Exercise 4 – Low-rank approximation techniques

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Prof. Dr. D. Kressner
Dr. L. Periša

Problem 1. Prove that the volume of a square matrix is equal to the product of its singular values.

Problem 2. For $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$, with $m > n$, let A_i be a $n \times (m-1)$ submatrix of A without i-th column, and B_i be $(m-1)\times n$ submatrix of B without i-th row. Prove that it holds that

$$\det(AB) = \frac{1}{m-n}\sum_{i=1}^{m} \det(A_i B_i).$$

Problem 3. This exercise is concerned with the lemma from slide 4 of Lecture 4.

1. Show that the inequality of the lemma cannot be improved in general by constructing a matrix $U = \begin{pmatrix} I_r \\ B \end{pmatrix}$ with $\max |b_{ij}| \le 1$ and $\|B\|_2 = \sqrt{(n-r)r}$.
2. Develop an extension of the result of the lemma. Prove that for an arbitrary $n \times r$ matrix U of rank r, there is an index set I with $\#I = r$ such that

$$\frac{1}{\sigma_{\min}(U(I,:))} \le \sqrt{r(n-r)+1}\,\frac{1}{\sigma_{\min}(U)}.$$

Problem 4. For the matrix Z_n defined on Slide 10, prove $(Z_n^{-1})_{ij} = 2^{j-i-1}$ for $j > i$.

Problem 5. Prove that any projector $\Pi \notin \{0, I\}$ satisfies $\|I - \Pi\|_2 = \|\Pi\|_2$.

Problem 6. *Quite likely, this problem is very hard.* Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite and let $k \in [1, n]$. Prove or give a counterexample for the conjecture

$$\begin{aligned} & \max\{|\det(A(I,J))| : I, J \subset \{1,\dots,n\}, \#I = \#J = k\} \\ = \; & \max\{|\det(A(I,I))| : I \subset \{1,\dots,n\}, \#I = k\}. \end{aligned}$$

In particular, this would imply that there is a symmetric positive definite $k \times k$ submatrix of maximal volume.

Problem 7. MATLAB exercise.

1. Implement the greedy method from Slide 9.
2. Given an $m \times n$ matrix A and an integer $r \le \min\{m, n\}$, implement the following method for obtaining a rank-r approximation:
   - Apply r steps of the greedy method to left singular vectors U_r of A to determine I.
   - Apply r steps of the greedy method to right singular vectors V_r of A to determine J.
   - Return rank-r approximation $A(:,J)(A(I,J))^{-1}A(I,:)$.

   Apply your implementation to the two matrices from Slide 34 for $r = 1,\dots,30$ (for Hilbert matrix) and $r = 1,\dots,100$ (for exponential). Plot the singular values and the obtained approximation error in the spectral norm.