Problem 1: $A \in \mathbb{R}^{m \times n}$, $m \geq n$, $k \leq n$

PROVE: $\max \{ \|Q^T A\|_F : Q \in \mathbb{R}^{m \times k}, Q^T Q = I_k \} = \sqrt{\sigma_1^2 + \dots + \sigma_k^2}$

$$\|Q^T A\|_F^2 = \langle Q^T A, Q^T A \rangle = tr(A^T Q Q^T A) = tr(Q Q^T A A^T) =$$

$$= \langle AA^T, QQ^T \rangle \underset{\uparrow}{\leq} \sigma_1(AA^T)\sigma_1(QQ^T) + \dots + \sigma_n(AA^T)\sigma_m(QQ^T)$$

Von Neumann

$$Q \; m \times k \rightarrow Q = U \Sigma V^T = U \begin{bmatrix} \Sigma_k \\ 0 \end{bmatrix} V^T$$

$$QQ^T = U \begin{bmatrix} \Sigma_k \\ 0 \end{bmatrix} V^T V [\Sigma_k \; 0] U^T = U \begin{bmatrix} \Sigma_k^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$\sigma_i(QQ^T) = 0, \; i = k+1, \dots, n$$

$$Q^T Q = I \Rightarrow V_k / \; V_k \Sigma_k U_k^T U_k \Sigma_k V_k^T = I \; / V_k^T$$

$$\Sigma_k^2 = I \Rightarrow \Sigma_k = I$$

$$\Rightarrow \sigma_i(QQ^T) = 1, \; i = 1, \dots, k$$

$$A = U_n \Sigma_n V_n^T \rightarrow AA^T = U_n \Sigma_n V_n^T V_n \Sigma_n U_n^T = U \Sigma_m^2 U_n^T$$

$$\Rightarrow \sigma_i(AA^T) = \sigma_i(A)^2, \; i = 1, \dots, n$$

$$\Rightarrow \|Q^T A\|_F \leq \sqrt{\sigma_1^2 + \dots + \sigma_k^2}$$

Problem 2.1: $\cos \measuredangle(x,y) = \dfrac{y^t x}{\|x\|_2 \|y\|_2}$, $\measuredangle(x,y) \in [0,\pi)$

- $|y^t x| \le \|x\|_2 \|y\|_2$ $\leftarrow$ Cauchy-Schwarz
- $\measuredangle(x,y) = 0 \iff \operatorname{span}(x) = \operatorname{span}(y)$

$(\Rightarrow)$ $y^t x = \|x\|_2 \|y\|_2$

$$\forall \alpha \in \mathbb{R}, \; \|x - \alpha y\|_2^2 = (x-\alpha y)^t (x - \alpha y)$$
$$= x^t x - 2\alpha y^t x + \alpha^2 y^t y$$
$$= \|x\|_2^2 - 2\alpha \|x\|_2 \|y\|_2 + \alpha^2 \|y\|_2^2$$
$$= \left(\|x\|_2 - \alpha \|y\|_2 \right)^2$$

$$\alpha = \frac{\|x\|_2}{\|y\|_2} \Rightarrow x = \alpha y \Rightarrow \operatorname{span}(x) = \operatorname{span}(y)$$

$(\Leftarrow)$ $\operatorname{span}(x) = \operatorname{span}(y)$

$\Rightarrow (\exists \alpha \in \mathbb{R})\, x = \alpha y$

$$\Rightarrow (y^t x)^2 = \left(\sum_{i=1}^n x_i y_i \right)^2 = \left[\alpha \sum_{i=1}^n y_i^2 \right]^2 =$$

$$= \sum_{i=1}^n y_i^2 \sum_{i=1}^n \alpha^2 y_i^2 = \|y\|_2^2 \cdot \|x\|_2^2$$

$$\Rightarrow y^t x = \|x\|_2 \|y\|_2 \Rightarrow \measuredangle(x,y) = 0$$

- $\sphericalangle(x,y) = \frac{\pi}{2} \iff x \perp y$

$$y^t x = 0 \iff x \perp y$$

- $\|x\|_2 = \|y\|_2 = 1$, $y^t x \geq 0 \implies \|y-x\|_2 = 2 \sin \frac{\sphericalangle(x,y)}{2}$

$$\cos \sphericalangle(x,y) = y^t x \geq 0$$

$$\gamma := \sphericalangle(x,y) \to \sin^2 \frac{\gamma}{2} = \frac{1}{2}(1 - \cos \gamma) = \frac{1}{2}(1 - y^t x)$$

$$\|y-x\|_2^2 = (y-x)^t(y-x) = y^t y - 2y^t x + x^t x = 2 - 2y^t x$$

$$= 2(1 - y^t x)$$

$$\implies \sin^2 \frac{\gamma}{2} = \frac{1}{4}\|y-x\|_2^2 \implies \|y-x\|_2 = 2 \sin \frac{\gamma}{2}$$

- $\|x\|_2 = \|y\|_2 = 1 \implies \sin \sphericalangle(x,y) = \|(I - xx^t)y\|_2 = \|(I - yy^t x)\|_2$

$$\gamma := \sphericalangle(x,y) \to \cos \gamma = y^t x$$

$$\sin^2 \gamma = 1 - \cos^2 \gamma = 1 - (y^t x)^2$$

$$\|(I - xx^t)y\|_2^2 = y^t(I - xx^t)(I - xx^t)y = (y^t - y^t x x^t)(y - x x^t y)$$

$$= y^t y - 2 y^t x x^t y + y^t x \cancel{x^t x} x^t y$$

$$= 1 - y^t x \underbrace{x^t y}_{= y^t x} = 1 - (y^t x)^2$$

$\|(I - yy^t)x\|_2^2$ in a same way

Problem 2.2: $\mathcal{Y} \subseteq \mathbb{R}^n$, $Y \in \mathbb{R}^{n \times k}$ ONB of $\mathcal{Y}$

$$\cos \measuredangle(x, \mathcal{Y}) = \frac{\|Y^T x\|_2}{\|x\|_2}$$

- $\|Y^T x\|_2 \le \|x\|_2$

$\cos \gamma \le 1 \;\forall \gamma \Rightarrow \|Y^T x\|_2 \le \|x\|_2$

- $\measuredangle(x, \mathcal{Y}) = 0 \Leftrightarrow x \in \mathcal{Y}$

$\|Y^T x\|_2 = \|x\|_2 \Leftrightarrow \|YY^T x\|_2 = \|x\|_2 \Leftrightarrow x \in \mathcal{Y}$

- $\measuredangle(x, \mathcal{Y}) = \frac{\pi}{2} \Leftrightarrow x \perp \mathcal{Y}$

$\|Y^T x\|_2 = 0 \Leftrightarrow \|YY^T x\|_2 = 0 \Leftrightarrow x \perp \mathcal{Y}$

- $\|x\|_2 = 1 \Rightarrow \sin \measuredangle(x, \mathcal{Y}) = \|(I - YY^T)x\|_2$

$\|x\|_2 = 1 \Rightarrow \sin^2 \measuredangle(x, \mathcal{Y}) = 1 - \|Y^T x\|_2^2$

$$\begin{aligned}
\|(I - YY^T)x\|_2^2 &= x^T (I - YY^T)(I - YY^T) x \\
&= x^T (I - YY^T - \cancel{YY^T} + \cancel{YY^T YY^T}) x \\
&= x^T (I - YY^T) x \\
&= x^T x - x^T YY^T x \\
&= 1 - \|Y^T x\|_2^2
\end{aligned}$$

$\measuredangle(x, \mathcal{Y}) \in [0, \pi) \Rightarrow \sin \measuredangle(x, \mathcal{Y}) \ge 0$

Problem 3: $\sin \Theta_1(\mathcal{X},\mathcal{Y}) = \|XX^T - YY^T\|_2$

X, Y $n\times k$ ONB for $\mathcal{X}, \mathcal{Y}$

1) $X = QR \rightarrow R = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$ + unitary invariance of $\|\cdot\|_2$

2) $X = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ Y_1 $k\times k$

$P = XX^T - YY^T$

$P^2 = (XX^T - YY^T)(XX^T - YY^T)$

$= XX^T - XX^TYY^T - YY^TXX^T + YY^T$

$XX^T = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$, $YY^T = \begin{bmatrix} Y_1Y_1^T & Y_1Y_2^T \\ Y_2Y_1^T & Y_2Y_2^T \end{bmatrix}$

$X^TY = [I_k \ 0]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = Y_1$

$\Rightarrow P^2 = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} Y_1Y_1^T & Y_1Y_2^T \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} Y_1Y_1^T & 0 \\ Y_2Y_1^T & 0 \end{bmatrix} + \begin{bmatrix} Y_1Y_1^T & Y_1Y_2^T \\ Y_2Y_1^T & Y_2Y_2^T \end{bmatrix}$

$= \begin{bmatrix} I_k - Y_1Y_1^T & 0 \\ 0 & Y_2Y_2^T \end{bmatrix}$

3) $\sigma_{max}(P) = \|XX^T - YY^T\|_2$

$\|$

$\sqrt{\lambda_{max}(P^2)} \overset{(*)}{=} \sqrt{1 - \lambda_{max}(Y_1Y_1^T)} = \sqrt{1 - \sigma_{max}^2(Y_1)} = \sqrt{1 - \sigma_{max}^2(X^TY)}$

$= \sqrt{1 - \cos^2 \Theta_1(\mathcal{X},\mathcal{Y})} = \sin \Theta_1(\mathcal{X},\mathcal{Y})$

(*)

$$\|P^2\|_2 = \max\{\underbrace{\|I_k - Y_1Y_1^T\|_2}_{1-\sigma_1^2(Y_1)}, \underbrace{\|Y_2Y_2^T\|_2}_{\sigma_1^2(Y_2)}\}$$

$$Y^TY = Y_1^TY_1 + Y_2^TY_2 = I \Rightarrow \sigma_1^2(Y_1) + \sigma_1^2(Y_2) = 1$$

$$\Rightarrow \sigma_1^2(Y_2) = 1 - \sigma_1^2(Y_1)$$

Problem 4.1: PROVE: $A \mapsto \text{rank}(A)$ is lower semi-continuous, $\forall A_0 \in \mathbb{R}^{m \times n}$

$f(A) = \text{rank}(A)$ lower-semi continuous if:

$(\forall A_0 \in \mathbb{R}^{m \times n})(\forall \varepsilon > 0)(\exists$ neighborhood $\mathcal{U}$ of $A_0)(\forall A \in \mathcal{U})$

$$f(A) \geq f(A_0) - \varepsilon$$

$A_0 \in \mathbb{R}^{m \times n}$

$A := A_0 + E, \quad \|E\|_2 \leq \varepsilon$

to prove: $\text{rank}(A_0 + E) \geq \text{rank}(A_0)$

$r_0 := \text{rank}(A_0)$

$r := \text{rank}(A_0 + E)$

$$\sigma_i(A_0) \leq |\sigma_i(A_0) - \sigma_i(A_0 + E)| + \sigma_i(A_0 + E)$$

$$\leq \varepsilon + \sigma_i(A_0 + E)$$

stability of SVD

$$\Rightarrow \sigma_i(A_0 + E) \geq \sigma_i(A_0) - \varepsilon \geq \sigma_i(A_0)$$

assume $r < r_0 \Rightarrow$ for $i = r+1, \ldots, r_0 \rightarrow \sigma_i(A_0) = 0$

$\Rightarrow$ ↯ with $r_0 = \text{rank}(A_0)$

$$\Rightarrow \underline{r \geq r_0}$$

Problem 4.2: find example for

$$\operatorname{rank}(A_0+E) > \operatorname{rank}(A_0) + \varepsilon \,, \quad \text{for some } E \text{ s.t. } \|E\|_2 \le \varepsilon$$

$$A_0 = \begin{bmatrix} \alpha_1 & & & & & \\ & \alpha_2 & & & & \\ & & \ddots & & & \\ & & & \alpha_{r_0} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \varepsilon & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}, \quad \varepsilon < 1$$

$r+1$

$r_0 = \operatorname{rank}(A_0)$

$\downarrow$

$\|E\|_2 = \varepsilon < 1$

$\operatorname{rank}(A_0+E) = r_0+1 > r + \varepsilon$

Problem 5 :

1) $A \in \mathbb{R}^{m \times n}$ $\quad A = [a_1 \ a_2 \cdots \ a_n]$

- PROVE ... $|\det(A)| \leq \|a_1\|_2 \cdot \|a_2\|_2 \cdot \cdots \|a_n\|_2$

$A = QR \Rightarrow a_j = \sum_{i=1}^{j} r_{ij} q_j$ mutually orthogonal

$\Rightarrow \|a_j\|_2^2 = \sum_{i=1}^{j} |r_{ij}|^2 \underbrace{\|q_j\|_2^2}_{=1} \geq |r_{jj}|^2 \quad , j = 1, 2, \ldots, n$

$\Rightarrow |r_{jj}| \leq \|a_j\|_2$

$|\det(A)| = |\underbrace{\det(Q)}_{=1} \det(R)| = |\prod_{j=1}^{n} r_{jj}| \leq \prod_{j=1}^{n} \|a_j\|_2$

- A ORTHOGONAL $\rightarrow$ $R = I \Rightarrow \|a_j\|_2 = |r_{jj}|$

$\Rightarrow |\det(A)| = \prod_{j=1}^{n} \|a_j\|_2$

2) $P \in \mathbb{R}^{m \times k}$ $\quad k \leq n < m$ $\quad P^T P = I$

$Q \in \mathbb{R}^{n \times k}$ $\quad Q^T Q = I$

$A = PSQ^T \quad , r < k$

PROVE: $\|A - P J_r(S) \cdot Q^T\|_F = \|A - J_r(A)\|_F$

$J_r(S) = U_r \Sigma_r V_r^T \Rightarrow \|A - \underbrace{P U_r}_{\text{orthogonal}} \Sigma_r \underbrace{(Q V_r)^T}_{} \|_F = \|A - J_r(A)\|_F$

3) $\underbrace{A}_{} = \underbrace{B}_{m \times R} \underbrace{C^T}_{R \times n}$ $\qquad R \leq n \leq m$

1. $\underbrace{C}_{n \times R} = \underbrace{Q}_{n \times R} \underbrace{R}_{R \times R}$ (qr factorization) $\longrightarrow O(nR^2)$

$\longrightarrow A = B R^T Q^T$

2. $D = \underbrace{B}_{m \times R} \underbrace{R^T}_{R \times R} \longrightarrow O(mR^2)$

3. $\underbrace{D}_{m \times R} = U \Sigma \tilde{V}^T$ (svd) $O(mR^2)$

$\longrightarrow A = U \Sigma \tilde{V}^T Q^T$

4. $V = \underbrace{Q}_{n \times R} \underbrace{\tilde{V}}_{R \times R} \longrightarrow O(nR^2)$

$\longrightarrow A = U \Sigma V^T$

$\Rightarrow O((m+n) R^2)$

4) $A_1 = U_1 \Sigma_1 V_1^T$ $m \times n$ $\qquad \Sigma_1, \Sigma_2$ $r \times r$

$A_2 = U_2 \Sigma_2 V_2^T$ $m \times n$

$$A_1 + A_2 = \underbrace{\underbrace{\begin{bmatrix} U_1 & U_2 \end{bmatrix}}_{m \times 2r} \underbrace{\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}}_{2r \times 2r}}_{B} \underbrace{\underbrace{\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}}_{2r \times n}}_{C^T} \qquad R := 2r$$

+ apply 3)