ÉCOLE POLYTECHNIQUE
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1 ▶ Euler and modified Euler method

(a) Write two MATLAB functions `expeuler` and `modeuler` which solve the intial value problem

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in [a, b], \\ y(a) = y_0 \end{cases}$$

for $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ using the explicit Euler method

$$y_{j+1} = y_j + hf(t_j, y_j)$$

and the improved Euler method

$$y_{j+1} = y_j + hf(t_j + h/2, y_j + h/2 f(t_j, y_j))$$

respectively, with stepsize $h = (b-a)/N$ (a, b and N are parameters). The function f should be supplied to your methods as a function handle.

(b) Apply both methods to the initial value problem

$$\begin{cases} y'(t) = -y(t), \quad t \in [0, 1], \\ y(0) = 1. \end{cases}$$

Record the absolute error to the exact solution for several values of N, and, using log-log plots, determine approximately the convergence order of both methods. Use the `polyfit` function to determine the slope in the log-log plot.

(c) Now integrate the differential equation

$$y'(t) = ty(t)(y(t) - 2), \quad t \in [0, 10],$$

using both methods with the initial values $y_0 = 1.9, 2.0, 2.1$. For $y_0 = 1.9$, produce a 2×2 `subplot` where you plot the numerical solutions of both methods for the four different numbers of stepsizes $N = 25, 50, 100, 500$. Compare the results to the exact solution.

2 ▶ Charged particle in electromagnetic field

The movement of a charged particle $\mathbf{x}(t) \in \mathbb{R}^3$ with mass m and charge q is driven by the Lorentz force

$$\mathbf{f}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) = q(\mathbf{E}(t, \mathbf{x}(t)) + \mathbf{B}(t, \mathbf{x}(t)) \times \dot{\mathbf{x}}(t)),$$

where $\mathbf{E}$ denotes the electric and $\mathbf{B}$ the magnetic filed, respectively ($\times$ is the cross product). Thus, the equation of motion is

$$m\ddot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \dot{\mathbf{x}}(t_0) = \mathbf{v}_0.$$

We consider the case of a unit mass ($m = 1$) with charge $q = 1$ in a magnetic field of a dipole,

$$\mathbf{B}(\mathbf{x}) = \frac{3(\mathbf{x}^T \mu)\mathbf{x} - \mu r^2}{r^5}, \quad r = \|\mathbf{x}\|_2,$$

with torque $\mu = (0, 0, 100)^T$, and no electric field, i.e., $\mathbf{E} = 0$.

Your task is to integrate the equation of motion in the time interval $[0, 10]$, starting from $\mathbf{x}_0 = (1, 1, 1)^T$ with velocity $\mathbf{v}_0 = (1, 1, 1)^T$. For this, transform it into an initial value problem for the six-dimensional function $\mathbf{y} = \binom{\mathbf{x}}{\dot{\mathbf{x}}}$, and use the MATLAB method `ode45` (this is a Runge-Kutta method we will derive later in the lecture). Use the function `comet3` for an animation.

3 ► Gronwall's lemma

(a) (Discrete version) Prove the following:

Let (u_n), (w_n) *be nonnegative sequences satisfying*

$$u_n \leq \alpha + \sum_{k=0}^{n-1} u_k w_k \quad \textit{for all } n \geq 0.$$

Then for all n *it holds*

$$u_n \leq \alpha \exp\Big(\sum_{k=0}^{n-1} w_k\Big).$$

(i) Verify the identity

$$1 + \sum_{k=0}^{n-1} \Big(\prod_{l=0}^{k-1} (1 + w_l)\Big) w_k = \prod_{k=0}^{n-1} (1 + w_k).$$

(ii) Proof by induction that for all n it holds

$$u_n \leq \alpha \prod_{k=0}^{n-1} (1 + w_k).$$

(iii) Deduce Gronwall's lemma.

(b) (Continuous version) Use (a) to prove a continuous version:

Let $u, w \in C([t_0, T])$ *be nonnegative functions satisfying*

$$u(t) \leq \alpha + \int_{t_0}^{t} u(s)w(s)\,\mathrm{d}s \quad \textit{for all } t \in [t_0, T].$$

Then for all $t \in [t_0, T]$ *it holds*

$$u(t) \leq \alpha \exp\Big(\int_{t_0}^{T} w(s)\,\mathrm{d}s\Big).$$

(c) (An application) Let $f\colon [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function satisfying the Lipschitz condition

$$\|f(t, \boldsymbol{\eta}) - f(t, \tilde{\boldsymbol{\eta}})\| \leq L\|\boldsymbol{\eta} - \tilde{\boldsymbol{\eta}}\|$$

for all $t, \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}$. Let $\sigma\colon [a, b] \to \mathbb{R}^n$ be continuous.

Let $\mathbf{y}$ and $\tilde{\mathbf{y}}$ be the solutions of the initial value problems

$$\begin{cases} \mathbf{y}'(t) = f(t, \mathbf{y}(t)), & t \in [a, b], \\ \mathbf{y}(a) = \mathbf{y}_0 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\mathbf{y}}'(t) = f(t, \tilde{\mathbf{y}}(t)) + \sigma(t), & t \in [a, b], \\ \tilde{\mathbf{y}(a)} = \tilde{\mathbf{y}}_0. \end{cases}$$

Find an estimate for the maximal error between $\mathbf{y}$ and $\tilde{\mathbf{y}}$,

$$\sup_{t \in [a,b]} \|\mathbf{y}(t) - \tilde{\mathbf{y}}(t)\|,$$

that only depends on the initial conditions $\mathbf{y}_0$, $\tilde{\mathbf{y}}_0$, the Lipschitz constant L, the time interval $[a, b]$ and the strength of the perturbation $\sup_{t \in [a,b]} \|\sigma(t)\|$.