{rank} N =: n$. For all $1 \leq p \leq n-1$, we want to show that the composition

$$C^{p-1}(\Sigma, \Lambda^q) \xrightarrow{\delta^{p-1}} C^p(\Sigma, \Lambda^q) \xrightarrow{\delta^p} C^{p+1}(\Sigma, \Lambda^q)$$

is the zero map. It suffices to show this in these degrees, because if $C^p(\Sigma, \Lambda^q)$ is non zero then $0 \leq q \leq p \leq n$.

In particular, it suffices to show that the compositions:

$$\Lambda^q M(\tau) \hookrightarrow \bigoplus_{\tau' \in \Sigma(n-(p-1))} \Lambda^q M(\tau') \xrightarrow{\delta^p \circ \delta^{p-1}} \bigoplus_{\sigma' \in \Sigma(n-(p+1))} \Lambda^q M(\sigma') \twoheadrightarrow \Lambda^q M(\sigma)$$

are the zero maps (where the first map is the inclusion of a summand, for any fixed $\tau \in \Sigma(n-(p-1))$, and the last map is the projection on one of the summands, for any fixed $\sigma \in \Sigma(n-(p+1))$).

From the definition of the differentials summand by summand, this composition is given by:

$$\sum_{\substack{\sigma' \in \Sigma(n-p) \\ \sigma \preccurlyeq \sigma' \preccurlyeq \tau}} c_{\sigma',\sigma} c_{\tau,\sigma'} \iota^q_{\sigma',\sigma} \circ \iota^q_{\tau,\sigma'} = \left(\sum_{\substack{\sigma' \in \Sigma(n-p) \\ \sigma \preccurlyeq \sigma' \preccurlyeq \tau}} c_{\sigma',\sigma} c_{\tau,\sigma'} \right) \iota^q_{\tau,\sigma}$$

where, $\iota^q_{\tau,\sigma} : \Lambda^q M(\tau) \to \Lambda^q M(\sigma)$ denotes the map induced by the inclusion $M(\tau) = \tau^\perp \cap M \subseteq \sigma^\perp \cap M = M(\sigma)$ and the coefficients $c_{\tau,\sigma'}$ keep track of the orientation: for a chosen orientation on every cone in Σ, $c_{\tau,\sigma'}$ has value 1 if the orientation induced by our chosen orientation of τ on σ' agrees with the one we chose for σ', and -1 if it induces the other orientation (and 0 if σ' is not a face of τ). So we only have to show that the sum on the right hand side of the above equation vanishes. Note that this sum has exactly two terms. Indeed:
Claim: A face σ of codimension 2 in a strongly convex polyhedral cone τ is contained in exactly two facets of τ.

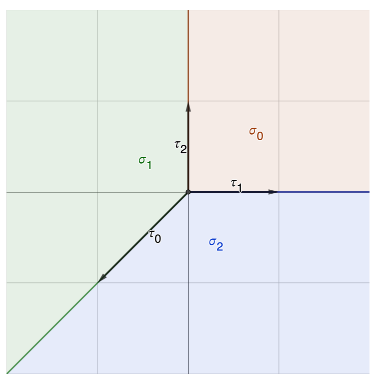
Proof of the claim: First of all, σ is contained in at least two facets, because any face of such a cone was the intersection of the facets containing this face. In particular, if σ was contained in only one facet, then it would be itself a facet, which contradicts the fact that σ has codimension 2 in τ. To show that σ cannot be contained in three distinct facets or more, recall that there is an inclusion-reversing bijection between the faces of τ and the faces of its dual cone $\tau^\vee$, that exchanges dimension and codimension. Assume by contradiction σ is contained in three facets of τ or more. Then, in the dual cone $\tau^\vee$, the dual face τ^* is a vertex, and σ^* is some face of dimension 2 containing τ^*. It has at least three edges, all containing τ^*, one for each facet of τ that contains σ. Embedding the two dimensional cone σ^* in the plane, it is now clear that it cannot have three edges all containing its vertex τ^*, because such cones in the plane either have one or two edges. This is a contradiction, and finishes the proof of our claim.

Let σ_1 and σ_2 be the two facets of τ that contain σ. Pick an $\mathbb{R}$-basis $\mathcal{B} \subseteq \sigma$ of $\operatorname{span}\sigma$, positively oriented. Let $u_i \in \sigma_i \setminus \sigma$ for $i = 1, 2$. In particular, $\mathcal{B} \cup \{u_i\} \subseteq \sigma_i$ is an $\mathbb{R}$-basis of $\operatorname{span}\sigma_i$ and since $\sigma_1 \neq \sigma_2$, we must have $u_2 \notin \sigma_1$ and therefore $\mathcal{B} \cup \{u_1, u_2\} \subseteq \sigma$ is an $\mathbb{R}$-basis of $\operatorname{span}\sigma$.

- Assume first that $c_{\sigma_1,\sigma} = 1$. This means that $(\mathcal{B}, u_1)$ is positively oriented with respect to the orientation on σ_1. Hence $c_{\sigma,\tau} = \operatorname{sgn}(\mathcal{B}, u_1, u_2)$ (the orientation of this basis). If on the contrary $c_{\sigma_1,\sigma} = -1$, then $-c_{\sigma_1,\tau} = \operatorname{sgn}(\mathcal{B}, u_1, u_2)$ (orientation of the basis $(\mathcal{B}, u_1, u_2)$). Hence $c_{\sigma_1,\sigma}c_{\sigma_1,\tau} = \operatorname{sgn}(\mathcal{B}, u_1, u_2)$ in both cases.
- Proceeding similarly with σ_2 we obtain $c_{\sigma_2,\sigma}c_{\sigma_2,\tau} = \operatorname{sgn}(\mathcal{B}, u_2, u_1)$.
- Therefore the sum we have to compute rewrites $\operatorname{sgn}(\mathcal{B}, u_1, u_2) + \operatorname{sgn}(\mathcal{B}, u_2, u_1) = 0$, because swapping two vectors in a basis reverses the orientation.

This concludes the proof; $(C^\bullet(\Sigma, \Lambda), \delta)$ is indeed a chain complex.

Solution 13.2. To fix notation, here is a representation of the fan of $\mathbb{P}^2$ described in the exercise, in $N_\mathbb{R} = \mathbb{R}^2$ for $N = \mathbb{Z}^2$.



(i) On the cones of dimension 2 in the fan, we choose the orientation induced by the standard orientation on $\mathbb{R}^2$, where (e_1, e_2) is positively oriented. On the cones of dimension 1 we choose the vectors e_1, e_2 and $-e_1 - e_2$ respectively as positively oriented basis. Here the rank of N is 2. Hence we only have to compute $C^p(\Sigma, \Lambda^q)$ when $0 \le q \le p \le 2$.

- $\boxed{q = 0:}$ By definition, for any $0 \le p \le 2$, $C^p(\Sigma, \Lambda^0) = \bigoplus_{\tau\in\Sigma(2-p)} \Lambda^0 M(\tau)$. Since for $\Lambda^0 A = \mathbb{Z}$ for any abelian group A, we obtain:

$$\begin{array}{ccccc} p=0 & & p=1 & & p=2 \\ \mathbb{Z}_{\sigma_0} \oplus \mathbb{Z}_{\sigma_1} \oplus \mathbb{Z}_{\sigma_2} & \xrightarrow{\delta^0} & \mathbb{Z}_{\tau_0} \oplus \mathbb{Z}_{\tau_1} \oplus \mathbb{Z}_{\tau_2} & \xrightarrow{\delta^1} & \mathbb{Z}_{\{0\}} \\ 1_{\sigma_0} & \longmapsto & 1_{\tau_1} - 1_{\tau_2} \quad 1_{\tau_0} & \longmapsto & 1 \\ 1_{\sigma_0} & \longmapsto & 1_{\tau_2} - 1_{\tau_0} \quad 1_{\tau_1} & \longmapsto & 1 \\ 1_{\sigma_0} & \longmapsto & 1_{\tau_0} - 1_{\tau_1} \quad 1_{\tau_2} & \longmapsto & 1 \end{array}$$

The facets of σ_0 are τ_1 and τ_2. We have $c_{\sigma_0,\tau_1} = 1$ and $c_{\sigma_0,\tau_2} = -1$: indeed the positively oriented basis $(e_1) \subseteq \tau_1$ of $\operatorname{span}\tau_1$ can be completed by $e_2 \in \sigma_0 \setminus \tau_1$ into the basis $(e_1, e_2) \subseteq \sigma_0$ of $\operatorname{span}\sigma_0$, which is positively oriented; whereas the positive basis (e_2) of τ_2 can be completed by $e_1 \in \sigma_0 \setminus \tau_2$ into the negatively oriented basis (e_2, e_1) of $\operatorname{span}\sigma_0$. The other orientation coefficients are computed in the same way. For δ^1, the orientation induced on $\{0\}$ necessarily agrees with the chosen one, the only degenerate one that exists.

- $\boxed{q = 1:}$ For all $0 \leq i \leq 2$, $\sigma_i^\perp = \{0\}$, hence $\Lambda^1 M(\sigma_i) = 0$. We also have $\tau_0^\perp \cap M = \mathbb{Z}(e_1 - e_2)$, $\tau_1^\perp \cap M = \mathbb{Z}e_2$, $\tau_2^\perp \cap M = \mathbb{Z}(-e_1)$. Finally, $\Lambda^1 M(\{0\}) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. We obtain:

$$\begin{array}{ccccc} p=0 & & p=1 & & p=2 \\ 0 & \xrightarrow{\delta^0} & \mathbb{Z}(e_1-e_2)\oplus\mathbb{Z}(e_2)\oplus\mathbb{Z}(-e_1) & \xrightarrow{\delta^1} & \mathbb{Z}e_1\oplus\mathbb{Z}e_2 \\ & & 1\cdot(e_1-e_2) & \longmapsto & 1\cdot e_1 \oplus (-1)\cdot e_2 \\ & & 1\cdot e_2 & \longmapsto & 0\cdot e_1 \oplus 1\cdot e_2 \\ & & 1\cdot(-e_1) & \longmapsto & (-1)\cdot e_1 \oplus 0\cdot e_2 \end{array}$$

  Again all orientation coefficients appearing in the definition of δ^1 are equal to 1, and the map is induced by the inclusions $\tau_i^\perp \subseteq \{0\}^\perp$.

- $\boxed{q = 2:}$ We only have to consider $p = 2$. We have

$$\Lambda^2 M(\{0\}) = \Lambda^2(\mathbb{Z}e_1 \oplus \mathbb{Z}e_2) \cong \mathbb{Z}(e_1 \wedge e_2)$$

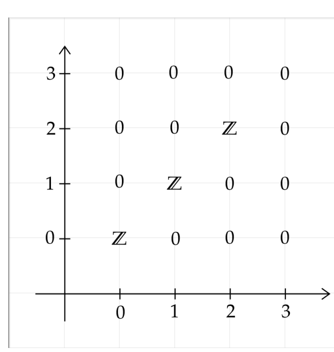
  and we obtain:

$$\begin{array}{ccccc} p=0 & & p=1 & & p=2 \\ 0 & \xrightarrow{\delta^0} & 0 & \xrightarrow{\delta^1} & \mathbb{Z}(e_1 \wedge e_2) \end{array}$$

(ii) By Theorem C.2.5 [CLS], a filtration of a topological space provides us with a spectral sequence converging to the compactly supported cohomology of the space. Applying this to the filtration for X_Σ, by Proposition 12.3.5 of [CLS], we have $C^p(\Sigma, \Lambda^q) = E_1^{p,q} \Rightarrow H_c^{p+q}(X;\mathbb{Z})$, and $E_2^{p,q} = H^p(C^\bullet(\Sigma, \Lambda^q))$. If we show that $E_2^{p,q} = 0$ for all $p \neq q$, we can conclude by exercise 4.a) below that:

$$\forall k \in \mathbb{N}, H^{2k}(X_\Sigma, \mathbb{Z}) = H^{2k}(\mathbb{P}^2, \mathbb{Z}) = E_2^{k,k}$$

and the other cohomology groups are zero (we can consider the usual cohomology instead of the compactly supported ones, because we know that the projective plane $\mathbb{P}^2$ is compact; one way to see this is to invoke a theorem saying that if the fan Σ is complete, which is the case here because $\sigma_0 \cup \sigma_1 \cup \sigma_2 = \mathbb{R}^2$, then the variety X_Σ is compact). Computing the sheet E_2 of this spectral sequence yields:



For $q = 2$, this is easy to see, since only one group in the complex $C^\bullet(X_\Sigma, \Lambda^2)$ does not vanish.

For $q = 1$, we have $E_2^{1,1} = H^1(C^\bullet(\Sigma, \Lambda^1)) = \ker(\delta^1) \cong \mathbb{Z}$ because this kernel is free abelian as a subgroup of a free abelian group, and has rank 1 by the null-rank theorem for abelian groups, because the domain has rank 3 and the image has rank 2 (indeed δ^1 is surjective because its image contains e_1 and e_2 (image of $(-1) \cdot (-e_1)$ and $1 \cdot e_2$ respectively)). Because of this surjectivity we also get $E_2^{2,1} = H^2(C^\bullet(\Sigma, \Lambda^1)) = \operatorname{coker} \delta^1 = 0$.

For $q = 0$, δ^1 is clearly surjective and hence as before $E_2^{2,0} = 0$. We also note that $\ker \delta^1 = \operatorname{Im} \delta^0$ hence $E_2^{1,0} = 0$. Finally, $\ker(\delta^0)$ is a free abelian group of rank 1 because the domain has rank 3 and the image of the map has rank 2; indeed $1_{\tau_1} - 1_{\tau_2}$ and $1_{\tau_2} - 1_{\tau_0}$ are $\mathbb{Z}$-linearly independent, and $1_{\tau_0} - 1_{\tau_1} = -(1_{\tau_2} - 1_{\tau_0}) - (1_{\tau_1} - 1_{\tau_2})$. Hence $E_2^{0,0} \cong \mathbb{Z}$.

So we conclude as explained above that

$$H^k(\mathbb{P}^2, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k \in \{0, 2, 4\} \\ 0 & \text{otherwise} \end{cases}$$

Solution 13.3. By Theorem 3.3.4 of [CLS], $\varphi_\ell\big|_{T_{X_\Sigma}}$ is given by the map $\overline{\varphi_\ell} \otimes \mathbb{C}^* : N \otimes_\mathbb{Z} \mathbb{C}^* \to N \otimes_\mathbb{Z} \mathbb{C}^*$, up to the isomorphism $T_{X_\Sigma} \cong N \otimes_\mathbb{Z} \mathbb{C}^*$. Choose an isomorphism $N \cong \mathbb{Z}^n$ and let $e_1, \dots, e_n$ be a $\mathbb{Z}$-basis of N corresponding to the canonical basis in $\mathbb{Z}^n$. Then, any $(t_1, \dots, t_n) \in T_{X_\Sigma} \cong N \otimes_\mathbb{Z} \mathbb{C}^*$, identified via this isomorphism with $\sum_{i=1}^n e_i \otimes t_i$, is mapped by φ_ℓ to

$$\begin{aligned} \sum_{i=1}^n (l \cdot e_i) \otimes t_i &= \sum_{i=1}^n (\underbrace{e_i \otimes t_i + \cdots + e_i \otimes t_i}_{\ell \text{ times}}) \\ &= \sum_{i=1}^n e_i \otimes t_i^\ell \qquad \text{(since we consider } \mathbb{C}^* \text{ with multiplicative structure)} \end{aligned}$$

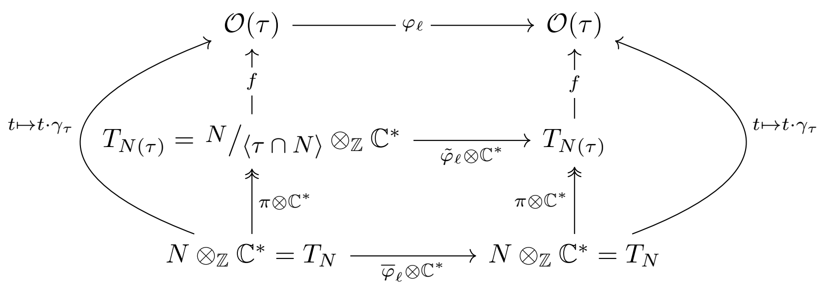
which corresponds via the same isomorphism to $(t_1^\ell, \dots, t_n^\ell) \in T_{X_\Sigma}$, as desired.

To consider the map φ_ℓ^* induced in cohomology by φ_ℓ, we first have to make sure that φ_ℓ preserves $\mathcal{O}(\tau)$. By Theorem 3.3.4, φ_ℓ is a toric map. Hence it is equivariant with respect to the action of the torus, so the image of any orbit is contained in another orbit. In particular, to check that $\varphi_\ell(\mathcal{O}(\tau)) \subseteq \mathcal{O}(\tau)$, it suffices to show that there exists $x \in \mathcal{O}(\tau)$ with $\varphi_\ell(x) \in \mathcal{O}(\tau)$. Let $x = \gamma_\tau$ be the distinguished point for τ. Then, pick any $u \in \operatorname{RelInt}(\tau)$ (the relative interior is empty if and only if τ is the zero cone. But then $\mathcal{O}(\tau) = \mathcal{O}(\{0\})$ is the torus of X_Σ, so we already know it is preserved by any toric morphism). By Proposition 3.2.2 in the book, we have $\lim_{t\to 0} \lambda^u(t) = \gamma_\tau$. Therefore, by continuity, $\varphi_\ell(\gamma_\tau) = \varphi_\ell(\lim_{t\to 0} \lambda^u(t) = \lim_{t\to 0} \varphi_\ell(\lambda^u(t))$. Because $\lambda^u(t)$ belongs to the torus of X_Σ for any $t \in \mathbb{C}^*$, by the first part φ_ℓ just raises it to the ℓ-th power, so that $\varphi_\ell(\gamma_\tau) = \lim_{t\to 0} \varphi_\ell(\lambda^u(t)) = \lim_{t\to 0} \lambda^{\ell \cdot u}(t) = \gamma_\tau$, because $\ell \cdot u \in \operatorname{RelInt}(\tau)$ since $\ell > 0$. Hence φ_ℓ preserves $\mathcal{O}(\tau)$.

Let $0 \le p \le n$ and consider $\tau \in \Sigma(n-p)$. Then, we have in cohomology the following commutative diagram:

$$\begin{array}{ccc} H_c^q(\mathcal{O}(\tau), \mathbb{Q}) & \xrightarrow{\varphi_\ell^*} & H_c^q(\mathcal{O}(\tau), \mathbb{Q}) \\ \downarrow f^* & & \downarrow f^* \\ H_c^q(T_{N(\tau)}, \mathbb{Q}) & \xrightarrow{\hat{\varphi}_\ell^*} & H_c^q(T_{N(\tau)}, \mathbb{Q}) \\ \downarrow \wr & & \downarrow \wr \\ (\bigwedge M(\tau))^{(q)} = \Lambda^q M(\tau) & \xrightarrow[((\ell\cdot)^\vee)^{\Lambda^q}]{} & (\bigwedge M(\tau))^{(q)} = \Lambda^q M(\tau) \end{array}$$

with $\hat{\varphi}_\ell$ the map induced on $T_{N(\tau)}$, by the map of lattices $\tilde{\varphi}_\ell : N(\tau) \to N(\tau)$ induced by $\overline{\varphi_\ell}$ (since $N(\tau)$ is a quotient of N). The vertical isomorphisms on the second row come from the fact that $H_c^*(T_{N(\tau)}, \mathbb{Q}) \cong \bigwedge M(\tau)$ as algebras, and the exponent (q) denotes the degree q part. The vertical map f^* is the map induced in cohomology by the isomorphism $f : T_{N(\tau)} \cong \mathcal{O}(\tau)$. For this diagram to commute we have to check that $\varphi_\ell \circ f = f \circ \varphi_\ell$. The situation is the following:



The top square commutes if and only if the precomposition of the two different possible maps by the surjective map $\pi \otimes \mathbb{C}^*$ are equal. This holds because the lower square commutes (mutliplication by ℓ on the lattice commutes with taking the quotient) and the outer rectangle too (the map φ_ℓ obtained from Theorem 3.3.4 is toric and hence equivariant under the action of the torus, and we have seen $\varphi_\ell(\gamma_\tau) = \gamma_\tau$).

The map $(\ell\cdot)^\vee)$ appearing on the previous diagram is dual to the multiplication by ℓ on $N(\tau)$, hence it is also multiplication by ℓ on $M(\tau)$. We are taking its q-th exterior power, hence it multiplies each of the q terms in a form by ℓ, by multilinearity we obtain that the map $((\ell\cdot)^\vee)^{\Lambda^q}$ is multiplication by ℓ^q. Hence also φ_ℓ^* on the top of the diagram is multiplication by ℓ^q.

Solution 13.4.

(i) Denote the filtrations with respect to which we have convergence by

$$0 = F^{k+1}H^k \subseteq F^k H^k \subseteq \cdots \subseteq F^1 H^k \subseteq F^0 H^k = H^k \text{ for all } k \in \mathbb{N}.$$

Since the E_2 sheet has only non-zero modules on the diagonal, the 0-th homology of the complexes it defines consists exactly of these modules on the diagonal, and 0 elsewhere. In particular the sequence degenerates at sheet E_2 and $E_\infty^{p,q} = E_2^{p,q} \cong F^p H^{p+q} \big/ F^{p+1} H^{p+q}$ for all $p, q \in \mathbb{Z}$.

Fix $k \in \mathbb{N}$. Then, for all $j \leq 2k+1$, $j \neq k$, it holds that

$$F^j H^{2k} \big/ F^{j+1} H^{2k} = E_\infty^{j,2k-j} = E_2^{j,2k-j} = 0$$

since $j \neq 2k - j$. In particular,

$$F^{2k+1} H^{2k} = F^{2k} H^{2k} = \cdots = F^{k+1} H^{2k} = 0$$

$$H^{2k} = F^0 H^{2k} = F^1 H^{2k} = \cdots = F^k H^{2k} = F^k H^{2k} \big/ F^{k+1} H^{2k} = E_\infty^{k,k} = E_2^{k,k}.$$

And for all $j \leq 2k+2$, we have

$$F^j H^{2k+1} \big/ F^{j+1} H^{2k+1} = E_\infty^{j,2k+1-j} = E_2^{j,2k+1-j} = 0$$

since $2k+1-j \neq j$ because $2k+1 \neq 2j$ for parity reasons. Hence all groups in the filtration are equal to $F^{2k+2} H^{2k+1} = 0$, and $H^{2k+1} = 0$.

(ii) By the way we defined the complexes $C^\bullet(\Sigma, \Lambda^\bullet)$, the modules $E_1^{p,q}$ in the first sheet E_1 are all of finite rank and vanish outside of the square $0 \leq p, q \leq n$. In particular, the sum introduced in the exercise is finite. For all $r \geq 1$, we compute, using the analog of the null-rank for abelian groups:

$$\begin{aligned}
\chi(E_{r+1}) &= \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} \operatorname{rank} E_{r+1}^{p,q} \\
&= \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} \operatorname{rank} H^0(E_r^{p+\bullet\cdot r, q-\bullet\cdot(r-1)}) && \text{(by definition)} \\
&= \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} \operatorname{rank} \left(\ker(E_r^{p,q} \to E_r^{p+r,q-r+1}) \middle/ \operatorname{Im}(E_r^{p-r,q-(r-1)} \to E_r^{p,q}) \right) \\
&= \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} (\operatorname{rank}(\ker(E_r^{p,q} \to E_r^{p+r,q-r+1})) - \operatorname{rank}(\operatorname{Im}(E_r^{p-r,q-(r-1)} \to E_r^{p,q}))) \\
&= \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} (\operatorname{rank}(E_r^{p,q}) - \operatorname{rank}(\operatorname{Im}(E_r^{p,q} \to E_r^{p+r,q-r+1})) \\
&\qquad - \operatorname{rank}(\operatorname{Im}(E_r^{p-r,q-(r-1)} \to E_r^{p,q})))
\end{aligned}$$

$$\begin{aligned}
&= \chi(E_r) + \sum_{p,q\in\mathbb{Z}} (-1)^{p+q+1} \operatorname{rank}(\operatorname{Im}(E_r^{p,q} \to E_r^{p+r,q-r+1})) \\
&\quad - \sum_{p',q'\in\mathbb{Z}} (-1)^{p'+r+q'+r-1} \operatorname{rank}(\operatorname{Im}(E_r^{p',q'} \to E_r^{p'+r,q'-r+1})) \\
&\qquad\qquad \text{(by setting } p' = p-r \text{ and } q' = q-(r-1)) \\
&= \chi(E_r) \qquad \text{(because } (-1)^{p'+r+q'+r-1} = (-1)^{p'+q'+1}.)
\end{aligned}$$

(iii) Since the spectral sequence degenerates at E_2 (Proposition 12.3.10 [CLS]), it holds that:

$$\chi(E_\infty) := \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} \operatorname{rank} E_\infty^{p,q} = \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} \operatorname{rank} E_2^{p,q} = \chi(E_2) = \chi(E_1)$$

by the previous question.

Hence, since $E_1^{p,q} \Rightarrow H_c^{p+q}(X_\Sigma, \mathbb{Z}) =: H^{p+q}$, we have:

$$\begin{aligned}
\chi(X_\Sigma) &= \sum_{k=0}^{2n} (-1)^k \operatorname{rank} H_c^k(X_\Sigma, \mathbb{Z}) \\
&= \sum_{k=0}^{2n} (-1)^k \sum_{j=0}^{k} \operatorname{rank} \left(F^j H^k \middle/ F^{j+1} H^k \right) \\
&= \sum_{k=0}^{2n} (-1)^k \sum_{j=0}^{k} \operatorname{rank}(E_\infty^{j,k-j}) \\
&= \sum_{0\leq q,j\leq 2n} (-1)^{q+j} \operatorname{rank}(E_\infty^{j,q}) && \text{(setting } q = k-j) \\
&= \chi(E_\infty) && \text{(} E_\infty^{p,q} = 0 \text{ when } p \neq q \text{ or } p > n \text{ or } q > n)
\end{aligned}$$

In particular,

$$\begin{aligned}
\chi(X_\Sigma) = \chi(E_1) &= \sum_{p,q\in\mathbb{Z}} (-1)^{p+q}\text{rank}\,(C^p(\Sigma, \Lambda^q)) \\
&= \sum_{0\le q\le p\le n} (-1)^{p+q}\binom{p}{q}|\Sigma(n-p)| \\
&= \sum_{p=0}^{n}(-1)^p|\Sigma(n-p)|\sum_{q=0}^{p}(-1)^q\binom{p}{q} \\
&= \sum_{p=0}^{n}(-1)^p|\Sigma(n-p)|\delta_{p=0}(p) \\
&= |\Sigma(n)|
\end{aligned}$$

because for all $x \in \mathbb{R}$ and $p \geq 1$, $(x-1)^p = \sum_{q=0}^{p}\binom{p}{q}x^{p-q}(-1)^q$. Inserting $x = 1$, we obtain that $\sum_{q=0}^{p}(-1)^q\binom{p}{q}$ is equal to 0. For $p = 0$ the sum has a single term equal to 1. This finishes the proof.