Exercise 5 – Low-rank approximation techniques

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Problem 1. The goal of this exercise is to understand how sharp the bound from the theorem on slide 23 of Lecture 4 is. Let the SVD of an $n \times n$ matrix A be given as $A = U\Sigma V^T$, where

$$U = \begin{bmatrix} u_1 & U_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{\varepsilon} & \\ & \varepsilon I \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & V_2 \end{bmatrix},$$

with $v_1 = \begin{bmatrix} 1/\sqrt{n} & 1/\sqrt{n} & \cdots & 1/\sqrt{n} \end{bmatrix}^T$ and $\varepsilon = \frac{1}{n-1}$. From the theorem, we know that for $k = 1$ there exist a column c_j and a row r_i of A and a number s such that

$$\|A - c_j s r_i\|_2 \leq \varepsilon(1 + 4\sqrt{n}).$$

Show that for any column c_j and row r_i of A and any number s it holds that

$$\|A - c_j s r_i\|_2 \geq \sqrt{\frac{\varepsilon}{2}} - \varepsilon.$$

Compare the upper and the lower bound for $n = 20$.

Problem 2. In MATLAB, implement adaptive cross approximation - with full pivoting and with partial pivoting. Reproduce the results from slide 34 of Lecture 4.

Problem 3. Let the matrix

$$A = \begin{bmatrix} \alpha & b^T \\ b & C \end{bmatrix} \in \mathbb{R}^{n \times n},$$

with $\alpha > 0$, be symmetric and positive definite. Show that its Schur complement

$$S = C - \frac{1}{\alpha} b b^T \in \mathbb{R}^{(n-1)\times(n-1)}$$

is also symmetric and positive definite.

Problem 4. Show that the Lemma from slide 6 of Lecture 5 in Frobenius norm has the following form

$$\|A - QQ^T A\|_F^2 \leq \|A - \mathcal{T}_r(A)\|_F^2 + 2\sqrt{r}\|AA^T - CC^T\|_F.$$

Hint: Find bounds for $\left| \|A^T Q\|_F^2 - \|C^T Q\|_F^2 \right|$ and $\left| \|\mathcal{T}_r(A)\|_F^2 - \|C^T Q\|_F^2 \right|$.

Problem 5. Implement both sampling strategies discussed in Lecture 5 in Matlab: (1) sampling based on column norms on Slide 10 and (2) sampling based on $\|V_k(\ell,:)\|_2$. Try to find an example for which the strategy based on $\|V_k(\ell,:)\|_2$ performs much better. Hint: Make the norms $\|V_k(\ell,:)\|_2$ very different while keeping the column norms balanced.