Exercise 10 – Low-rank approximation techniques

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Problem 1: Symmetric tensors

An $n \times \cdots \times n$ tensor $\mathcal{X}$ of order d is called symmetric if

$$\mathcal{X}_{i_1,\ldots,i_d} = \mathcal{X}_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}$$

holds for every permutation $\sigma : \{1,\ldots,d\} \to \{1,\ldots,d\}$. What can you say about the multilinear rank of such a tensor?

Problem 2: Multiplication with orthogonal matrices

Show that for a tensor $\mathcal{X}$ of order 3, U_1, U_2, U_3 columnwise orthogonal matrices and $\mathcal{C} = U_1^T \circ_1 U_2^T \circ_2 U_3^T \circ_3 \mathcal{X}$, the following equality holds

$$\|\mathcal{X} - U_1 \circ_1 U_2 \circ_2 U_3 \circ_3 \mathcal{C}\|^2 = \|\mathcal{X}\|^2 - \|U_1^T \circ_1 U_2^T \circ_2 U_3^T \circ_3 \mathcal{X}\|^2.$$

Problem 3: Implementation of HOSVD and STHOSVD

1. Implement the HOSVD algorithm from the slides.
2. There is a more efficient variant, called the Sequentially Truncated HOSVD (STHOSVD), which proceeds as follows:

   a. Calculate SVD $X^{(1)} = \widetilde{U}_1 \widetilde{\Sigma}_1 \widetilde{V}_1^T$. Truncate $U_1 := \widetilde{U}_1(:, 1 : r_1)$ and update $\mathcal{X} \leftarrow U_1^T \circ_1 \mathcal{X}$.

   b. Calculate SVD $X^{(2)} = \widetilde{U}_2 \widetilde{\Sigma}_2 \widetilde{V}_2^T$. Truncate $U_2 := \widetilde{U}_2(:, 1 : r_2)$ and update $\mathcal{X} \leftarrow U_2^T \circ_2 \mathcal{X}$.

   c. Calculate SVD $X^{(3)} = \widetilde{U}_3 \widetilde{\Sigma}_3 \widetilde{V}_3^T$. Truncate $U_3 := \widetilde{U}_3(:, 1 : r_3)$ and update $\mathcal{X} \leftarrow U_3^T \circ_3 \mathcal{X}$.

   d. Set $\mathcal{C} = \mathcal{X}$.

   Any order of truncation is possible. It is usually most efficient to proceed from the largest to the smallest mode size.

   Implement STHOSVD and compare the performance of both algorithms for random tensors of different sizes.
3. Adjust both algorithms to work with predescribed accuracy ε (discard involved singular values lower than ε). Test on function-related tensors, e.g. from Problem 7 of Exercise set 8.
4. *Optional*: Extend the quasi-optimality results on the error of HOSVD in the lecture to STHOSVD.

Problem 4: Inner product of two tensors in TT decomposition

Consider two d-dimensional tensors $\mathcal{X}$ and $\mathcal{Y}$ of size $n \times n \times \cdots \times n$ which are written in TT decomposition with ranks $(R, R, \ldots, R)$ and $(S, S, \ldots, S)$ respectively. What is the complexity of computing the inner product $\langle \mathcal{X}, \mathcal{Y} \rangle$?

Problem 5: An upper bound on the TT rank

Show that a tensor of tensor rank R has TT rank at most $(R, \ldots, R)$ by converting its CP decomposition into a TT decomposition.