ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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1 ► Tangent cone and linearized feasible directions

Consider again the tangent cone $T_\Omega(\mathbf{x})$ from the last exercise sheet:

$$\mathbf{x} = (-1, 0, 0)^T, \quad \Omega = \{\mathbf{x} \in \mathbb{R}^3 \colon x_1^2 + x_2^2 \leq 1,\ x_3^2 = 0\}.$$

Sketch $\mathcal{F}(\mathbf{x})$. Modify the parametrization of Ω such that $T_\Omega(\mathbf{x}) = \mathcal{F}(\mathbf{x})$ holds.

2 ► Lagrange multipliers

Show that the Lagrange multipliers λ^* and μ^* in Theorem 5.11 are unique.

3 ► KKT conditions

Find the point(s) on the parabola $y = \frac{1}{5}(x-1)^2$ which is/are closest to $(x, y) = (1, 2)$ in the Euclidian norm, i.e., solve

$$\min_{x,y} (x-1)^2 + (y-2)^2 \quad \text{subject to} \quad (x-1)^2 = 5y.$$

Find all KKT points, and the solution. Check that replacing $(x-1)^2$ in the objective function by $5y$ and solving the unconstrained problem for y does not work.

4 ► KKT conditions

For a fixed parameter t consider

$$\min_{\mathbf{x} \in \mathbb{R}^2} \left(x_1 - \frac{3}{4}\right)^2 + (x_2 - t)^4 \quad \text{subject to} \quad \begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} \geq 0.$$

(a) For what values of t, if there are any, does $\mathbf{x} = (1, 0)^T$ satisfy the KKT conditions?

(b) Show that when $t = 1$, only the first constraint is active at the solution, and find the solution.

5 ► Geometric and arithmetic mean

(a) Use the KKT conditions to find the global solution of

$$\min_{\mathbf{x} \in \mathbb{R}^n} \sum_{i=1}^n x_i \quad \text{subject to} \quad \prod_{i=1}^n x_i = 1,\ \mathbf{x} \geq 0.$$

Hint. First show that the global minimizer satisfies $\mathbf{x} > 0$.

(b) Use (a) to prove the inequality between the geometric and the arithmetic mean:

$$\left(\prod_{i=1}^n x_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n x_i \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \text{ with } \mathbf{x} \geq 0.$$