Problem 1

QR factorization: $A = Q_A R_A$, $B = Q_B R_B \Rightarrow A \otimes B =$

$$= \underbrace{(Q_A \otimes Q_B)}_{\text{this is orthogonal}} \cdot \underbrace{(R_A \otimes R_B)}_{\text{this is upper triangular}}$$

this is orthogonal:

$$(Q_A \otimes Q_B)^T \cdot (Q_A \otimes Q_B) = (Q_A^T \otimes Q_B^T) \cdot (Q_A \otimes Q_B) =$$
$$= (Q_A^T Q_A) \otimes (Q_B^T Q_B) = I \otimes I = I$$

this is upper triangular

Singular value decomposition: $A = U_A \Sigma_A V_A^T \quad B = U_B \Sigma_B V_B^T$

$$\Rightarrow A \otimes B = \underbrace{(U_A \otimes U_B)}_{\text{orthogonal}} \cdot \underbrace{(\Sigma_A \otimes \Sigma_B)}_{\text{diagonal}} \cdot \underbrace{(V_A \otimes V_B)}_{\text{orthogonal}}^T$$

diagonal. Note that the elements are <u>not</u> in decreasing order, in general.

Problem 2

(a) The (i,j)th entry of the $p \times m$ matrix M corresponding to $(A \otimes B)v$ is

$A \otimes B$ is $mp \times nq$ matrix, $(A \otimes B)v$ is a vector of length mp that can be reshaped into a $p \times m$ matrix M. It is suff. to prove that $M = BVA^T$.

$$M_{ij} = \left[(A \otimes B)v\right]_{(j-1)p+i} = \sum_{\alpha=1}^{nq} (A \otimes B)_{(j-1)p+i,\alpha} \cdot v_\alpha =$$

$$= \sum_{k=1}^{q} \sum_{h=1}^{n} (A \otimes B)_{(j-1)p+i,\,(h-1)q+k} \cdot v_{(h-1)q+k} =$$

$$= \sum_{k=1}^{q} \sum_{h=1}^{n} A_{jh} B_{ik} \underbrace{V_{kh}}_{V \text{ is the } q\times n \text{ matrix obtained by reshaping } v \in \mathbb{R}^{qn}} = \overbrace{B(i,:)}^{i\text{-th row of } B} \cdot V \cdot \underbrace{A(j,:)}_{j\text{-th row of } A \,=\, j\text{-th column of } A^T} = B(i,:)\, V\, (A^T)(:,j) = (BVA^T)_{ij} \quad \square$$

(b) $A \otimes B$ has dimension $mp \times nq$, so if we form the matrix explicitly then $(A \otimes B)v$ costs $O(mnpq)$. If instead we use the property of point (a), we need to compute $B \cdot V$ (cost $O(pnq)$) and then multiply the result by A^T (cost $O(pnm)$). The "reshaping operation" only costs $O(mp)$. So the total cost is $O(pnq + pnm + mp)$.

Problem 3

We rewrite the Lyapunov equation as

$$A X I + I X A^T = C$$

Using the property of Problem 2, taking vectorizations we get

$$(I \otimes A)\, vec(X) + (A \otimes I)\, vec(X) = vec(C),$$

which corresponds to the linear system $\mathcal{M}x = c$, where

$$x = vec(X)$$
$$c = vec(C)$$
$$\mathcal{M} = I \otimes A + A \otimes I.$$

Problem 4

Let $B_j \in \mathbb{R}^{m_B \times n_B}$ and $C_j \in \mathbb{R}^{m_C \times n_C}$. Consider the map

$$\mathcal{M} : [m_C m_B \times n_C n_B \text{ matrices}] \to [m_C n_C \times m_B n_B \text{ matrices}]$$

that takes $C_j \otimes B_j$ into $vec(C_j) \cdot vec(B_j)^T$. This is a "reshaping" operation (the entries of the matrices are permuted but not changed). Therefore,

$$\| A - \sum_{i=1}^{r} C_j \otimes B_j \|_F = \| \mathcal{M}(A) - \sum_{i=1}^{r} vec(C_j) \cdot vec(B_j)^T \|_F .$$

Considering now an SVD of $A = U\Sigma V^T$, it is sufficient to take

$$vec(C_j) = U(:, j)\Sigma_{jj} \text{ and } vec(B_j) = V(:, j).$$

Problem 5

- $\mathcal{Y} = \mathcal{X} \circ_\mu U \Rightarrow Y^{(\mu)} = U X^{(\mu)} \overset{\text{multiply by } U^T \text{ on the left}}{\Rightarrow} U^T Y^{(\mu)} = \underbrace{U^T U}_{= I \text{ because } U \text{ has orthonormal columns}} X^{(\mu)} = X^{(\mu)} \Rightarrow \mathcal{X} = \mathcal{Y} \circ_\mu U^T$

- $\| \mathcal{X} \circ_\mu U \|_F = \| U X^{(\mu)} \|_F = \| X^{(\mu)} \|_F = \| \mathcal{X} \|_F .$

Problem 6

Let $A = U\Sigma V^T$. Then $A = \Sigma \circ_1 U \circ_2 V$:

What is the matrix $Y := \Sigma \circ_1 U$? $\Sigma^{(1)} = \Sigma \Rightarrow (\Sigma \circ_1 U)^{(1)} = U\Sigma \Rightarrow Y = U\Sigma$

What is the matrix $Y \circ_2 V$? $Y^{(2)} = Y^T = \Sigma^T U^T = \Sigma U^T \Rightarrow (Y \circ_2 V)^{(2)} = V \Sigma U^T$

$$\Rightarrow \Sigma \circ_1 U \circ_2 V = Y \circ_2 V = (V\Sigma U^T)^T = U \Sigma V^T = A. \quad \square$$

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% Exercise 7

clear all
n = 10;
x = [0.1:0.1:n/10];
y = [0.1:0.1:n/10];
z = [0.1:0.1:n/10];
f = @(x,y,z,gamma) 1/((x+y+z).^gamma);
gamma = 1;
X = zeros(n,n,n);
for i = 1:n
    for j = 1:n
        for k = 1:n
            X(i,j,k) = f(x(i), y(j), z(k), gamma);
        end
    end
end

X1 = tenmat(X,1);
X2 = tenmat(X,2);
X3 = tenmat(X,3);
[U, S1, V] = svd(X1);
[U, S2, V] = svd(X2);
[U, S3, V] = svd(X3);
plot(1:n, diag(S1));
hold on;
plot(1:n, diag(S2));
plot(1:n, diag(S3));
legend("svd(X1)","svd(X2)","svd(X3)")
hold off


%n#mode matricization
function Xn = tenmat(X, n)
    N = ndims(X);
    sz = size(X);
    m = setdiff(1:N, n);
    X = permute(X,[n m]);
    Xn = reshape(X, sz(n), prod(sz(m)));
end

%n#mode multiplication
function Y = ttm(X, A, n)
    Y = tenmat(X, n);
    Y = A * Y;
    sz = size(X);
    sz2 = [size(A,1), sz(1:(n-1)), sz((n+1):end)];
    Y = reshape(Y, sz2);
    perm = [2:n, 1, (n+1):length(sz)];
    Y = permute(Y, perm);
end
```