Exercise 2 – Low-rank approximation techniques

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Problem 1. Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and $k \leq n$. Use von Neumann's trace inequality to prove that

$$\max\{\|Q^T A\|_F : Q \in \mathbb{R}^{m\times k},\ Q^T Q = I_k\} = \sqrt{\sigma_1^2 + \cdots + \sigma_k^2},$$

where $\sigma_1, \dots, \sigma_k$ are the singular values of A.

Problem 2: Properties of the Schatten norms

Let A and B be $n \times n$ matrices.

1. Prove that $\|A\|_{(1)} \geq \|A\|_{(p)} \geq \|A\|_{(q)} \geq \|A\|_{(\infty)}$ for all $1 < p < q$.
2. Prove that, for $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|A\|_{(p)} = \max_{\|C\|_{(q)}=1} \langle A, C\rangle.$$

3. Prove that the Schatten norms are actually norms, that is, $\|A\|_{(p)} \geq 0$ with equality if and only if $A = 0$, $\|\alpha A\|_{(p)} = |\alpha|\|A\|_{(p)}$ for all $\alpha \in \mathbb{R}$, and the triangular inequality $\|A + B\|_{(p)} \leq \|A\|_{(p)} + \|B\|_{(p)}$ holds.

Problem 3: Orthogonal projections

A matrix $P \in \mathbb{R}^{n\times n}$ is an *orthogonal projection* onto a subspace S if $\text{range}(P) = S$, $P^2 = P$ and $P^T = P$.

1. Prove that the orthogonal projection onto S is unique.

   Hint: Prove that if P_1 and P_2 are orthogonal projections, then $\|(P_1 - P_2)z\|_2^2 = (P_1 z)^T (I - P_2)z + (P_2 z)^T (I - P_1)z$ for all $z \in \mathbb{R}^n$.

2. Prove that, if X is a basis of S, the orthogonal projection on S can be written as $X(X^T X)^{-1} X^T$; if X is an orthonormal basis, then the orthogonal projection is XX^T.
3. Let $A \in \mathbb{R}^{n\times n}$ be a matrix with n distinct singular values. Is the solution of the problem

$$\min_{P \text{ orthogonal projection on a rank-}k\text{ subspace}} \|(I - P)A\|_F$$

   unique? What about the spectral norm?

Problem 4: Density of full-rank matrices

Prove that for all matrices $A \in \mathbb{R}^{n\times n}$ there exists a sequence $\{A_k\}_{k\in\mathbb{N}}$ of matrices of rank n such that $\|A - A_k\|_2 \to 0$ for $k \to \infty$.

Problem 5: Computing best rank-k approximations Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Compute an SVD of A, and best rank-1 approximations in the spectral norm and in the Frobenius norm.

Problem 6: Singular values of a submatrix

Let $A \in \mathbb{R}^{n\times m}$ and let B be an $(n-1) \times m$ matrix obtained by eliminating a row from A. In Matlab / Julia / Python, compare the largest singular value of A and B. What do you notice? Prove your conjecture.