ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Prof. D. Kressner
M. Steinlechner

1 ▶ Constrained least squares problem

Let $\mathbf{y} \in \mathbb{R}^N$, $A \in \mathbb{R}^{N \times r}$, $r \leq N$, rank $A = r$. Consider the constrained minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^N} \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|_2^2, \quad A^T\mathbf{x} = 0, \quad 1 - \|\mathbf{x}\|_2^2 \geq 0.$$

(a) Write down the KKT conditions.

(b) Assume $\|\mathbf{y}\|_2 < 1$. Show that the inequality constraint is inactive at a global solution $\mathbf{x}^*$ and deduce that

$$\mathbf{x}^* = (I - A(A^TA)^{-1}A^T)\mathbf{y}.$$

2 ▶ Constrained optimization

*Note: This is an old exam question (without part **(a)**).*

Consider the set

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid g(\mathbf{x}) \geq 0\} \quad \text{where} \quad g(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ 1 - x_1 - x_2 \end{pmatrix}.$$

Solve the constrained optimization problem

$$\min_{\mathbf{x} \in \Omega} f(\mathbf{x}) \quad \text{where} \quad f(\mathbf{x}) = (x_1 - 3/2)^2 + (x_2 - 2)^2 \tag{1}$$

by following these steps:

(a) Plot the constraint set defined by $g(\mathbf{x})$ in the (x_1, x_2) plane and add contour lines of the cost function. Can you already guess the optimal solution?

(b) Write down the KKT conditions for problem (1).

(c) Show that LICQ holds for all $\mathbf{x} \in \Omega$.

(d) Find all KKT points for which at most one constraint is active.

(e) Find all $\mathbf{x} \in \Omega$ at which at least two constraints are active.

(f) Select the solution of the problem from the points in (d) and (e).