Exercise 6 – Low-rank approximation techniques

Prof. Dr. D. Kressner
A. Cortinovis

Problem 1: Sharpness of the existence result on CUR decomposition

Let the SVD of an $n \times n$ matrix A be given as $A = U\Sigma V^T$, where

$$U = \begin{bmatrix} u_1 & U_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{\varepsilon} & \\ & \varepsilon I \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & V_2 \end{bmatrix},$$

with $v_1 = \begin{bmatrix} 1/\sqrt{n} & 1/\sqrt{n} & \cdots & 1/\sqrt{n} \end{bmatrix}^T$ and $\varepsilon = \dfrac{1}{n-1}$. From the existence result on CUR decomposition, we know that for $k = 1$ there exist a column c_j and a row r_i of A and a number s such that

$$\|A - c_j s r_i\|_2 \leq \varepsilon(1 + 4\sqrt{n}).$$

Show that for any column c_j and row r_i of A and any number s it holds that

$$\|A - c_j s r_i\|_2 \geq \sqrt{\frac{\varepsilon}{2}} - \varepsilon.$$

Compare the upper and the lower bound for $n = 20$.

Problem 2: ACA with partial and complete pivoting

In MATLAB, implement adaptive cross approximation - with full pivoting and with partial pivoting. Apply it to the Hilbert and exponential matrix and reproduce the two graphs from the slides.

Problem 3: Schur complements preserve positive definiteness

Let the matrix

$$A = \begin{bmatrix} \alpha & b^T \\ b & C \end{bmatrix} \in \mathbb{R}^{n \times n},$$

with $\alpha > 0$, be symmetric and positive definite. Show that its Schur complement

$$S = C - \frac{1}{\alpha} b b^T \in \mathbb{R}^{(n-1)\times(n-1)}$$

is also symmetric and positive definite.

This implies that, when doing ACA on a symmetric positive definite matrices with diagonal pivoting, the remainders R_k are symmetric positive definite.

Problem 4: Maximum volume in symmetric positive definite matrices

Let $A \in \mathbb{R}^{n \times n}$ be a nonzero symmetric positive definite matrix and let $k \in \{1, \ldots, n\}$. The goal of this exercise is to prove that

$$\begin{aligned} &\max\{|\det(A(I,J))| : I, J \subset \{1, \ldots, n\}, \#I = \#J = k\} \\ &\qquad = \max\{|\det(A(I,I))| : I \subset \{1, \ldots, n\}, \#I = k\}. \end{aligned} \tag{1}$$

As principal submatrices of a symmetric positive definite matrices are symmetric positive definite, this implies that A has a $k \times k$ symmetric positive definite submatrix of maximal volume.

1. Prove (1) for $k = 1$.
2. For $X \in \mathbb{R}^{n \times n}$, the k-th *compound matrix* $C_k(X)$ is defined as the $\binom{n}{k} \times \binom{n}{k}$ matrix containing the determinants of all the $k \times k$ submatrices of X, with rows and columns indexed by the subsets of cardinality k of $\{1, 2, \ldots, n\}$ in lexicographical order.

   The diagonal elements of $C_k(X)$ correspond to which submatrices of X?
3. Prove that for any $n \times n$ matrices X and Y we have $C_k(XY) = C_k(X)C_k(Y)$.

   Hint: Use Cauchy-Binet formula.
4. Prove that for a symmetric positive definite matrix A, for all $k \in \{1, \ldots, n\}$ its compound matrix $C_k(A)$ is symmetric positive definite.

   Hint: Consider a Cholesky decomposition of A.
5. Combine the previous points to prove (1) for general k.