**Exercise 9 – Low-rank approximation techniques**

**Problem 1: Rank-1 tensors**

Let $X$ and $Y$ be $n_1 \times \cdots \times n_d$ tensors of tensor rank 1:

$$X = x_1 \circ \cdots \circ x_d, \quad Y = y_1 \circ \cdots \circ y_d.$$  

Show that $X + Y$ is of rank 1 if and only if all but at most one of the components $x_i$ and $y_i$ are equal (within a scalar multiple). For the “only if” statement prove only the case $d = 2$ and $n_1 = n_2 = 2$.

**Problem 2: Complexity of inner product**

What is the complexity of computing the inner product of two tensors of tensor rank at most $R$ given in CP decompositions (with $R$ terms)?

**Problem 3: A bound on the multilinear rank**

The aim of this exercise is to prove the upper bound

$$\text{rank}(X) \leq \min\{R_2 R_3, R_1 R_3, R_1 R_2\},$$

where $(R_1, R_2, R_3)$ is the multilinear rank of $X \in \mathbb{R}^{n_1 \times n_2 \times n_3}$. Without loss of generality, the proof can be restricted to showing $\text{rank}(X) \leq R_2 R_3$.

a. Show that $\text{rank}(X) \leq n_2 n_3$.

*Hint: One way to see this is to consider $X^{(1)}$ column by column.*

b. Establish a relation between the tensor rank of $X$ and the tensor rank of the core tensor in a Tucker decomposition of $X$.

c. Combine Points a and b to conclude the result.

**Problem 4: Properties of the Khatri-Rao product**

Show that the following relations hold:

- $(A \odot B) \odot C = A \odot (B \odot C)$,
- $(A \odot B)^T (A \odot B) = A^T A \ast B^T B$,

where $\ast$ denotes the elementwise product, and one assumes that all involved matrix sizes are suitably chosen.

**Problem 5: Implementation of ALS for CP decomposition**

Implement the Alternating Least Squares procedure from the slides. Compare its convergence for a random tensor (of full generic rank) and a random tensor of exact tensor rank $R$ (generated by $[A, B, C]$ for random matrices $A, B, C$).