Exam	
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE	**Advanced Numerical Analysis** **Teacher:** Prof. Dr. Daniel Kressner **Date:** 02.07.2013 **Duration:** 3h (08:15 - 11:15)

Sciper:	**Student:**	**Section:**

Score Table

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Problem 7	Total

Please read carefully:

- You are only allowed to have one A4 page of hand-written notes (no photocopies).
- Please put your CAMIPRO card on your desk before you start the exam as it will be checked during the exam.
- You must write your sciper number, your name, and section on this page before you start the exam.
- Calculators and all other electronical devices are forbidden.
- Do not use your own paper for writing down the solutions; use the blank pages after each exercise. You can request additional paper from the assistants.
- Except for Problems 1 and 7, do not only write down the final result or answer, but also some explanations and justification of the result. Results without justification are not counted.

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Problem 1 **8 points**

Choose one answer to each of the following questions. Every correct answer gives 1 point, no answer gives 0 points, and every wrong answer gives −1 points. However, you cannot get less than zero points in total.

(a) The approximate solution $\mathbf{y}_n$ obtained from applying any A-stable method to a general IVP always converges to zero for $n \to \infty$.

true ☐ **false** ☐

(b) The approximate solution $\mathbf{y}_n$ obtained from applying the implicit Euler method to the linear IVP $\mathbf{y}'(t) = G\mathbf{y}(t)$, with initial value $\mathbf{y}(0) = \mathbf{y}_0$ and a general matrix $G \in \mathbb{R}^{n\times n}$, always converges to zero for $n \to \infty$.

true ☐ **false** ☐

(c) The linear system

$$\mathbf{y}_1 = \mathbf{y}_0 + hG\mathbf{y}_1,$$

which needs to be solved in the first step of the implicit Euler method applied to the linear IVP $\mathbf{y}'(t) = G\mathbf{y}(t)$ with initial value $\mathbf{y}(0) = \mathbf{y}_0$ and a general matrix $G \in \mathbb{R}^{n\times n}$, always has a unique solution $\mathbf{y}_1$.

true ☐ **false** ☐

(d) The statement of (c) holds if $h > 0$ is sufficiently small.

true ☐ **false** ☐

(e) The linear system that needs to be solved in the first step of an implicit Runge-Kutta method applied to the linear IVP $\mathbf{y}'(t) = G\mathbf{y}(t)$, with initial value $\mathbf{y}(0) = \mathbf{y}_0$ and a general matrix $G \in \mathbb{R}^{n\times n}$, always has a unique solution if $h > 0$ is sufficiently small.

true ☐ **false** ☐

(f) Consider a method of maximal consistency order p for solving IVPs $\mathbf{y}'(t) = f(t, \mathbf{y}(t))$, $\mathbf{y}(t_0) = \mathbf{y}_0$. Then the following statement holds: For all $(p+1)$-times continuously differentiable functions f there exists a constant $c > 0$ such that the local error $\mathbf{e}_1 = \|\mathbf{y}_1 - \mathbf{y}(t_1)\|$ of the first step satisfies $\mathbf{e}_1 \geq ch^{p+1}$.

true ☐ **false** ☐

(g) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and $\mathbf{x}, \mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{p}^T \nabla f(\mathbf{x}) < 0$. Then there exists $\alpha^* > 0$ such that

$$f(\mathbf{x} + \alpha\mathbf{p}) < f(\mathbf{x}) \quad \text{for all } 0 < \alpha < \alpha^*.$$

true ☐ **false** ☐

(h) The statement of c. holds if $h > 0$ is sufficiently small.

true ☐ **false** ☐

(i) The linear system that needs to solved in the first step of an implicit Runge-Kutta method applied to the linear IVP $\mathbf{y}'(t) = G\mathbf{y}(t)$, with initial value $\mathbf{y}(0) = \mathbf{y}_0$ and a general matrix $G \in \mathbb{R}^{n\times n}$, always has a unique solution $\mathbf{y}_1$ if $h > 0$ is sufficiently small.

true ☐ **false** ☐

(j) Consider a method of maximal consistency order p for solving IVPs $\mathbf{y}'(t) = f(t, \mathbf{y}(t))$, $\mathbf{y}(t_0) = \mathbf{y}_0$. Then the following statement holds: For all $(p+1)$-times continuously differentiable functions f there exists a constant $c > 0$ such that the local error is bounded from below by ch^{p+1}.

true ☐ **false** ☐

(k) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and bounded from below. At a point $\mathbf{x}$ let $\mathbf{p}$ be a descent direction. Then there exists a largest number $\alpha > 0$ (strict inequality!) with the property $f(\mathbf{x} + \alpha\mathbf{p}) \leq f(\mathbf{x}) + \alpha\mathbf{p}^T\nabla f(\mathbf{x})$.

true ☐ **false** ☐

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Problem 2 **5 points**

Use the table with Runge-Kutta order conditions (on the next sheet) to find all explicit autonomization-invariant order-three Runge-Kutta methods of the form

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & a_{21} & & \\ 2/3 & a_{31} & a_{32} & \\ \hline & b_1 & b_2 & b_3 \end{array}.$$

Table with order conditions

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Problem 3 **4 points**

For a fixed parameter θ, consider the following method for solving IVPs:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + hf(t_n + \theta h, (1-\theta)\mathbf{y}_n + \theta \mathbf{y}_{n+1}).$$

Show that the method is A-stable if and only if $\theta \geq 1/2$.

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Problem 4 **4 points**

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Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be two times continuously differentiable and bounded from below. At a point $\mathbf{x}$ let $\mathbf{p}$ be a descent direction and assume that the Hessian $H(\mathbf{x})$ is negative definite. Show that there exists a largest number $\alpha > 0$ (strict inequality!) with the property $f(\mathbf{x} + \alpha\mathbf{p}) \leq f(\mathbf{x}) + \alpha\mathbf{p}^T\nabla f(\mathbf{x})$.

Problem 5 **6 points**

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Consider the quadratic minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) = \min_{\mathbf{x}\in\mathbb{R}^2} \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + \mathbf{c}$$

with

$$A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \mathbf{c} = 2.$$

(a) Show that A is positive definite.

(b) Determine $\nabla f(\mathbf{x})$ and the Hessian $H(\mathbf{x})$.

(c) Find all local minima using the necessary and sufficient second order conditions. Which one is the global minimum?

(d) From the starting point $\mathbf{x}_0 = (0,0)^T$, determine the exact line search parameter α^* that minimizes $\alpha \mapsto f(\mathbf{x}_0 - \alpha\nabla f(\mathbf{x}_0))$.

(e) Calculate one step of the steepest descent method from $\mathbf{x}_0 = (0,0)^T$ using the exact step length α^* from (d), and one step from $\mathbf{x}_0 = (0,0)^T$ using the Newton direction with step length $\alpha = 1$.

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Problem 6 **6 points**

Consider the set

$$\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid g(\mathbf{x}) \geq 0\} \quad \text{where} \quad g(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ 1 - x_1 - x_2 \end{pmatrix}.$$

Solve the constrained optimization problem

$$\min_{\mathbf{x} \in \Omega} f(\mathbf{x}) \quad \text{where} \quad f(\mathbf{x}) = (x_1 - 3/2)^2 + (x_2 - 2)^2 \tag{1}$$

by following these steps:

(a) Write down the KKT conditions for problem (1).

(b) Show that LICQ holds for all $\mathbf{x} \in \Omega$.

(c) Find all KKT points for which at most one constraint is active.

(d) Find all $\mathbf{x} \in \Omega$ at which at least two constraints constraints are active.

(e) Select the solution of the problem from the points in (c) and (d).

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Problem 7 **4 points**

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Complete the MATLAB code according to the description. Your hand-written code must be syntactically correct, that is, it will not produce an error message when executed in MATLAB.

(a) The code below implements the steepest descent algorithm using the Armijo rule, that is, the step-size at iteration k is given by the largest $\alpha_k \in \{1, \beta, \beta^2, \beta^3, \dots\}$ such that $f(\mathbf{x}_k + \alpha_k \mathbf{p}_k) - f(\mathbf{x}_k) \leq c_1 \alpha_k \nabla f(\mathbf{x}_k)^T \mathbf{p}_k$ (here $\beta, c_1 \in]0, 1[$). The input `x0` is a column vector, and also `df(x)` returns the gradient of f at x as a column vector.

```
function x = steepdesc(f,df,x0,tol,c1,beta)

x = x0;
while norm(df(x)) > tol
    p = - df(x);
    alpha = 1;
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Insert Armijo backtracking here

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Insert Armijo backtracking loop here









    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    % Perform steepest descent step
    x = x + alpha*p;
end
end
```

Please turn the page!

(b) The code below implements the implicit Euler method for an autonomous ODE $\mathbf{y}'(t) = f(\mathbf{y})$, $\mathbf{y}(t_0) = \mathbf{y}_0$. The nonlinear system $\mathbf{y}_{n+1} = \mathbf{y}_n + hf(\mathbf{y}_{n+1})$ in each step is solved by applying the Newton method to the function $g(\mathbf{y}) = \mathbf{y} - \mathbf{y}_n - hf(\mathbf{y})$ with starting guess $\mathbf{y}^{(0)} = \mathbf{y}_n$ and stopping criterion $\|g(\mathbf{y})\| \leq$ `tol`. The function values and the Jacobian of f at a point `y` are evaluated by calling `f(y)` and `df(y)`, respectively.

```
function Y = impeuler(f,df,tspan,y0,N,tol)

dim = length(y0);
h = (tspan(2) - tspan(1))/N;
Y(:,1) = y0;
for n=1:N
    y1 = y0;
    while norm(y1 - y0 - h*f(y1)) > tol

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % Insert Newton step for g(y1) = y1 - y0 - h*f(y1) = 0 here









    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    end

% Perform updates
Y(:,n+1) = y1;
y0 = y1;
end
end
```