Problem 1: $\mathcal{X} = x_1 \circ \dots \circ x_d$

$\mathcal{Y} = y_1 \circ \dots \circ y_d$

PROVE: $\mathcal{X}+\mathcal{Y}$ is rank-1 $\iff$ all but most one of the components x_i and y_i are equal

$(\Leftarrow)$ w.l.o.g.: $x_i = y_i$, $i = 2, \dots, d$

$$\mathcal{X}+\mathcal{Y} = (x_1 \circ \dots \circ x_d) + (y_1 \circ \dots \circ y_d)$$

$$\mathrm{vec}(\mathcal{X}+\mathcal{Y}) = \mathrm{vec}(\mathcal{X}) + \mathrm{vec}(\mathcal{Y})$$

$$= (x_d \otimes \dots \otimes x_2) \otimes x_1 + (x_d \otimes \dots \otimes x_2) \otimes y_1$$

$$= (x_d \otimes \dots \otimes x_2) \otimes (x_1 + y_1)$$

$$\Rightarrow \mathcal{X}+\mathcal{Y} = (x_1+y_1) \circ x_2 \circ \dots \circ x_d$$

$(\Rightarrow)$ $d = 2$, $n_1 = n_2 = 2$

$$\mathcal{X} = \begin{bmatrix} a \\ b \end{bmatrix} \circ \begin{bmatrix} c \\ d \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} e \\ f \end{bmatrix} \circ \begin{bmatrix} g \\ h \end{bmatrix}, \quad \mathcal{X}+\mathcal{Y} = \begin{bmatrix} p \\ q \end{bmatrix} \circ \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\mathrm{vec}(\mathcal{X}) = \begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathrm{vec}(\mathcal{Y}) = \begin{bmatrix} g \\ h \end{bmatrix} \otimes \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\mathrm{vec}(\mathcal{X}+\mathcal{Y}) = \begin{bmatrix} r \\ s \end{bmatrix} \otimes \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\mathrm{vec}(X) + \mathrm{vec}(Y) = \mathrm{vec}(X+Y)$$

$$\begin{bmatrix} ca \\ cb \\ da \\ db \end{bmatrix} + \begin{bmatrix} ge \\ gf \\ he \\ hf \end{bmatrix} = \begin{bmatrix} rp \\ rq \\ sp \\ sq \end{bmatrix}$$

$$\left.\begin{aligned} ca + ge &= rp \\ cb + gf &= rq \end{aligned}\right\} \quad \frac{p}{q} = \frac{ca+ge}{cb+gf}$$

$$\left.\begin{aligned} da + he &= sp \\ db + hf &= sq \end{aligned}\right\} \quad \frac{p}{q} = \frac{da+he}{db+hf}$$

$$(ca+ge)(db+hf) = (cb+gf)(da+he)$$

$$\cancel{cadb} + cahf + gedb + \cancel{gehf} = \cancel{cbda} + cbhe + gfda + \cancel{gfhe}$$

$$ch(af-be) + dg(be-af) = 0$$

$$(af-be)(ch-dg) = 0$$

- $af = be \rightarrow \begin{bmatrix} a & e \\ b & e \end{bmatrix} = \begin{bmatrix} a & e \\ a & f \end{bmatrix}$
- $ch = dg \rightarrow \begin{bmatrix} ch \\ dh \end{bmatrix} = \begin{bmatrix} dg \\ dh \end{bmatrix}$

✓

Problem 2 :

$$\mathcal{X} = \sum_{r=1}^{R} a_1^{(r)} \circ a_2^{(r)} \circ a_3^{(r)}$$

$$\mathcal{Y} = \sum_{p=1}^{R} b_1^{(p)} \circ b_2^{(p)} \circ b_3^{(p)}$$

$$\left.\right\} n_1 \times n_2 \times n_3$$

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \mathrm{vec}(\mathcal{X})^T \mathrm{vec}(\mathcal{Y}) =$$

$$= \sum_{r=1}^{R} \left(a_3^{(r)} \otimes a_2^{(r)} \otimes a_1^{(r)}\right)^T \sum_{p=1}^{R} \left(b_3^{(p)} \otimes b_2^{(p)} \otimes b_1^{(p)}\right)$$

$$= \sum_{r=1}^{R} \sum_{p=1}^{R} \underbrace{\left(a_3^{(r)T} b_3^{(p)}\right)}_{O(n_3)} \underbrace{\left(a_2^{(r)T} b_2^{(p)}\right)}_{O(n_2)} \underbrace{\left(a_1^{(r)T} b_1^{(p)}\right)}_{O(n_1)}$$

$$\Rightarrow O\left(R^2 (n_1 + n_2 + n_3)\right)$$

Problem 3 : PROVE : $\mathrm{rank}(\mathcal{X}) \leq \min\{R_2R_3, R_1R_3, R_1R_2\}$

$R_\mu = \mathrm{rank}(X^{(\mu)})$, $\mu = 1,2,3$, $\mathcal{X}$ $n_1 \times n_2 \times n_3$

W.L.O.G. PROVE : $\mathrm{rank}(\mathcal{X}) \leq R_2R_3$

1. $X^{(1)}$ $n_1 \times n_2n_3$

$$X^{(1)} = [x_1 \dots x_{n_2n_3}] = \sum_{i=1}^{n_2n_3} x_i e_i^T$$

$$\mathrm{vec}(\mathcal{X}) = \mathrm{vec}(X^{(1)}) = \sum_{i=1}^{n_2n_3} e_i \otimes x_i = \sum_{i=1}^{n_2n_3} (g_i \otimes h_i \otimes x_i)$$

$e_i = g_i \otimes h_i$, for appropriate g_i, h_i

$\Rightarrow \mathrm{rank}(\mathcal{X}) \leq n_2n_3$

2. $R = \mathrm{rank}(\mathcal{C}) \Rightarrow (\exists\, a_i, b_i, c_i)\ \mathrm{vec}(\mathcal{C}) = \sum_{i=1}^{R} c_i \otimes b_i \otimes a_i$

$$\mathrm{vec}(\mathcal{X}) = (W \otimes V \otimes U)\,\mathrm{vec}(\mathcal{C}) = (W \otimes V \otimes U) \sum_{i=1}^{R} c_i \otimes b_i \otimes a_i$$

$$= \sum_{i=1}^{R} (Wc_i) \otimes (Vb_i) \otimes (Ua_i)$$

$\Rightarrow \mathrm{rank}(\mathcal{X}) \leq \mathrm{rank}(\mathcal{C}) \leq R_2R_3$

Problem 4 :

1) $(A \odot B) \odot C = A \odot (B \odot C)$

$A \quad n_1 \times R$

$B \quad n_2 \times R$

$C \quad n_3 \times R$

$A \odot B = [a_1 \otimes b_1 \quad \dots \quad a_R \otimes b_R]$, $B \odot C = [b_1 \otimes c_1 \quad \dots \quad b_R \otimes c_R]$

$(A \odot B) \odot C = [a_1 \otimes b_1 \otimes c_1 \quad \dots \quad a_R \otimes b_R \otimes c_R]$

$A \odot (B \odot C) = [a_1 \otimes b_1 \otimes c_1 \quad \dots \quad a_R \otimes b_R \otimes c_R]$

2) $(A \odot B)^t (A \odot B) = A^t A * B^t B$

$$(A \odot B)^t (A \odot B) = \begin{bmatrix} a_1^t \otimes b_1^t \\ \vdots \\ a_R^t \otimes b_R^t \end{bmatrix} [a_1 \otimes b_1 \quad \dots \quad a_R \otimes b_R]$$

$$= \begin{bmatrix} (a_1^t \otimes b_1^t)(a_1 \otimes b_1) & \dots & (a_1^t \otimes b_1^t)(a_R \otimes b_R) \\ \vdots & & \\ (a_R^t \otimes b_R^t)(a_1 \otimes b_1) & \dots & (a_R^t \otimes b_R^t)(a_R \otimes b_R) \end{bmatrix}$$

$$= \begin{bmatrix} a_1^t a_1 \cdot b_1^t b_1 & \dots & a_1^t a_R \cdot b_1^t b_R \\ \vdots & & \\ a_R^t a_1 \cdot b_R^t b_1 & \dots & a_R^t a_R \cdot b_R^t b_R \end{bmatrix} = A^t A * B^t B$$

Problem 6 : $\mathcal{X}$ $n \times n \times \dots \times n$

$\mathcal{X}$ symm $\longrightarrow$ $\mathcal{X}_{i_1 \dots i_d} = \mathcal{X}_{\sigma(i_1), \dots, \sigma(i_d)}$, $\forall$ permutation $\sigma : \{1, \dots, d\} \to \{1, \dots, d\}$

multilinear rank $(\mathcal{X}) = ?$

$\mathcal{X}$ symmetric $\Rightarrow$ $X^{(1)} = X^{(2)} = X^{(3)}$ $\Rightarrow$

multilinear rank $(\mathcal{X}) = (R, R, R)$,

$R = \text{rank}(X^{(n)})$

Problem 7 :

$$\|\mathcal{X} - U_1 \circ_1 U_2 \circ_2 U_3 \circ_3 C\|^2 =$$

$$= \|\mathcal{X}\|^2 + \|U_1 \circ_1 U_2 \circ_2 U_3 \circ_3 C\|^2 - 2\langle \mathcal{X}, U_1 \circ_1 U_2 \circ_2 U_3 \circ_3 C\rangle$$

$$= \|\mathcal{X}\|^2 + \|C\|^2 - 2\langle \underbrace{U_1^t \circ_1 U_2^t \circ_2 U_3^t \circ_3 \mathcal{X}}_{C}, C\rangle$$

$$= \|\mathcal{X}\|^2 - \|C\|^2$$

$$= \|\mathcal{X}\|^2 - \|U_1^t \circ_1 U_2^t \circ_2 U_3^t \circ \mathcal{X}\|^2$$

Problem 8.4.

Same results hold for STHOSVD.

Problem 5.

```
clear all
X=rand(2,2,2);
[A,B,C]=als(X,3,1e-6);
norm(tenmat(X,1)-A*khatri_rao(C,B)','fro') %tenmat from Exercise 6

n1=20;
n2=30;
n3=25;
R=10;
A=rand(n1,R);
B=rand(n2,R);
C=rand(n3,R);
X=matten(A*khatri_rao(C,B)',1,[n1,n2,n3]);
[A,B,C]=als(X,R,1e-6);
norm(tenmat(X,1)-A*khatri_rao(C,B)','fro')
```

```
%inverse of tenmat - reshapes matrix back to tensor by a given mode
function Aten = matten(A,n,dims)
if dims(n)≠size(A,1)
    error("Dimensions mismatch.")
end
m = setdiff(1:length(dims),n);
if prod(dims(m))≠size(A,2)
    error("Dimensions mismatch.")
end
Aten = reshape(A,[dims(n),dims(m)]);
Aten = ipermute(Aten,[n,m]);
end
```

```
function C=khatri_rao(A,B)
[m,n]=size(A);
[p,q]=size(B);
if n≠q
    error("Dimension mismatch.")
end
C=zeros(m*p,n);
for i=1:n
    C(:,i)=kron(A(:,i),B(:,i));
end
end
```

```
function [A,B,C]=als(X,R,tol)
[¬,n2,n3]=size(X);
B=rand(n2,R);
C=rand(n3,R);
B=nrm(B);
res=1;
maxit=1000;
it=1;
while res>tol && it<maxit
    C=nrm(C);
    A=tenmat(X,1)*khatri_rao(C,B)/((C'*C).*(B'*B));
    A=nrm(A);
    B=tenmat(X,2)*khatri_rao(C,A)/((C'*C).*(A'*A));
    B=nrm(B);
    C=tenmat(X,3)*khatri_rao(B,A)/((B'*B).*(A'*A));
    res=norm(tenmat(X,1)-A*khatri_rao(C,B)','fro');
    it=it+1;
```

```
end
end

function Anorm=nrm(A)
[m,n]=size(A);
Anorm=zeros(m,n);
for i=1:n
    Anorm(:,i)=A(:,i)./norm(A(:,i));
end
end
```

Problem 8.

```
function [C,U1,U2,U3]=hosvd(X,R,tol)

Xmat=tenmat(X,1);
[U,S,¬]=svd(Xmat);
if tol==0
    U1=U(:,1:R);
else
    K=find(diag(S)>tol);
    U1=U(:,K);
end

C=ttm(X,U1',1);

Xmat=tenmat(X,2);
[U,S,¬]=svd(Xmat);
if tol==0
    U2=U(:,1:R);
else
    K=find(diag(S)>tol);
    U2=U(:,K);
end

C=ttm(C,U2',2);

Xmat=tenmat(X,3);
[U,S,¬]=svd(Xmat);
if tol==0
    U3=U(:,1:R);
else
    K=find(diag(S)>tol);
    U3=U(:,K);
end

C=ttm(C,U3',3);

end
```

```
function [X,U1,U2,U3]=sthosvd(X,R,tol)
Xmat=tenmat(X,1);
[U,S,¬]=svd(Xmat);
if tol==0
    U1=U(:,1:R);
else
    K=find(diag(S)>tol);
    U1=U(:,K);
end
X=ttm(X,U1',1);

Xmat=tenmat(X,2);
[U,S,¬]=svd(Xmat);
if tol==0
    U2=U(:,1:R);
else
    K=find(diag(S)>tol);
    U2=U(:,K);
end


X=ttm(X,U2',2);
```

```

Xmat=tenmat(X,3);
[U,S,¬]=svd(Xmat);
if tol==0
    U3=U(:,1:R);
else
    K=find(diag(S)>tol);
    U3=U(:,K);
end

X=ttm(X,U3',3);
end
```

```
clear all
X=rand(6,6,6);
[C,U1,U2,U3]=hosvd(X,6,0);
T=ttm(C,U1,1);
T=ttm(T,U2,2);
T=ttm(T,U3,3);

norm(X(:)—T(:))


[C,U1,U2,U3]=sthosvd(X,6,0);
T=ttm(C,U1,1);
T=ttm(T,U2,2);
T=ttm(T,U3,3);
norm(X(:)—T(:))/norm(X(:))

f = @(x,y,z,gamma) 1./((x+y+z).^gamma);
gamma=0.5;
n=50;
x=[0.1:0.1:n/10];
y=[0.1:0.1:n/10];
z=[0.1:0.1:n/10];
X=zeros(n,n,n);
for i=1:n
    for j=1:n
        for k=1:n
            X(i,j,k)=f(x(i),y(j),z(k),gamma);
        end
    end
end
[C,U1,U2,U3]=hosvd(X,0,1e—8);
T=ttm(C,U1,1);
T=ttm(T,U2,2);
T=ttm(T,U3,3);

norm(X(:)—T(:))


[C,U1,U2,U3]=sthosvd(X,0,1e—8);
T=ttm(C,U1,1);
T=ttm(T,U2,2);
T=ttm(T,U3,3);
norm(X(:)—T(:))/norm(X(:))
```