Exercise 11 – Low-rank approximation techniques

Problem 1: Sum of tensors in TT decomposition
Given two tensors \( X \) and \( Y \) in TT decomposition, derive a TT decomposition for \( X + Y \).

Problem 2: TT ranks and multilinear rank
Given a tensor in TT decomposition, derive upper bounds for its multilinear rank.

Problem 3: Implementation of TT-SVD
Implement the TT-SVD algorithm from the slides. Adjust the algorithm to work with prescribed accuracy \( \varepsilon > 0 \) for which it determines the TT ranks adaptively such that \( \|X - X_{\text{SVD}}\| \leq \varepsilon \). (Use the theorem from the slides.)

Problem 4: Left/right orthogonalization of TT tensors
Let \( X \) be a tensor in TT decomposition with TT cores \( U_1, \ldots, U_d \).

1. Show that \( \|X\| = \|U_d\| \) holds for a left-orthogonal TT decomposition. What can you say about the singular values of \( X^{<d-1>} \)?
2. Show that \( \|X\| = \|U_1\| \) holds for a right-orthogonal TT decomposition. What can you say about the singular values of \( X^{<1>} \)?
3. In Matlab, implement the left and right orthogonalization process as described in the slides and check the conjectures from Points 1 and 2.

Problem 5: Computing the average of a TT tensor
Using tensor contractions, develop an efficient method for computing the mean,

\[
\bar{X} = \frac{1}{n_1 \cdots n_d} \sum_{i_1, \ldots, i_d} X(i_1, \ldots, i_d),
\]

of a tensor in TT decomposition.