Exam	
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE	**Advanced Numerical Analysis**
	Teacher: Prof. Dr. Daniel Kressner
	Date: 02.07.2013
	Duration: 3h (08:15 - 11:15)

Sciper:	**Student:**	**Section:**

Score Table

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Problem 7	Total

Please read carefully:

- You are only allowed to have one A4 page of hand-written notes (no photocopies).
- Please put your CAMIPRO card on your desk before you start the exam as it will be checked during the exam.
- You must write your sciper number, your name, and section on this page before you start the exam.
- Calculators and all other electronical devices are forbidden.
- Do not use your own paper for writing down the solutions; use the blank pages after each exercise. You can request additional paper from the assistants.
- Except for Problems 1 and 7, do not only write down the final result or answer, but also some explanations and justification of the result. Results without justification are not counted.

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Problem 1 **8 points**

Choose one answer to each of the following questions. Every correct answer gives 1 point, no answer gives 0 points, and every wrong answers gives -1 points. However, you cannot get less than zero points in total.

(a) The approximate solution $\mathbf{y}_n$ obtained from applying any A-stable method to a general IVP always converges to zero for $n \to \infty$.

true ☐ **false** ☒

(b) The approximate solution $\mathbf{y}_n$ obtained from applying the implicit Euler method to the linear IVP $\mathbf{y}'(t) = G\mathbf{y}(t)$, with initial value $\mathbf{y}(0) = \mathbf{y}_0$ and a general matrix $G \in \mathbb{R}^{n \times n}$, always converges to zero for $n \to \infty$.

true ☐ **false** ☒

(c) The linear system

$$\mathbf{y}_1 = \mathbf{y}_0 + hG\mathbf{y}_1,$$

which needs to be solved in the first step of the implicit Euler method applied to the linear IVP $\mathbf{y}'(t) = G\mathbf{y}(t)$ with initial value $\mathbf{y}(0) = \mathbf{y}_0$ and a general matrix $G \in \mathbb{R}^{n \times n}$, always has a unique solution $\mathbf{y}_1$.

true ☐ **false** ☒

(d) The statement of (c) holds if $h > 0$ is sufficiently small.

true ☒ **false** ☐

(e) The linear system that needs to be solved in the first step of an implicit Runge-Kutta method applied to the linear IVP $\mathbf{y}'(t) = G\mathbf{y}(t)$, with initial value $\mathbf{y}(0) = \mathbf{y}_0$ and a general matrix $G \in \mathbb{R}^{n \times n}$, always has a unique solution if $h > 0$ is sufficiently small.

true ☒ **false** ☐

(f) Consider a method of maximal consistency order p for solving IVPs $\mathbf{y}'(t) = f(t, \mathbf{y}(t))$, $\mathbf{y}(t_0) = \mathbf{y}_0$. Then the following statement holds: For all $(p+1)$-times continuously differentiable functions f there exists a constant $c > 0$ such that the local error $\mathbf{e}_1 = \|\mathbf{y}_1 - \mathbf{y}(t_1)\|$ of the first step satisfies $\mathbf{e}_1 \geq ch^{p+1}$.

true ☐ **false** ☒

(g) Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and $\mathbf{x}, \mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{p}^T \nabla f(\mathbf{x}) < 0$. Then there exists $\alpha^* > 0$ such that

$$f(\mathbf{x} + \alpha\mathbf{p}) < f(\mathbf{x}) \quad \text{for all } 0 < \alpha < \alpha^*.$$

true ☒ **false** ☐

(h) Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and bounded from below. At a point $\mathbf{x}$ let $\mathbf{p}$ be a descent direction. Then there exists a largest number $\alpha > 0$ (strict inequality!) with the property $f(\mathbf{x} + \alpha\mathbf{p}) \leq f(\mathbf{x}) + \alpha\mathbf{p}^T \nabla f(\mathbf{x})$.

true ☐ **false** ☒

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Problem 2 **5 points**

Use the table with Runge-Kutta order conditions to find all explicit autonomization-invariant order-three Runge-Kutta methods of the form

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & a_{21} & & \\ 2/3 & a_{31} & a_{32} & \\ \hline & b_1 & b_2 & b_3 \end{array}.$$

Solution

The *conditions for order three* read $\sum_i b_i \mathcal{A}_i^{(\beta)} = 1/\beta!$ for all trees of order $\#\beta \leq 3$. According to the table this means

$$\sum_i b_i = 1, \quad \sum_i b_i c_i = \frac{1}{2}, \quad \sum_i b_i c_i^2 = \frac{1}{3}, \tag{1}$$

and

$$\sum_i \sum_j a_{ij} c_j = \frac{1}{6}. \tag{2}$$ ①

Plugging in the given values for the c_i, (1) gives a linear system for the b_i:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 2/3 \\ 0 & 1/4 & 4/9 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \end{pmatrix}.$$

It is equivalent to (substract the second row from two times the third row)

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 2/3 \\ 0 & 0 & 2/9 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 1/6 \end{pmatrix}.$$

and has the unique solution

$$b_3 = \frac{3}{4}, \qquad b_2 = 0, \qquad b_1 = \frac{1}{4}.$$ ②

Next, using $c_1 = 0$, and $a_{ij} = 0$ for $j \geq i$, (2) boils down to $b_3 a_{32} c_2 = 1/6$, i.e., since $c_2 = 1/2$ and $b_3 = 3/4$,

$$a_{32} = \frac{4}{9}.$$ ①

The other two values of A can be obtained from the *conditions for invariance under autonomization*:

$$\sum_j a_{ij} = c_j.$$

Namely,

$$a_{21} = \frac{1}{2}, \qquad a_{31} = \frac{2}{9}.$$ ①

Hence the *only* explicit autonomization-invariant RK-method of order three with these values c_j is

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ 2/3 & 2/9 & 4/9 & \\ \hline & 1/4 & 0 & 3/4 \end{array}.$$

Problem 3 **4 points**

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For a fixed parameter θ, consider the following method for solving IVPs:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + hf(t_n + \theta h, (1-\theta)\mathbf{y}_n + \theta \mathbf{y}_{n+1}).$$

Show that the method is A-stable if and only if $\theta \geq 1/2$.

Solution

For the IVP $\mathbf{y}'(t) = G\mathbf{y}(t)$ the method reads

$$\mathbf{y}_{n+1} = \mathbf{y}_n + (1-\theta)hG\mathbf{y}_n + \theta hG\mathbf{y}_{n+1}.$$ ①

If h is sufficiently small, we can solve for $\mathbf{y}_{n+1}$:

$$\mathbf{y}_{n+1} = (I - \theta hG)^{-1}(I + (1-\theta)hG)\mathbf{y}_n = S(hG)\mathbf{y}_n.$$

The eigenvalues of $S(h\lambda)$ are of form

$$S(z) = \frac{1 + (1-\theta)z}{1 - \theta z}$$ ①

(with $z = h\lambda$, where λ are the eigenvalues of G). The method is A-stable, if $|S(z)| \leq 1$ for all $z \in \mathbb{C}^-$, that is, if

$$|1 + (1-\theta)z|^2 \leq |1 - \theta z|^2 \quad \text{for all } z = a + ib \text{ with } a \leq 0.$$

In terms of a, b this inequality reads

$$(1 + (1-\theta)a)^2 + (1-\theta)^2 b^2 \leq (1 - \theta a)^2 + \theta^2 b^2 \quad \text{for all } z = a + ib \text{ with } a \leq 0. \tag{3}$$

Since $a \leq 0$, we always have for $\theta \geq 0$ that it holds

$$|1 + (1-\theta)a| = |1 - \theta a + a| \leq |1 - \theta a| \quad \Leftrightarrow \quad (1 + (1-\theta)a)^2 \leq (1 - \theta a)^2.$$

If $\theta \geq 1/2$, we also have

$$(1-\theta)^2 b^2 \leq \theta b^2$$ ①

so that (3) holds.

If $\theta < 1/2$, (3) does not hold if $a = 0$ and $b \neq 0$, since $(1-\theta)^2 > \theta^2$ then.

①

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Problem 4 **4 points**

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be two times continuously differentiable and bounded from below. At a point $\mathbf{x}$ let $\mathbf{p}$ be a descent direction and assume that the Hessian $H(\mathbf{x})$ is negative definite. Show that there exists a largest number $\alpha^* > 0$ (strict inequality!) with the property $f(\mathbf{x} + \alpha^*\mathbf{p}) \leq f(\mathbf{x}) + \alpha^*\mathbf{p}^T\nabla f(\mathbf{x})$.

Solution

Since $\mathbf{p}^T\nabla f(\mathbf{x}) < 0$ and f is bounded from below, the function values of

$$g(\alpha) = f(\mathbf{x} + \alpha\mathbf{p}) - f(\mathbf{x}) - \alpha\mathbf{p}^T\nabla f(\mathbf{x})$$

will tend to $+\infty$ for $\alpha \to +\infty$. Thus, the set ①

$$\mathcal{Z} = \{\alpha \geq 0\colon g(\alpha) \leq 0\}$$

is bounded and hence compact (it is closed because g is continuous). Therefore, it contains a largest element

$$\alpha^* = \max \mathcal{Z} < +\infty.$$

On the other hand, it holds by Taylor's theorem that ①

$$\frac{g(\alpha)}{\alpha^2} = \frac{1}{2}\underbrace{\mathbf{p}^T H(\mathbf{x})\mathbf{p}}_{<0} + \underbrace{o(|\alpha|)}_{\to 0},$$

①

which is negative for sufficiently small $\alpha > 0$, i.e., $g(\alpha)$ is negative. Consequently, it holds $\alpha^* > 0$.

①

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Problem 5 **6 points**

Consider the quadratic minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) = \min_{\mathbf{x}\in\mathbb{R}^2} \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mathbf{c}$$

with

$$A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \mathbf{c} = 2.$$

(a) Show that A is positive definite.

(b) Determine $\nabla f(\mathbf{x})$ and the Hessian $H(\mathbf{x})$.

(c) Find all local minima using the necessary and sufficient second order conditions. Which one is the global minimum?

(d) From the starting point $\mathbf{x}_0 = (0,0)^T$, determine the exact line search parameter α^* that minimizes $\alpha \mapsto f(\mathbf{x}_0 - \alpha\nabla f(\mathbf{x}_0))$.

(e) Calculate one step of the steepest descent method from $\mathbf{x}_0 = (0,0)^T$ using the exact step length α^* from (d), and one step from $\mathbf{x}_0 = (0,0)^T$ using the Newton direction with step length $\alpha = 1$.

Solution

(a) For $\mathbf{x} \neq 0$ it holds$\mathbf{x}^T A\mathbf{x} = 3x_1^2 - 4x_1x_2 + 6x_2^2 = (x_1 - 2x_2)^2 + 2x_1^2 + 2x_2^2 > 0.$ ①

(b) We know from several occasions that for this kind of quadratic function we have

$$\nabla f(\mathbf{x}) = A\mathbf{x} - \mathbf{b} = \begin{pmatrix} 3x_1 - 2x_2 - 4 \\ -2x_1 + 6x_2 - 2 \end{pmatrix}, \quad H(\mathbf{x}) = A. \tag{4}$$ ①

(c) The only solution to the necessary condition $0 = \nabla f(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$ is

$$\mathbf{x}^* = A^{-1}\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$ ①

It has to be a global minimum, because f possesses a global minimum. This has to be the case, since every level set $\mathcal{L}_{\mathbf{x}_0} = \{\mathbf{x} \colon f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$ is bounded (hence compact). This in turn follows from

$$f(\mathbf{x}) \geq \lambda_{\min}(A)\|\mathbf{x}\|_2^2 - \|\mathbf{b}\|_2\|\mathbf{x}\|_2 + 2 \to +\infty \quad \text{for } \|\mathbf{x}\|_2 \to +\infty.$$

(d) Let $g(\alpha) = f(\mathbf{x}_0 - \alpha\nabla f(\mathbf{x}_0))$. For $\mathbf{x}_0 = 0$ it holds (see (4)) $\nabla f(\mathbf{x}_0) = -\mathbf{b}$. Hence, $g(\alpha) = f(\alpha\mathbf{b})$, and (see again (4))

$$g'(\alpha) = \mathbf{b}^T\nabla f(\alpha\mathbf{b}) = \alpha\mathbf{b}^T A\mathbf{b}^T - \mathbf{b}^T\mathbf{b}.$$

Hence the minimum of g (which is a convex parabola) is attained for

$$\alpha^* = \frac{\mathbf{b}^T\mathbf{b}}{\mathbf{b}^T A\mathbf{b}} = \frac{20}{48 - 32 + 24} = \frac{1}{2}.$$ ①

(e) The steepest descent with α^* step is

$$\mathbf{x}_1 = \mathbf{x}_0 - \alpha^*\nabla f(\mathbf{x}^*) = 0 + \frac{1}{2}\mathbf{b} = \mathbf{x}^*!$$ ①

The Newton step is

$$\mathbf{x}_1 = \mathbf{x}_0 - H(\mathbf{x}_0)^{-1}\nabla f(\mathbf{x}_0) = A^{-1}\mathbf{b} = \mathbf{x}^*.$$ ①

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Problem 6 **6 points**

Consider the set

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid g(\mathbf{x}) \geq 0\} \quad \text{where} \quad g(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ 1 - x_1 - x_2 \end{pmatrix}.$$

Solve the constrained optimization problem

$$\min_{\mathbf{x} \in \Omega} f(\mathbf{x}) \quad \text{where} \quad f(\mathbf{x}) = (x_1 - 3/2)^2 + (x_2 - 2)^2 \tag{5}$$

by following these steps:

(a) Write down the KKT conditions for problem (5).

(b) Show that LICQ holds for all $\mathbf{x} \in \Omega$.

(c) Find all KKT points for which at most one constraint is active.

(d) Find all $\mathbf{x} \in \Omega$ at which at least two constraints constraints are active.

(e) Select the solution of the problem from the points in (c) and (d).

Solution

(a) We have

$$\nabla f(\mathbf{x}) = \begin{pmatrix} 2(x_1 - 3/2) \\ 2(x - 2) \end{pmatrix}, \quad \nabla g_1(\mathbf{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \nabla g_2(\mathbf{x}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \nabla g_3(\mathbf{x}) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Therefore the main KKT equation $\nabla f(\mathbf{x}) = \sum_i \nabla \lambda_i g_i(\mathbf{x})$ reads

$$\begin{pmatrix} 2(x_1 - 3/2) \\ 2(x_2 - 2) \end{pmatrix} = \begin{pmatrix} \lambda_1 - \lambda_3 \\ \lambda_2 - \lambda_3 \end{pmatrix}.$$

Additionally, one requires

$$g(\mathbf{x}) \geq 0, \quad \lambda \geq 0, \quad \lambda_i g_i(\mathbf{x}) = 0 \quad \text{for } i = 1, 2, 3.$$ ①

(b) There is no point in Ω where all three constraints are active. Since any two vectors out of $\nabla g_1(\mathbf{x})$, $\nabla g_2(\mathbf{x})$, $\nabla g_3(\mathbf{x})$ are linearly independent, LICQ always holds. ①

(c) If we assume that at most one constraint is active, then by the complimentary condition at least two Lagrange multipliers have to vanish. However, if $\lambda_1 = \lambda_3 = 0$ or $\lambda_2 = \lambda_3 = 0$ we deduce from the first KKT equation $x_1 = 3/2$ or $x_2 = 2$, respectively, which both would violate $g(\mathbf{x}) \geq 0$. Thus, the only option is $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$. This gives

$$x_1 - 3/2 = -\lambda_3/2 = x_2 - 2.$$

Since g_3 is active, we have the additional equation

$$x_1 + x_2 = 1.$$

The unique solution of both equations is$\mathbf{x}^* = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$. ②

(d) Only at three points two constraints are active: $\mathbf{x}_1 = 0$ with $f(\mathbf{x}_1) = 25/4$, $\mathbf{x}_2 = (1, 0)^T$ with $f(\mathbf{x}_2) = 17/4$, and $\mathbf{x}_3 = (0, 1)^T$ with $f(\mathbf{x}_3) = 13/4$. ①

(e) Since all function values in (e) are larger than $f(\mathbf{x}^*) = 25/8$, since $\mathbf{x}^*$ is the unique KKT point with at most one active constraint, and since the problem must have a solution, the solution is $\mathbf{x}^*$. ①

Problem 7 **4 points**

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Complete the MATLAB code according to the description. Your hand-written code must be syntactically correct, that is, it will not produce an error message when executed in MATLAB.

(a) The code below implements the steepest descent algorithm using the Armijo rule, that is, the step-size at iteration k is given by the largest $\alpha_k \in \{1, \beta, \beta^2, \beta^3, \dots\}$ such that $f(\mathbf{x}_k + \alpha_k \mathbf{p}_k) - f(\mathbf{x}_k) \le c_1 \alpha_k \nabla f(\mathbf{x}_k)^T \mathbf{p}_k$ (here $\beta, c_1 \in]0,1[$). The input `x0` is a column vector, and also `df(x)` returns the gradient of f at x as a column vector.

```
function x = steepdesc(f,df,x0,tol,c1,beta)

x = x0;
while norm(df(x)) > tol
    p = - df(x);
    alpha = 1;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Insert Armijo backtracking loop here

    while f(x + alpha*p) - f(x) > c1*alpha*p'*df(x)        ①

        alpha = alpha*beta;                                 ①

    end


    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    % Perform steepest descent step
    x = x + alpha*p;
end
end
```

Please turn the page!

(b) The code below implements the implicit Euler method for an autonomous ODE $\mathbf{y}'(t) = f(\mathbf{y})$, $\mathbf{y}(t_0) = \mathbf{y}_0$. The nonlinear system $\mathbf{y}_{n+1} = \mathbf{y}_n + hf(\mathbf{y}_{n+1})$ in each step is solved by applying the Newton method to the function $g(\mathbf{y}) = \mathbf{y} - \mathbf{y}_n - hf(\mathbf{y})$ with starting guess $\mathbf{y}^{(0)} = \mathbf{y}_n$ and stopping criterion $\|g(\mathbf{y})\| \leq$ `tol`. The function values and the Jacobian of f at a point `y` are evaluated by calling `f(y)` and `df(y)`, respectively.

```
function Y = impeuler(f,df,tspan,y0,N,tol)

dim = length(y0);
h = (tspan(2) - tspan(1))/N;
Y(:,1) = y0;
for n=1:N
    y1 = y0;
    while norm(y1 - y0 - h*f(y1)) > tol

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Insert Newton step for g(y1) = y1 - y0 - h*f(y1) = 0 here

        y1 = y1 - (eye(dim) - h* df(y1)) \ (y1 - y0 - h*f(y1));

        %Alternative, following the lecture:
        %z = y1 - y0;
        %z = z - (eye(dim) - h*df(y0+z)) \ (z - h*f(y0+z));
        %y1 = y0+z;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    end

% Perform updates
Y(:,n+1) = y1;
y0 = y1;
end
end
```

②