Problem 1 : $A \in \mathbb{R}^{n \times n}$

$\mathrm{Vol}(A) = |\det(A)|$

$$|\det(A)| = \prod_{i=1}^{n} \sigma_i(A)$$

$$A = U \Sigma V^T \to \det(A) = \det(U \Sigma V^T) = \det(U) \det(\Sigma) \det(V^T)$$

$$U^T U = I \Rightarrow \det(U^T U) = \det^2(U) = \det(I) = 1$$

$$\Rightarrow \det(U) = \pm 1$$

$$V^T V = I \Rightarrow \det(V) = \pm 1$$

$$\det(\Sigma) = \prod_{i=1}^{n} \sigma_i(A)$$

$$\Rightarrow |\det(A)| = \prod_{i=1}^{n} \sigma_i(A).$$

Problem 2 : $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times n}$, $m > n$

A_i $n \times (m-1)$

B_i $(m-1) \times n$

$$\det(AB) = \frac{1}{m-n} \sum_{i=1}^{m} \det(A_i B_i).$$

Cauchy-Binet $\to$ $\det(AB) = \sum_{I} \det(A(:,I)) \det(B(I,:))$, $\# I = n$

- $A(:,I)$ submatrix of A_i, $i \notin I$ $\to$ there are $m-n$ different values of i such that $A(:,I)$ is a submatrix of A_i
- same for $B(I,:)$

$$\Rightarrow \sum_{i=1}^{m} \det(A_i B_i) = (m-n) \sum_{I} \det(A(:,I)) \det(B(I,:))$$

$$= (m-n) \det(AB)$$

Problem 3:

1) $\max |b_{ij}| \leq 1 \Rightarrow \|B\|_2 \leq \sqrt{(n-r)\cdot r}$

$B \quad (n-r)\times r$

$$B = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & & 1 \end{bmatrix}}_{(n-r)\times r} \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{r\times 1} = \begin{bmatrix} r \\ \vdots \\ r \end{bmatrix} = r \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{(n-r)\times 1}$$

$$\|B\|_2 = \max \frac{\|Bx\|_2}{\|x\|_2} \underset{x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}{\geq} \frac{\sqrt{r^2(n-r)}}{\sqrt{r}} = \sqrt{r(n-r)}$$

$$\Rightarrow \|B\|_2 = \sqrt{(n-r)\cdot r}$$

$$\Rightarrow \|U\|_2 = \sqrt{1+(n-r)\cdot r}$$

2) · for general U $n\times r$ and $U(I_1:)$ such that $|\det(U(I_1:))|$ is maximized :

$$\tilde{U} = U \cdot U(I_1:)^{-1} = \begin{pmatrix} I_r \\ B \end{pmatrix}$$

$$\|\tilde{U}\|_2 \leq \sqrt{1+(n-r)\cdot r}$$

PSEUDO-INVERSE

$$\tilde{U} = U \cdot U(I_1:)^{-1} \Rightarrow U(I_1:)^{-1} = U^+\tilde{U}$$

$$\Rightarrow \|U(I_1:)^{-1}\|_2 \leq \|U^+\|_2 \|\tilde{U}\|_2$$

$$\Rightarrow \frac{1}{\sigma_{min}(U(I_1:))} \leq \frac{1}{\sigma_{min}(U)} \sqrt{1+(n-r)\cdot r}$$

Problem 4:

$$Z_n = \begin{bmatrix} 1 & -1 & \cdots & -1 \\ & 1 & \ddots & \vdots \\ & & \ddots & -1 \\ & & & 1 \end{bmatrix} \qquad (Z_n^{-1})_{ij} = 2^{j-i-1}, \; j > i$$

- $n = 2$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$Z_2^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \to 2^{2-1-1} = 2^0$$

- assume for any n $\quad Z_n^{-1} = \begin{bmatrix} 1 & 1 & 2 & \cdots & 2^{n-2} \\ & \ddots & & & \vdots \\ & & & & 2 \\ & & & & 1 \\ & & & & 1 \end{bmatrix}$

- $$Z_{n+1} = \left[\begin{array}{c|c} Z_n & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \to Z_{n+1}^{-1} = \begin{bmatrix} Z_n^{-1} & -Z_n^{-1} b \\ 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} Z_n & b \\ \hline 0 & 1 \end{array}\right]$$

$$-Z_n^{-1} b = \begin{bmatrix} 1 & 1 & 2 & \cdots & 2^{n-2} \\ & \ddots & & & \\ & & & & 2 \\ & & & & 1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 2^{n-1} \\ \vdots \\ 2^2 \\ 2 \\ 1 \end{bmatrix}$$

Problem 5 : $\| I - P \|_2 = \| P \|_2$, P projector

Let $\|u\|_2 = 1$ arbitrary. $\mathcal{X} = \mathcal{R}(P)$, $\mathcal{Y} = \mathcal{N}(P)$

$x = Pu \in \mathcal{X}$

$y = (I-P)u \in \mathcal{Y}$ $\rightarrow$ $u = x + y$

$$\|u\|_2^2 = \|x\|_2^2 + \|y\|_2^2 + 2\langle x, y\rangle = 1$$

$$\tilde{x} := \frac{\|y\|_2}{\|x\|_2} x \in \mathcal{X}, \quad \tilde{y} := \frac{\|x\|_2}{\|y\|_2} y \in \mathcal{Y}$$

$$w := \tilde{x} + \tilde{y} \Rightarrow \|w\|_2^2 = \|\tilde{x}\|_2^2 + \|\tilde{y}\|_2^2 + 2\langle \tilde{x}, \tilde{y}\rangle$$

$$= \|y\|_2^2 + \|x\|_2^2 + 2\langle x, y\rangle = 1$$

$Pw = \tilde{x}$ $\rightarrow$ $\tilde{y} = w - Pw$

$$\|Pu\|_2 = \|x\|_2 = \|\tilde{y}\|_2 = \|(I-P)w\|_2 \leq \|I-P\|_2$$

$$\Rightarrow \|P\|_2 \leq \|I-P\|_2$$

by symmetry: $\|I-P\|_2 \leq \|P\|_2$

$$\Rightarrow \|I-P\|_2 = \|P\|_2$$

Problem 7.

```
n=100;
A=hilb(n);
R=30;

figure(1)
lowrank_approx(A,R);

A=zeros(n,n);
gamma=0.1;
for i=1:n
    for j=1:n
        A(i,j)=exp(-gamma*abs(i-j)/n);
    end
end
R=100;

figure(2)
lowrank_approx(A,R);
```

```
function lowrank_approx(A,R)

[U,S,V]=svd(A,'econ');
s=diag(S);

err_greedy=zeros(R,1);
for r=1:R

    Ur=U(:,1:r);
    I=greedy(Ur);

    Vr=V(:,1:r);
    J=greedy(Vr);
    Ar=A(:,J)/A(I,J)*A(I,:);
    err_greedy(r)=norm(A-Ar);
end
semilogy(1:R,err_greedy)
hold on
semilogy(1:R,s(1:R))
legend("err","svdvals")
hold off
end
```

```
function I=greedy(U)
r=size(U,2);
I=[];
for k=1:r
    [¬, imax]=max(abs(U(:,k)));
    imax=imax(1);
    I=[I;imax];
    U=U-1/U(imax,k)*U(:,k)*U(imax,:);
end
end
```