$A \in \mathbb{R}^{m \times n}$

Problem 1:

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2$$

a) • $n = 1 \longrightarrow A = [a] \in \mathbb{R}^m$

$$\|A\|_2 = \max_{\|x\|_2=1} \|\underbrace{A}_{m \times 1} \underbrace{x}_{\in \mathbb{R}}\|_2 = \max \{\|-a\|_2, \|a\|_2\} = \|a\|_2$$

$$\Rightarrow \|x\|_2 = \sqrt{x^2} = |x| = 1 \iff x \in \{-1, 1\}$$

• $m = 1 \longrightarrow A = [a^T] \in \mathbb{R}^{1 \times n}$

$$\|A\|_2 = \max_{\|x\|_2=1} \|\underbrace{A}_{1 \times n} \underbrace{x}_{n \times 1}\|_2 = \max_{\|x\|_2=1} \|a^T x\|_2$$

$$= \max_{\|x\|_2=1} |\langle a | x \rangle| \underset{\text{Cauchy-Schwarz}}{\leq} \max_{\|x\|_2=1} \|a\|_2 \|x\|_2 = \|a\|_2$$

$$x = \frac{a}{\|a\|_2} \longrightarrow \left|\left\langle a, \frac{a}{\|a\|_2} \right\rangle\right| = \|a\|_2 \Rightarrow \max_{\|x\|_2=1} |\langle a|x\rangle| = \|a\|_2$$

$$\Rightarrow \|A\|_2 = \|a\|_2$$

b) $$\|\operatorname{diag}(\lambda_1, \dots, \lambda_n)\|_2 = \max_{\|x\|_2=1} \left\| \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix} \right\|_2 =$$

$$= \max_{\|x\|_2=1} \sqrt{\lambda_1^2 x_1^2 + \dots + \lambda_n^2 x_n^2} \leq \max_{\|x\|_2=1} \sqrt{\lambda_{max}^2 x_1^2 + \dots + \lambda_{max}^2 x_n^2}$$

$$= |\lambda_{max}| \max_{\|x\|_2=1} \sqrt{x_1^2 + \dots + x_n^2} = |\lambda_{max}|$$

$$i_{max} = \arg\max_{i=1,\dots,n} |\lambda_i| \longrightarrow \|\operatorname{diag}(\lambda_1, \dots, \lambda_n) \cdot e_{i_{max}}\|_2 = \left\| \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_{max} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|_2 = |\lambda_{max}|$$

Problem 2: $A \in \mathbb{R}^{m\times n}$, $m \geq n$

$\rightarrow$ $\sigma_1(A), \dots, \sigma_n(A)$ singular values

- $X = \begin{pmatrix} A & I_m \end{pmatrix}$ $\quad m \times (m+n)$

$$XX^t = \begin{pmatrix} A & I_m \end{pmatrix} \begin{pmatrix} A^t \\ I_m \end{pmatrix} = AA^t + I_m$$

$$\sigma_i(X) = \sqrt{\lambda_i(XX^t)} = \sqrt{\lambda_i(AA^t + I_m)} = \sqrt{\lambda_i(AA^t) + 1}$$

$$= \sqrt{\sigma_i^2(A) + 1} \quad , i = 1, 2, \dots, m$$

(for $i = n+1, \dots, m$ $\rightarrow$ $\sigma_i(X) = \sqrt{1} = 1$)

- $X = \begin{pmatrix} A \\ I_n \end{pmatrix}$ $\quad (m+n) \times n$

$$X^t X = \begin{pmatrix} A^t & I_n \end{pmatrix} \begin{pmatrix} A \\ I_n \end{pmatrix} = A^t A + I_n$$

again, $\sigma_i(X) = \sqrt{\sigma_i^2(A) + 1}$ $\quad , i = 1, 2, \dots, n$

- $X = \begin{pmatrix} A & I_m \\ I_n & 0 \end{pmatrix}$ $\quad (m+n) \times (m+n)$

$$X^t X = \begin{pmatrix} A^t & I_n \\ I_m & 0 \end{pmatrix} \begin{pmatrix} A & I_m \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} A^t A + I_n & A^t \\ A & I_m \end{pmatrix}$$

$$X^t X \begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix} \rightarrow \begin{pmatrix} A^t A + I_n & A^t \\ A & I_m \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \lambda \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\rightarrow \quad A^t A y + y + A^t z = \lambda y$$
$$\underline{Ay + z = \lambda z}$$

$$(\lambda - 1) A^t z + y + A^t z = \lambda y \Rightarrow A^t z = \frac{\lambda - 1}{\lambda} y$$

$$\Rightarrow A^t A y + y + \frac{\lambda-1}{\lambda} y = \lambda y$$

$$A^t A y = \frac{(\lambda-1)^2}{\lambda} y$$

$\cdot\ \lambda \neq 0$ eigenvalue of $X^t X \Rightarrow \frac{(\lambda-1)^2}{\lambda}$ eigenvalue of $A^t A$ $\quad \left(\lambda = 0 \Rightarrow \begin{bmatrix} y \\ z \end{bmatrix} = 0\right)$

$\Rightarrow \mu$ eigenvalue of $A^t A \Rightarrow \mu = \frac{(\lambda-1)^2}{\lambda}$, for $\lambda \neq 0$ eigenvalue of $X^t X$

$$\Rightarrow \lambda\mu = \lambda^2 - 2\lambda + 1 \Rightarrow \lambda^2 - (\mu+2)\lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{(\mu+2) \pm \sqrt{\mu^2 + 4\mu + 4 - 4}}{2} = \frac{(\mu+2) \pm \sqrt{\mu^2 + 4\mu}}{2}$$

$\rightarrow$ singular values of X are given by

$$\sqrt{\frac{1}{2}\left(\sigma_i^2(A) + 2 \pm \sqrt{\sigma_i^4(A) + 4\sigma_i^2(A)}\right)}, \quad i = 1, 2, \dots, n$$

$2n$ of them

$+$ additional $\sqrt{\frac{1}{2} \cdot 2} = 1$ $\quad (m-n)$ of them

Problem 3: $A, B \in \mathbb{R}^{m \times n}$

prove: $tr(B^t A) = tr(AB^t) = vec(B)^t vec(A)$

$A = [a_1 \cdots a_n]$
$B = [b_1 \cdots b_n]$ } column partitions

$\to B^t = \begin{bmatrix} b_1^t \\ \vdots \\ b_n^t \end{bmatrix}$

$tr(B^t A) = b_1^t a_1 + \cdots + b_n^t a_n$

$vec(B)^t vec(A) = [b_1^t \cdots b_n^t] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = b_1^t a_1 + \cdots + b_n^t a_n$

$$tr(\underbrace{AB^t}_{m \times m}) = \sum_{i=1}^{m} (AB^t)_{ii} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ij} a_{ij} =$$

$$= \sum_{j=1}^{n} (B^t A)_{jj} = tr(B^t A)$$

Problem 4 : $A \in \mathbb{R}^{n \times n}$

PROVE : $tr(A) = n$ and $\|A\|_2 \leq 1 \Rightarrow A = I_n$

$tr(A) = n \Rightarrow a_{11} + a_{22} + \dots + a_{nn} = n$

$a_{ii} = \langle e_i, Ae_i \rangle \leq \underbrace{\|e_i\|_2}_{=1} \|Ae_i\|_2 \leq \|A\|_2 \leq 1, \quad \forall i = 1, 2, \dots, n$

$\Rightarrow a_{ii} = 1, \forall i = 1, \dots, n$

Suppose $a_{ij} \neq 0$, for some $j \in \{1, \dots, n\} \Rightarrow$

$\|Ae_j\|_2 = \sqrt{a_{ij}^2 + 1} > 1 \quad \Rightarrow\Leftarrow$ with $\|A\|_2 \leq 1 \quad \Rightarrow a_{ij} = 0, \forall i \neq j$

$\downarrow$

e.g. $\begin{bmatrix} 1 & a & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & & \\ a & 1 & & \\ 0 & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ a \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Problem 5: $A, B \in \mathbb{R}^{m\times n}$, $m \geq n$

$$\max \{ tr(P^t M Q) \mid P \in \mathbb{R}^{m\times k}, Q \in \mathbb{R}^{n\times k}, P^tP = Q^tQ = I_k \} = \sigma_1 + \dots + \sigma_k$$

PROVE: $\sigma_1(A+B) + \dots + \sigma_k(A+B) \leq \sigma_1(A) + \dots + \sigma_k(A) + \sigma_1(B) + \dots + \sigma_k(B)$

$$\sum_{i=1}^{k} \sigma_i(A+B) = \max \{ tr(P^t(A+B)Q) \mid P \in \mathbb{R}^{m\times k}, Q \in \mathbb{R}^{n\times k}, P^tP = Q^tQ = I_k \}$$

$$= \max \{ tr(P^tAQ + P^tBQ) \mid -\text{''}- \}$$

$$= \max \{ tr(P^tAQ) + tr(P^tBQ) \mid -\text{''}- \}$$

$$\leq \max \{ tr(P^tAQ) \mid \dots \} + \max \{ tr(P^tBQ) \mid \dots \}$$

$$= \sigma_1(A) + \dots + \sigma_k(A) + \sigma_1(B) + \dots + \sigma_k(B)$$

Problem 6: $A, B \in \mathbb{R}^{m \times n}$

$C = A * B$

Prove: $\mathrm{rank}(C) \leq \mathrm{rank}(A) \cdot \mathrm{rank}(B)$

$$r_A = \mathrm{rank}(A) \Rightarrow \quad A = \sum_{i=1}^{r_A} p_i q_i^T \qquad (A = \underset{m \times r_A}{P} \, \underset{r_A \times n}{Q^T})$$

$$r_B = \mathrm{rank}(B) \rightarrow \quad B = \sum_{j=1}^{r_B} s_j t_j^T \qquad (B = \underset{m \times r_B}{S} \, \underset{r_B \times n}{T^T})$$

$$A * B = \left(\sum_{i=1}^{r_A} p_i q_i^T \right) * \left(\sum_{j=1}^{r_B} s_j t_j^T \right)$$

$$= \sum_{i=1}^{r_A} \sum_{j=1}^{r_B} (p_i q_i^T) * (s_j t_j^T)$$

$$= \sum_{i=1}^{r_A} \sum_{j=1}^{r_B} (p_i * s_j)(q_i * t_j)^T$$

$\Rightarrow$ $A * B$ is a sum of $r_A \cdot r_B$ rank-one matrices $\rightarrow$

$$\mathrm{rank}(A * B) \leq \mathrm{rank}(A) \cdot \mathrm{rank}(B)$$