

Problem 1: QR and SVD of a Kronecker product

Express the QR factorization/SVD of $A \otimes B$ in terms of the QR factorizations/SVDs of A and B .

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Problem 2: Useful property on vectorization of products

- (a) Prove that, for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$ and $v \in \mathbb{R}^{nq}$, we have

$$(A \otimes B)v = \text{vec}(BVA^T), \quad v = \text{vec}(V).$$

- (b) What is the complexity, in terms of memory and number of operations, of calculating $(A \otimes B)v$ directly and using this property?
- (c) In Matlab, plot the times needed to evaluate $(A \otimes B)v$ in both ways, with $A, B \in \mathbb{R}^{n \times n}$ and $v \in \mathbb{R}^{n^2}$, for $n = 10, 20, \dots, 100$.

Problem 3: Lyapunov equation as a linear system

The Lyapunov matrix equation, which plays an important role in dynamical systems and control, takes the form $AX + XA^T = C$, where A and C are $n \times n$ known coefficient matrices and X is the $n \times n$ solution matrix. Using Kronecker products, reformulate this matrix equation as a linear system.

Problem 4: Approximation by Kronecker products

Given $A \in \mathbb{R}^{n^2 \times n^2}$ we aim to find matrices $B_j, C_j \in \mathbb{R}^{n \times n}$, $j = 1, \dots, r$ such that

$$\left\| A - \sum_{j=1}^r C_j \otimes B_j \right\|_F$$

is minimized. Show how the SVD of a suitable permutation of A can be used to solve this problem.

Hint: First try to solve this by yourself. Note that for $A = I_{n^2}$ only $r = 1$ is needed. If you get stuck, have a look at [Van Loan, C. F. The ubiquitous Kronecker product. J. Comput. Appl. Math. 123 (2000), no. 1-2, 85–100.]

Problem 5: Properties of μ -mode multiplication

Show that for any matrix U with orthonormal columns, and tensors \mathcal{X} and \mathcal{Y} of appropriate size, the following properties hold

- $\mathcal{Y} = \mathcal{X} \circ_{\mu} U \Rightarrow \mathcal{X} = \mathcal{Y} \circ_{\mu} U^T$,
- $\|\mathcal{X}\|_F = \|\mathcal{X} \circ_{\mu} U\|_F$.

Problem 6: μ -mode multiplication for matrices

Let $X = U\Sigma V^T$ be an SVD of a matrix X . Considering X as a tensor of second order, rewrite its SVD in terms of μ -mode multiplication.

Problem 7: Implementing matricization and μ -mode multiplication

In MATLAB, do the following.

1. Create a function for the μ -mode matricization of a $n_1 \times n_2 \times \dots \times n_d$ tensor \mathcal{X} .
2. Create a function for the μ -mode multiplication of a $n_1 \times n_2 \times \dots \times n_d$ tensor \mathcal{X} and a $m \times n_{\mu}$ matrix A .
3. Create a $n \times n \times n$ tensor \mathcal{X} by evaluating the function

$$f(x, y, z) = \frac{1}{(x + y + z)^{\gamma}}, \quad \gamma \in \mathbb{R}^+,$$

on the grid $\{0.1, 0.2, \dots, n/10\}^3$. For different values of γ , plot the singular values of the matricizations $X^{(1)}$, $X^{(2)}$, $X^{(3)}$.