EXERCISE 9 – Low-rank approximation techniques

Prof. Dr. D. Kressner
A. Cortinovis

Problem 1: Rank-1 tensors

Let $\mathcal{X}$ and $\mathcal{Y}$ be $n_1 \times \cdots \times n_d$ tensors of tensor rank 1:

$$\mathcal{X} = x_1 \circ \cdots \circ x_d, \quad \mathcal{Y} = y_1 \circ \cdots \circ y_d.$$

Show that $\mathcal{X} + \mathcal{Y}$ is of rank 1 if and only if all but at most one of the components x_i and y_i are equal (within a scalar multiple). For the "only if" statement prove only the case $d = 2$ and $n_1 = n_2 = 2$.

Problem 2: Complexity of inner product

What is the complexity of computing the inner product of two tensors of tensor rank at most R given in CP decompositions (with R terms)?

Problem 3: A bound on the multilinear rank

The aim of this exercise is to prove the upper bound

$$\operatorname{rank}(\mathcal{X}) \leq \min\{R_2R_3, R_1R_3, R_1R_2\},$$

where (R_1, R_2, R_3) is the multilinear rank of $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$. Without loss of generality, the proof can be restricted to showing $\operatorname{rank}(\mathcal{X}) \leq R_2R_3$.

a. Show that $\operatorname{rank}(\mathcal{X}) \leq n_2n_3$.

Hint: One way to see this is to consider $X^{(1)}$ column by column.

b. Establish a relation between the tensor rank of $\mathcal{X}$ and the tensor rank of the core tensor in a Tucker decomposition of $\mathcal{X}$.

c. Combine Points **a** and **b** to conclude the result.

Problem 4: Properties of the Khatri-Rao product

Show that the following relations hold:

- $(A \odot B) \odot C = A \odot (B \odot C)$,
- $(A \odot B)^T(A \odot B) = A^TA * B^TB$,

where $*$ denotes the elementwise product, and one assumes that all involved matrix sizes are suitably chosen.

Problem 5: Implementation of ALS for CP decomposition

Implement the Alternating Least Squares procedure from the slides. Compare its convergence for a random tensor (of full generic rank) and a random tensor of exact tensor rank R (generated by $[\![A, B, C]\!]$ for random matrices A, B, C).