Exercise 6 – Low-rank approximation techniques

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Problem 1. Express the QR factorization/SVD of $A \otimes B$ in terms of the QR factorizations/SVDs of A and B.

Problem 2. The Lyapunov matrix equation, which plays an important role ni dynamical systems and control, takes the form $AX + XA^T = C$, where A and C are $n \times n$ known coefficient matrices and X is the $n \times n$ solution matrix. Using Kronecker products, reformulate this matrix equation as a linear system.

Problem 3. For $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$ and $v \in \mathbb{R}^{nq}$, the following property holds

$$(A \otimes B)v = \text{vec}(BVA^T), \quad v = \text{vec}(V).$$

Derive the complexity in terms of memory and number of operations of calculating $(A \otimes B)v$ directly and using this property. In MATLAB, plot the times needed to evaluate $(A \otimes B)v$ both ways, with $A, B \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^{n^2}$, for $n = 10, 20, \ldots, 100$.

Problem 4. Given $A \in \mathbb{R}^{n^2\times n^2}$ we aim to find matrices $B_j, C_j \in \mathbb{R}^{n\times n}$, $j = 1, \ldots, r$ such that

$$\Big\| A - \sum_{j=1}^{r} C_j \otimes B_j \Big\|_F$$

is minimized. Show how the SVD of a suitable permutation of A can be used to solve this problem. (Hint: First try to solve this by yourself. Note that for $A = I_{n^2}$ only $r = 1$ is needed. If you get stuck, have a look at [Van Loan, C. F. The ubiquitous Kronecker product. J. Comput. Appl. Math. 123 (2000), no. 1-2, 85–100.])

Problem 5. Show that for any matrix U with orthonormal columns, and tensors $\mathcal{X}$ and $\mathcal{Y}$ of appropriate size, the following properties hold

- $\mathcal{Y} = U \circ_\mu \mathcal{X} \quad \Rightarrow \quad \mathcal{X} = U^T \circ_\mu \mathcal{Y}$,
- $\|\mathcal{X}\|_F = \|U \circ_\mu \mathcal{X}\|_F$.

Problem 6. Let $X = U\Sigma V^T$ be an SVD of a matrix X. Considering X as a tensor of second order, rewrite its SVD in terms of μ-mode multiplication.

Problem 7. In MATLAB, do the following.

1. Create a function for the μ-mode matricization of a $n_1 \times n_2 \times \cdots \times n_d$ tensor $\mathcal{X}$.
2. Create a function for the μ-mode multiplication of a $n_1 \times n_2 \times \cdots \times n_d$ tensor $\mathcal{X}$ and a $m \times n_\mu$ matrix A.
3. Create a $n \times n \times n$ tensor $\mathcal{X}$ by evaluating the function

$$f(x, y, z) = \frac{1}{(x + y + z)^\gamma}, \quad \gamma \in \mathbb{R}^+,$$

on the grid $\{0.1, 0.2, \ldots, n/10\}^3$. For different values of γ, plot the singular values of the matricizations $X^{(1)}$, $X^{(2)}$, $X^{(3)}$.