Non-unique ergodiciy for deterministic and stochastic 3D Navier–Stokes and Euler equations

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Joint work with Martina Hofmanová, Umberto Pappalettera and Xiangchan Zhu

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Introduction

Consider the Navier-Stokes/Euler equations on $\mathbb{T}^3$:

$$\begin{aligned} \partial_t u + u \cdot \nabla u =& \nu \Delta u - \nabla p + f, \quad \mathrm{div} u = 0 \\ u(0) =& u_0 \end{aligned} \tag{1}$$

- $u(t,x) \in \mathbb{R}^3$: the velocity field at time t and position x,
- $p(t,x)$: the pressure,
- viscosity $\nu \geq 0$ – Navier–Stokes equations $\nu > 0$ and Euler equations $\nu = 0$
- f: random noise/ deterministic force

Derivation of Navier-Stokes system: Newton's law

Suppose $u = u_\nu(t, x(t))$ and ρ: the density

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\nu(t) = \underbrace{\partial_t u_\nu}_{\text{variation}} + \underbrace{u_\nu \cdot \nabla u_\nu}_{\text{convection}} = \underbrace{\nu\Delta u_\nu}_{\text{Diffusion}} - \underbrace{\nabla p}_{\text{Internal source}} + \underbrace{f}_{\text{External source}},$$

$$\underbrace{\partial_t \rho + \nabla \cdot (\rho u_\nu) = 0}_{\text{mass conservation}} \Rightarrow^{\text{if } \rho = \text{constant}} \mathrm{div} u_\nu = 0$$

$$u_\nu(0) = u_0.$$

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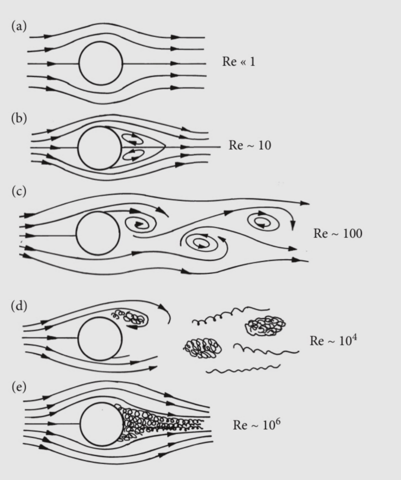
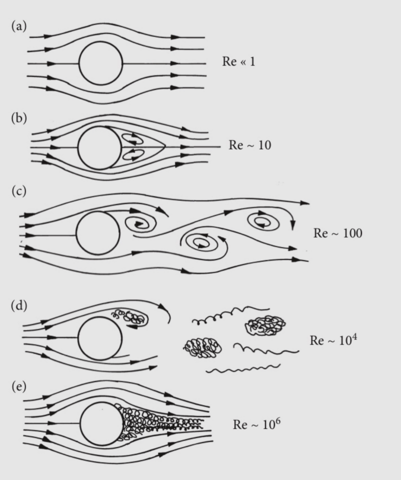
Motiviations for random force:

- stochastic reduction
- regularization by noise
- Turbulence: dynamical behavior (high sensitivity) – statistical law
- Kolmogorov 1941 turbulence theory



- $$\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + f, \quad \mathrm{Re} \sim 1/\nu,$$



- $$\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + f, \quad \mathrm{Re} \sim 1/\nu,$$
- high Reynolds number limit $\nu \to 0$ – highly turbulent regime

Solution theory: [Leray34], [Kato, Fujita62], [Temam84], [Constantin, Foias88] [Cafarelli,Kohn, Nirenberg84], [Fefferman 00], [Koch, Tataru01],...

- The global existence of **weak** solutions has been obtained in all dimensions.
- Existence and smoothness of solutions in the three dimensional case remains open (the Millennium Prize problem)./ Small initial data
- [Buckmaster, Vicol 19]: Non-uniqueness of analytic weak solutions: Construction of solutions with given energy
- [Buckmaster, Colombo, Vicol 20] connect two arbitrary strong solutions via a weak solution
- [Albritton, E. Brué, M. Colombo. 22] non-uniqueness of Leray solutions for some force

Stochastic Navier-Stokes equations

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- Martingale solutions/Probabilistically weak solutions: probability measure on the canonical space $C([0,\infty):H^{-3})$
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Known results:

- Leray martingale Markov solutions to stochastic 3D Navier–Stokes have been constructed [Da Prato, Debussche 03, Flandoli, Romito 08]

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- Leray martingale Markov solutions to stochastic 3D Navier–Stokes have been constructed [Da Prato, Debussche 03, Flandoli, Romito 08]
- Nonuniqueness in law for Stochastic 3D Navier–Stokes/Euler equations [Hofmanová, Zhu, Z. 19, 20]
- Nonuniqueness of Markov solutions/Global probabilistically strong solutions to stochastic 3D Navier–Stokes equations [Hofmanová, Zhu, Z. 21]

Ergodic hypothesis

- basic assumption in turbulence theory:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F(u(t))dt=\int F d\nu$$

- the measure is invariant – Statistically stationary solutions: Law$[u(t+\cdot)]=$Law$[u(\cdot)]$
- ergodic stationary solution

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Results related to ergodicity

- 2d Navier-Stokes: Uniqueness of invariant measure [Hairer, Mattingly 06]
- 3d Navier-Stokes with non-degenerate noise: Every Markov selection has a unique invariant measure [Da Prato-Debussche 03, Flandoli, Romito 08]

Aim

- Existence and (non)uniqueness of ergodic stationary solutions u_ν to the stochastic Navier–Stokes equations
- Relative compactness of stationary solutions u_ν and convergence to a stationary solution to the Euler equations
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- Kolmogorov law of turbulence along the vanishing viscosity limit $\nu \to 0$

Nonuniqueness of stationary solutions for the stochastic NS/ Euler equations

Main results: Nonuniqueness of stationary solutions

$$\begin{aligned} \mathrm{d}u + u\cdot\nabla u\mathrm{d}t =& \nu\Delta u\mathrm{d}t - \nabla p\mathrm{d}t + \mathrm{d}B(t), \quad \mathrm{div} u = 0 \\ u(0) =& u_0 \end{aligned} \tag{2}$$

Theorem (Hofmanová, Zhu, Z. 22)

There exist

1. *infinitely many stationary solutions;*
2. *infinitely many ergodic stationary solutions;*

to the stochastic 3D Navier–Stokes and Euler equations.

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For any $\nu_n \to 0$, $\exists$ stationary solutions u_n to (2) with $\nu = \nu_n$ so that $\mathcal{L}[u_n]$, $n \in \mathbb{N}$, is tight in $C(\mathbb{R}; L^2_\sigma)$ and every accumulation point is a stationary solution to the stochastic Euler equations.

Idea of proof: stochastic convex integration

- Stochastic convex integration: for some $\vartheta > 0$ and $r > 1$

$$\sup_{\nu} \sup_{s\in\mathbb{R}} \Big(\mathbf{E} \sup_{s\le t\le s+1} \|u(t)\|_{H^\vartheta}^{2r} + \mathbf{E}\|u\|_{C^\vartheta([s,s+1];L^2)}^{2r}\Big) < \infty.$$

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- decomposition $u = z + v$

$$\mathrm{d}z - (\nu\Delta - 1)z\mathrm{d}t = \mathrm{d}B, \quad \mathrm{div} z = 0$$

$$\partial_t v - \nu\Delta v - z + \mathrm{div}((v+z)\otimes(v+z)) + \nabla p = 0, \quad \mathrm{div} v = 0.$$

- For any $p \ge 1$

$$\sup_{s\ge 0} \mathbf{E}\left[\sup_{s\le t\le s+1} \|z(t)\|_{H^{1-\delta}}^{p} + \|z\|_{C^{1/2-\delta}_{[s,s+1]}L^2}^{p}\right] \le (p-1)^{p/2} L^p,$$

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- iteration scheme

$$\partial_t v_q - \nu\Delta v_q - z + \mathrm{div}((v_q+z)\otimes(v_q+z)) + \nabla p_q = \mathrm{div}\mathring{R}_q, \quad \mathrm{div} v_q = 0.$$

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- Key step: Let $w_{q+1} = v_{q+1} - v_q$, then we have

$$\mathrm{div}\mathring{R}_{q+1} = \underbrace{-\nu\Delta w_{q+1} + \partial_t w_{q+1} + \mathrm{div}((v_q + z)\otimes w_{q+1} + w_{q+1}\otimes(v_q + z))}_{\text{linear error}}$$
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- Choose

$$w_{q+1} \sim \sum_{\xi} a_\xi(\mathring{R}_q) W_\xi$$

with W_ξ intermittent jets from [Buckmaster, Colombo, Vicol19]

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- The space concentration ensure the linear error is small in L^1
- $\int W_\xi \otimes W_\xi \simeq 1$ and $a_\xi(\mathring{R}_q) \approx \sqrt{-\mathring{R}_q}$ oscillates slowly

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- iterative estimates: $r>1$ fixed, any $m\in\mathbb{N}$

$$\sup_{t\geq 0}\mathbf{E}\left[\sup_{t\leq s\leq t+1}\|\mathring{R}_q(s)\|_{L^1}^{r}\right]\leq\delta_{q+1}\to 0,\quad q\to\infty$$

$$\sup_{t\geq 0}\mathbf{E}\left[\sup_{t\leq s\leq t+1}\|\mathring{R}_q(s)\|_{L^1}^{m}\right]\leq(6^q\cdot 4mL^2)^{(6^q)}$$

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- use small factors to absorb the blow up

Idea of proof: Krylov–Bogoliubov

- instead of Markov semigroup, work with shifts on trajectories

$$S_t(u, B)(\cdot) = (u(t+\cdot), B(t+\cdot) - B(t))$$

on path space $C(\mathbb{R}; L^2) \times C(\mathbb{R}; L^2)$ continuity for free (cf. Feller property)

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- the bounds good enough to apply Krylov–Bogoliubov – existence of stationary solutions

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- nonuniqueness by choosing different energy
- bounds uniform in ν-the results apply to the stochastic Euler equations and vanishing viscosity limit

Kolmogorov $4/5$ law for the forced NS equations

Energy inequality

Consider the forced NS equations

$$\begin{aligned} \partial_t u_\nu + u_\nu \cdot \nabla u_\nu =& \nu \Delta u_\nu - \nabla p_\nu + f_\nu, \quad \mathrm{div} u_\nu = 0 \\ u_\nu(0) =& u_0 \end{aligned}$$

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Formal energy inequality

- Assume u_ν is smooth – test the equation by u_ν

$$\langle \partial_t u_\nu, u_\nu \rangle + \langle u_\nu \cdot \nabla u_\nu, u_\nu \rangle + \langle \nabla p, u_\nu \rangle = \nu \langle \Delta u_\nu, u_\nu \rangle + \langle f_\nu, u_\nu \rangle$$
$$\Rightarrow \frac{1}{2} \partial_t \| u_\nu \|_{L^2}^2 + \nu \| \nabla u_\nu \|_{L^2}^2 = \langle f_\nu, u_\nu \rangle$$

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- energy conservation for Euler equations $\frac{1}{2}\partial_t \|u\|_{L^2}^2 = \langle f, u \rangle$

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- energy conservation for Euler equations $\frac{1}{2}\partial_t \|u\|_{L^2}^2 = \langle f, u \rangle$
- vanishing viscosity limit in a class of smooth solutions would imply

$$\lim_{\nu \to 0} \nu \|\nabla u_\nu\|_{L^2}^2 = 0$$

Kolmogorov ('41) theory for turbulence

- Zeroth law of turbulence (Anomalous dissipation): the inviscid limit $\nu \to 0$

$$\varepsilon = \liminf_{\nu \to 0} \left(\nu \langle |\nabla u_\nu|^2 \rangle \right) > 0,$$

$\langle \cdot \rangle$: integration w.r.t. spatial variable

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- 4/5 law: The third order longitudinal structure function

$$S_3^{||}(\ell) = \int_{\mathbb{S}^2}\int_{\mathbb{T}^3}((u_\nu(t,x+\ell\hat{n}) - u_\nu(t,x))\cdot\hat{n})^3 \mathrm{d}x\mathrm{d}S(\hat{n}) \simeq -\frac{4}{5}\varepsilon\ell,$$

  for $\ell \in (\ell_I, \ell_D)$. [Bedrossian, Coti Zelati, Punshon-Smith, Weber19/ Dudley 23]

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- The pth order absolute structure function

$$S_p(\ell) = \int_{\mathbb{S}^2}\int_{\mathbb{T}^3}|u_\nu(t,x+\ell\hat{n}) - u_\nu(t,x)|^p \mathrm{d}x\mathrm{d}S(\hat{n}) \sim (\varepsilon\ell)^{p/3}.$$

in the infinite Reynolds number limit. $p = 3$ is verified by all experiments.

Assumption

- [Brue, De Lellis 22/ Brue, Colombo, Crippa, De Lellis, Sorella 22] constructed forces f_ν and initial data such that the Leray solution to the forced NS equation satisfy the anomalous dissipation:

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- [Brue, De Lellis 22/ Brue, Colombo, Crippa, De Lellis, Sorella 22] constructed forces f_ν and initial data such that the Leray solution to the forced NS equation satisfy the anomalous dissipation:

$$\varepsilon := \limsup_{\nu \to 0} \nu \int_0^1 \|\nabla u_\nu\|_{L^2}^2 \mathrm{d}t > 0.$$

- (**H**) Assume that u_ν is a Leray–Hopf solution on $[0,1] \times \mathbb{T}^3$ with force f_ν such that there exist $\sigma > 0$, $\alpha > 0$ such that

$$\sup_{\nu \in (0,1)} \left(\|u_\nu\|_{L^\infty(0,1;L^2)} + \|u_\nu\|_{L^1(0,1;H^\alpha)} + \|f_\nu\|_{L^{1+\sigma}(0,1;L^2)} \right) < \infty.$$

- Assumption (**H**) is satisfied by the solutions constructed in [Brue, Colombo, Crippa, De Lellis, Sorella 22]

Structure function

Let

$$\delta_h u_\nu(x) = u_\nu(x+h) - u_\nu(x), \quad h \in \mathbb{R}^3.$$

The average of the cube of the longitudinal velocity increment given by

$$S_{\|}^{\nu}(t,\ell) := \frac{1}{4\pi}\int_{\mathbb{S}^2}\int_{\mathbb{T}^3} (\delta_{\ell\hat{n}} u_\nu(t,x)\cdot\hat{n})^3 \mathrm{d}x\mathrm{d}S(\hat{n}),$$

The averaged structure function

$$S_{0}^{\nu}(t,\ell) := \frac{1}{4\pi}\int_{\mathbb{S}^2}\int_{\mathbb{T}^3} |\delta_{\ell\hat{n}} u_\nu(t,x)|^2 \delta_{\ell\hat{n}} v_\nu(t,x)\cdot\hat{n} \mathrm{d}x\mathrm{d}S(\hat{n}).$$

The third order absolute structure function is

$$S_{3}^{\nu}(t,\ell) := \frac{1}{4\pi}\int_{\mathbb{S}^2}\int_{\mathbb{T}^3} |\delta_{\ell\hat{n}} u_\nu(t)|^3 \mathrm{d}x\mathrm{d}S(\hat{n}).$$

Main results: Kolmogorov 4/5 law

Theorem (Hofmanová, Pappalettera, Zhu, Z. 23)

Let u_ν, $\nu \in (0,1)$, *be Leray-Hopf solutions to the forced Navier–Stokes system satisfying* (**H**). *Then there exists* $\ell_D = \ell_D(\nu)$ *with* $\lim_{\nu\to 0} \ell_D = 0$ *such that for any* $p \in [1,\infty)$

$$\lim_{\ell_I\to 0} \limsup_{\nu\to 0} \sup_{\ell\in[\ell_D,\ell_I]} \left\| \int_0^{\cdot} \frac{S_{\parallel}^{\nu}(r,\ell)}{\ell} \mathrm{d}r + \frac{4}{5}\varepsilon_\nu(\cdot) \right\|_{L^p(0,1)} = 0,$$

$$\lim_{\ell_I\to 0} \limsup_{\nu\to 0} \sup_{\ell\in[\ell_D,\ell_I]} \left\| \int_0^{\cdot} \frac{S_{0}^{\nu}(r,\ell)}{\ell} \mathrm{d}r + \frac{4}{3}\varepsilon_\nu(\cdot) \right\|_{L^p(0,1)} = 0,$$

with

$$\varepsilon_\nu(t) = \frac{1}{2}\|u_\nu(t)\|_{L^2}^2 - \frac{1}{2}\|u_\nu(0)\|_{L^2}^2 + \int_0^t \langle f_\nu, u_\nu\rangle \mathrm{d}s.$$

If the energy equality holds true then $\varepsilon_\nu(t) = \nu \int_0^t \|\nabla u_\nu(s)\|_{L^2}^2 \mathrm{d}s$.

Main results: Kolmogorov 4/5 law – probabilistic interpretation

Theorem (Hofmanová, Pappalettera, Zhu, Z. 23)

Let u_ν, $\nu \in (0,1)$, be solutions to the forced Navier–Stokes system satisfying the hypothesis (**H**)*. There exists $\ell_D = \ell_D(\nu)$ with $\lim_{\nu\to 0} \ell_D = 0$ such that for every $p \in [1,\infty)$, $\kappa > 0$ and $K > 0$*

$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_D, \ell_I]} \sup_{\mathfrak{t}} \left\langle \left| \int_0^{\mathfrak{t}} \frac{S_{\|}^{\nu}(r,\ell)}{\ell} \mathrm{d}r + \frac{4}{5} \varepsilon_\nu(\mathfrak{t}) \right|^p \right\rangle = 0,$$

and

$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_D, \ell_I]} \sup_{\mathfrak{t}} \left\langle \left| \int_0^{\mathfrak{t}} \frac{S_{0}^{\nu}(r,\ell)}{\ell} \mathrm{d}r + \frac{4}{3} \varepsilon_\nu(\mathfrak{t}) \right|^p \right\rangle = 0.$$

Here $\mathfrak{t}$ are arbitrary random times taking values in $[0,1]$ whose law is absolutely continuous with respect to the Lebesgue measure with a density ψ satisfying $\|\psi\|_{L^{1+\kappa}(0,1)} \le K$ and the bracket $\langle\cdot\rangle$ denotes the ensemble average.

Idea of Proof: Kármán–Howarth–Monin (KHM) relation

$$\underbrace{\int_{\mathbb{R}^3} \eta(h) : \Gamma_\nu(t,h)\mathrm{d}h}_{\sim\frac{2}{3}\|u_\nu(t)\|_{L^2}^2} - \underbrace{\int_{\mathbb{R}^3} \eta(h) : \Gamma_\nu(0,h)\mathrm{d}h}_{\sim\frac{2}{3}\|u_\nu(0)\|_{L^2}^2}$$

$$= -\underbrace{\frac{1}{2}\sum_{k=1}^{3}\int_0^t \int_{\mathbb{R}^3} \partial_k\eta(h) : D_\nu^k(r,h)\mathrm{d}h\mathrm{d}r}_{\sim\frac{S_0^\nu(\ell)}{\ell}} + \underbrace{2\nu\int_0^t \int_{\mathbb{R}^3} \Delta\eta(h) : \Gamma_\nu(r,h)\mathrm{d}h\mathrm{d}r}_{\sim 0}$$

$$+ \underbrace{2\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \eta(h) : f_\nu \otimes T_h u_\nu \mathrm{d}x\mathrm{d}h\mathrm{d}r}_{\sim\langle f_\nu, u_\nu\rangle},$$

where $T_h v_\nu(t,x) = u_\nu(t,x+h)$,

$$\Gamma_\nu(t,h) := \int_{\mathbb{T}^3} u_\nu(t,x) \otimes u_\nu(t,x+h)\mathrm{d}x,$$

$$D_\nu^k(t,h) = \int_{\mathbb{T}^3} (\delta_h u_\nu(t,x) \otimes \delta_h u_\nu(t,x))\delta_h u_\nu^k(t,x)\mathrm{d}x.$$

Main results: the third order absolute structure function

Theorem (Hofmanová, Pappalettera, Zhu, Z. 23)

Let $\alpha \in (0, 1/3)$ *be given and let* u_ν, $\nu \in (0,1)$, *be the solutions to the forced Navier–Stokes equations satisfying*

$$\sup_{\nu\in(0,1)} \left(\|u_\nu\|_{L^3(0,1;C^\alpha)} + \|u_\nu\|_{L^\infty(0,1;L^\infty)} \right) < \infty.$$

Then the third order absolute structure function exponents

$$\zeta_3 := \liminf_{\ell_I \to 0} \inf_{\nu\in(0,1)} \inf_{\ell\in[\ell_D,\ell_I]} \frac{\log(\int_0^1 S_3^\nu(r,\ell)\mathrm{d}r)}{\log \ell},$$

$$\bar{\zeta}_3 := \limsup_{\ell_I \to 0} \liminf_{\nu\to 0} \sup_{\ell\in[\ell_D,\ell_I]} \frac{\log(\int_0^1 S_3^\nu(r,\ell)\mathrm{d}r)}{\log \ell}.$$

satisfy

$$3\alpha \le \zeta_3 \le \bar{\zeta}_3 \le 1.$$

Thank you !