Exercise 1 – Low-rank approximation techniques

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Problem 1.

The spectral norm of a matrix $A \in \mathbb{R}^{m\times n}$ is defined as

$$\|A\|_2 := \max_{\|x\|_2=1} \|Ax\|_2. \tag{1}$$

a) Using the definition (1), show that the spectral norm coincides with the Euclidean norm if A is a row or column vector.

b) Using the definition (1), show that

$$\|\operatorname{diag}(\lambda_1,\dots,\lambda_n)\|_2 = \max_{i=1,\dots,n} |\lambda_i|,$$

with $\lambda_1,\dots,\lambda_n \in \mathbb{R}$.

Problem 2. Express the singular values for each of the three following matrices

$$\begin{pmatrix} A & I_m \end{pmatrix}, \quad \begin{pmatrix} A \\ I_n \end{pmatrix}, \quad \begin{pmatrix} A & I_m \\ I_n & 0 \end{pmatrix},$$

in terms of the singular values of $A \in \mathbb{R}^{m\times n}$, with $m \geq n$.

Problem 3. Prove that for $A, B \in \mathbb{R}^{m\times n}$ it holds that

$$\operatorname{trace}(B^T A) = \operatorname{trace}(AB^T) = \operatorname{vec}(B)^T \operatorname{vec}(A).$$

Problem 4. Show that the only matrix $A \in \mathbb{R}^{n\times n}$ satisfying $\operatorname{trace}(A) = n$ and $\|A\|_2 \leq 1$ is $A = I_n$.

Problem 5. For matrix $M \in \mathbb{R}^{m\times n}$, $m \geq n$, with singular values $\sigma_1,\dots,\sigma_n$ it holds that for every $k = 1,\dots,n$,

$$\max\left\{ \operatorname{trace}(P^T M Q) \mid P \in \mathbb{R}^{m\times k}, Q \in \mathbb{R}^{n\times k}, P^T P = Q^T Q = I_k \right\} = \sigma_1 + \cdots + \sigma_k.$$

Using this statement prove that for $A, B \in \mathbb{R}^{m\times n}$, $m \geq n$,

$$\sigma_1(A+B) + \cdots + \sigma_k(A+B) \leq \sigma_1(A) + \cdots + \sigma_k(A) + \sigma_1(B) + \cdots + \sigma_k(B)$$

holds for every $k = 1,\dots,n$.

Problem 6. The aim of this exercise is to understand the rank of the Hadamard product of two low-rank matrices. Let $A, B \in \mathbb{R}^{n\times n}$ be matrices with ranks r_A and r_B, respectively, and $C = A * B$ their Hadamard (element-wise) product. Using MATLAB/Julia/Python create matrices A and B and calculate the rank of C for

a) $n = 100$ and A and B random matrices with $r_A, r_B \in [1, 10]$ (try different combinations),

b) $n = 100$ and A and B random matrices with $r_A = r_B = 15$,

c) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

What can you conclude? BONUS: Set and prove the conjecture.