Exercise 5 – Low-rank approximation techniques

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Problem 1: Volume and singular values

Prove that the volume of a square matrix is equal to the product of its singular values.

Problem 2: Tightness & extension of lemma "Maximal volume yields good submatrix"

1. Show that the inequality in the lemma "Maximal volume yields good submatrix" cannot be improved in general by constructing a matrix $U = \begin{pmatrix} I_r \\ B \end{pmatrix}$ with $\max |b_{ij}| \leq 1$ and $\|B\|_2 = \sqrt{(n-r)r}$.
2. Prove the following extension of the result of the lemma: For an arbitrary $n \times r$ matrix U of rank r, there is an index set I with $\#I = r$ such that
$$\frac{1}{\sigma_{\min}(U(I,:))} \leq \sqrt{r(n-r)+1}\frac{1}{\sigma_{\min}(U)}.$$

Problem 3: The inverse of a matrix used for counterexamples

For the $n \times n$ matrix

$$Z_n = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ & 1 & -1 & \cdots & -1 \\ & & 1 & \cdots & -1 \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}$$

prove that $(Z_n^{-1})_{ij} = 2^{j-i-1}$ for $j > i$.

Problem 4: Knuth's iterative exchanges of rows improve the volume

Let $U \in \mathbb{R}^{n\times r}$, let $I = \{1, 2, \dots, r\}$ and consider the matrix $\tilde{U} := UU(I,:)^{-1}$. Take $(i^*, j^*) := \arg\max |\tilde{u}_{ij}|$ and define a new index set $J := I \cup \{i^*\} \backslash \{j^*\}$.

Prove that $\mathrm{Vol}(U(J,:)) = |\tilde{u}_{i^*j^*}| \cdot \mathrm{Vol}(U(I,:))$ and conclude that using the algorithm in the slide "Row selection beyond greedy" the volume of the selected submatrix improves by a factor $\geq \mu$ after each row exchange.

Problem 5: Norm of oblique projectors

Prove that any projector $P \notin \{0, I\}$ satisfies $\|I - P\|_2 = \|P\|_2$.

Hint: Consider a unit vector u such that $\|Pu\|_2 = \|P\|_2$ and let

$$w := \frac{\|(I-P)u\|_2}{\|Pu\|_2} Pu + \frac{\|Pu\|_2}{\|(I-P)u\|_2}(I-P)u.$$

Prove that w has unit norm and use it to show that $\|P\|_2 \leq \|I - P\|_2$.

Problem 6: Matlab exercise

1. Implement the algorithm "Greedy row selection from orthonormal basis".
2. Given an $m \times n$ matrix A and an integer $r \leq \min\{m, n\}$, implement the following method for obtaining a rank-r approximation:
   - Apply r steps of the greedy method to left singular vectors U_r of A to determine I.
   - Apply r steps of the greedy method to right singular vectors V_r of A to determine J.
   - Return rank-r approximation $A(:,J)(A(I,J))^{-1}A(I,:)$.
3. Apply your implementation to the two 100×100 matrices $A_{ij} = \frac{1}{i+j-1}$ for $r = 1, \dots, 30$ and $A_{ij} = \exp\left(-\frac{|i-j|}{1000}\right)$ for $r = 1, \dots, 100$. Plot the singular values and the obtained approximation error in the spectral norm.