Solution 6 – Advanced Numerical Analysis

ÉCOLE POLYTECHNIQUE
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1 ▶ Quiz

2 ▶ Implementing steepest descent

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function ex6problem2

    A = [5, -4;-4,5];
    b = [-1;2];
    f = @(x) 0.5*x'*(A*x) - b'*x;
    g = @(x) A*x - b;

    opts = struct{'alpha', 0.1, 'beta', 0.5, 'method', 'backtrack'};

    %set to 1 for armijo, 2 for optimal step size, 3 for fixed
    c = 1e-4;
    tol = 1e-6;

    %Run steepest descent
    x = steepdesc(f, g, [3;4.5], c, tol);

    % Visualization
    xrange = linspace(0,5,100);
    yrange = linspace(0,5,100);
    [X,Y] = meshgrid( xrange, yrange );
    vecX = X(:); vecY = Y(:);
    for i = 1:length(vecX)
        z(i,1) = f( [vecX(i); vecY(i)] );
    end
    Z = reshape(z, 100, 100);

    contourf(X,Y,Z,20,'linestyle','None');
    axis square

    hold on
    plot( x(1,:), x(2,:), '-wo', 'linewidth',2)

    function x = steepdesc(f,g,x0,c,tol)
        x(:,1) = x0;
        k=1; btc = 0;

        while norm(g(x(:,k))) > tol
            xk = x(:,k); gk = g(x(:,k));
            if strcmpi(opts.method, 'backtrack')
                % start backtracking
                alpha = 1;
                while f(xk - alpha*gk) > f(xk) - c*alpha*gk'*gk
                alpha = alpha*beta;
                btc = btc + 1;
                end
            elseif strcmpi(opts.method, 'optimal')
                alpha = (gk'*gk) / (gk'*A*gk);
            elseif strcmpi(opts.method, 'fixed')
                alpha = opts.alpha;
            else
                error('Unknown␣method!␣Choose␣either␣backtrack,␣optimal␣or␣fixed')
            end
            %Perform step
            x(:,k+1) = xk - alpha*gk;
            k = k+1;
        end

        if strcmpi(opts.method, 'backtrack')
```

```
            disp(sprintf('%i_global_steps,_using_%i_backtracking_steps', k-1, btc));
        elseif strcmpi(opts.method, 'optimal')
            disp(sprintf('%i_steps_with_optimal_choice',k-1));
        elseif strcmpi(opts.method, 'fixed')
            disp(sprintf('%i_steps_with_stepsize_%g', k-1, alpha));
        end
    end

end
```

3 ▶ Descent directions are not enough

Problem 3

For $f(x) = \frac{1}{2}\|x\|^2$ we have $\nabla f(x) = x$.

Therefore

$$x_{k+1} = x_k + \alpha_k \left(q_k - \frac{1}{2^{k+3}} x_k\right) = \left(1 - \frac{\alpha_k}{2^{k+2}}\right) x_k + \alpha_k q_k$$

Since $x_k \perp q_k$ and $\|p_k\|^2 = \|\nabla f(x_k)\|^2 = 1$, it holds

$$\begin{aligned}
\|x_{k+1}\|^2 &= \left(1 - \frac{\alpha_k}{2^{k+3}}\right)^2 \|x_k\|^2 + \alpha_k^2 \|q_k\|^2 \\
&= \left[\left(1 - \frac{\alpha_k}{2^{k+3}}\right)^2 + \alpha_k^2 \left(1 - \left(\frac{1}{2^{k+3}}\right)^2\right)\right] \|x_k\|^2 \\
&= \left[1 + \alpha_k^2 - \frac{\alpha_k}{2^{k+2}}\right] \|x_k\|^2
\end{aligned}$$

To generate descent we need $\alpha_k \left(\alpha_k - \frac{1}{2^{k+2}}\right) \leq 0$, i.e. $\alpha_k \in \left(0, \frac{1}{2^{k+2}}\right)$.

But then

$$\begin{aligned}
\|x_{k+1}\|^2 &\geq \left[1 - \left(\tfrac{1}{4}\right)^{k+2}\right] \|x_k\|^2 \\
&\geq \underbrace{\left(\prod_{i=1}^{k+1} \left(1 - \left(\tfrac{1}{4}\right)^{i+1}\right)\right)}_{c_k} \|x_0\|^2
\end{aligned}$$

The assertion follows if we show that $c_k \geq c > 0$.

To this end, consider

$$\ln c_k = \sum_{i=1}^{k+1} \ln\left(1 - \left(\tfrac{1}{4}\right)^{i+1}\right) \overset{\text{Taylor}}{=} \sum_{i=1}^{k+1} - \frac{1}{\xi_i} \left(\tfrac{1}{4}\right)^{i+1}, \quad \xi_i \in \left(1 - \left(\tfrac{1}{4}\right)^{i+1}, 1\right]$$

which is bounded from below. ∎