The role of permutations in mixing phenomena

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Mixing automorphisms

$(K = [0,1]^2, \mathcal{B}(K), |\cdot|)$.

Group of automorphisms

$$G(K) = \{T : K \to K \text{ invertible and measure-preserving}\}.$$

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Weakly mixing

Let $T \in G(K)$. T is *weakly mixing* if $\forall A, B \in \mathcal{B}(K)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (|T^j(A) \cap B| - |A||B|)^2 = 0. \tag{1}$$

Strongly mixing

Let $T \in G(K)$. T is *strongly mixing* if $\forall A, B \in \mathcal{B}(K)$

$$\lim_{n\to\infty} |T^n(A) \cap B| = |A||B|. \tag{2}$$

Mixing in Ergodic Theory

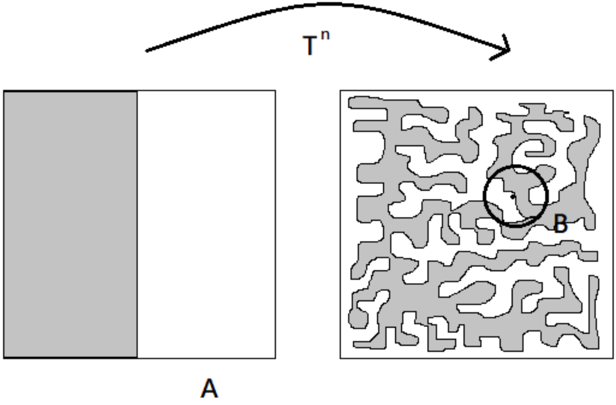

Figure: Action of a mixing map.

Why permutations?

In the simplest case $(\Omega = \{1, 2, \dots, N\}, \mu)$ in which every point has the same mass $\frac{1}{N}$ the invertible measure-preserving maps are the **permutations**.

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Permutations are building blocks for mixing.

Regular Lagrangian Flow for rough vector fields

Our setting

$(K=[0,1]^2, \mathcal{B}(K), |\cdot|)$ and $b:[0,1]\times K \to \mathbb{R}^2$,
$b_t \in \mathrm{BV}(\mathbb{R}^2)\cap\{\text{supp } b_t \subset K\}$, $b \in L^\infty_t \mathrm{BV}_x$, **divergence-free**.

[1] Ambrosio '04, Sobolev vector fields DiPerna-Lions '89

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There exists a unique[1] **Regular Lagrangian Flow** (RLF) $X_t : K \to K$ such that

- for a.e. x the flow X_t is an absolutely continuous integral solution of

$$\begin{cases} \dot{\gamma}(t) = b(t, \gamma(t)); \\ \gamma(0) = x; \end{cases}$$

- $|X_t^{-1}(A)| = |A| \ \forall t \in [0,1]$, $\forall A \in \mathcal{B}(K)$ (**measure-preserving**).

[1] Ambrosio '04, Sobolev vector fields DiPerna-Lions '89

Mixing vector fields

For every $t \in [0,1]$ fixed $X_t : K \to K$ is invertible and measure-preserving: bridge with Ergodic Theory.

Definition:

Let $b \in L_t^\infty \mathrm{BV}_x$ divergence-free. Then b is ergodic (weakly mixing, strongly mixing) if its unique RLF when evaluated at time $t = 1$, namely $X_{t=1} \in G(K)$, is ergodic (weakly mixing, strongly mixing).

Permutation vector fields

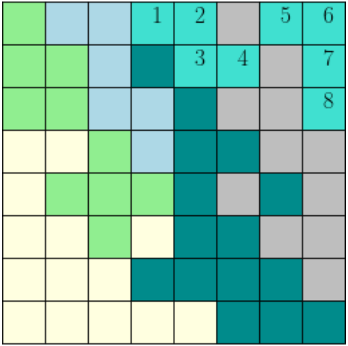

Figure: A permutation of squares realized by $X^p_{t=1}$ the RLF of a permutation vector field.

Approximation by permutations

1. **First approximation result**: approximation of a vector field by a **permutation vector field**;
2. **Second approximation result**: you can always assume the permutation has a **unique cycle**;
3. **Perturbation with a universal mixer.**

Approximation by Permutations

Theorem (First approximation result, Bianchini, Z., 2021)

Let $b \in L^\infty_t BV_x$ be a divergence-free vector field. Then for every $\epsilon > 0$ there exist $C_1, C_2 > 0$ positive constants, $D \in \mathbb{N}$ arbitrarily large and a divergence-free vector field $b^p \in L^\infty_t BV_x$ such that

1. *it holds*

$$\|b - b^p\|_{L^\infty(L^1)} \leq \epsilon, \quad ||TV(b^p)(K)||_\infty \leq C_1||TV(b)(K)||_\infty + C_2, \tag{3}$$

2. *the map $X^p\llcorner_{t=1}$ generated by b^p at time $t = 1$ translates each subsquare of the grid $\mathbb{N} \times \mathbb{N}\frac{1}{D}$ into a subsquare of the same grid, i.e. it is a permutation of squares.*

It is based on Shnirelman, A. *"The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid"*.

Shnirelman's Lemma (1985)

Let T be a measure-preserving diffeomorphism $T : K \to K$ of class $\mathcal{C}^3$ and such that $T = id$ in a neighborhood of ∂K. Assume that it is close to the identity, i.e. there exists $\delta > 0$ sufficiently small such that $||T - \mathrm{id}||_{\mathcal{C}^1} \leq \delta$.

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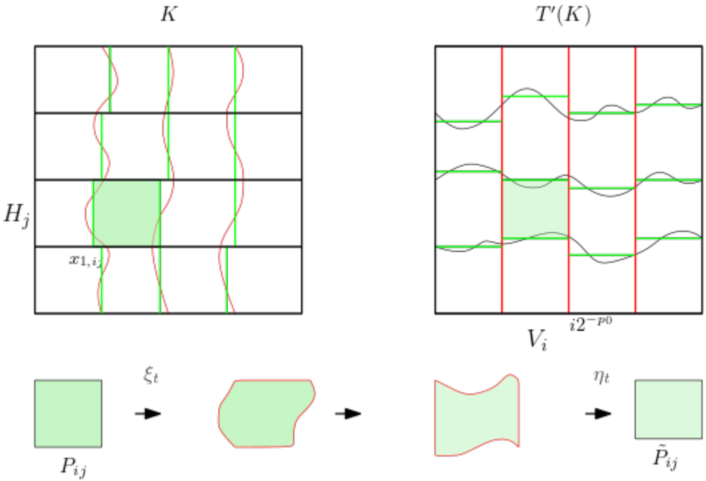
Lemma

There exist $N \in \mathbb{N}$, and a path of measure-preserving invertible maps $t \to \sigma_t$ piecewise smooth w.r.t. the time variable t such that $\sigma_0 = T$ and σ_1 maps arbitrarily small rational rectangles $P_{ij} \in \mathbb{N} \times \mathbb{N}\frac{1}{N}$ affinely onto rational rectangles $\tilde{P}_{ij} \in \mathbb{N} \times \mathbb{N}\frac{1}{N}$.

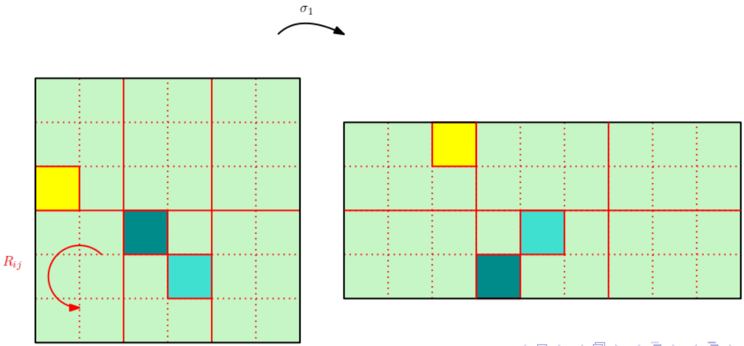
Shnirelman's Lemma (1985)



Rotations: from subsquares to subsquares

Lemma

There exist $M \in \mathbb{N}$ and a flow $\bar{R}_t : K \to K$ invertible, measure-preserving and piecewise smooth such that the map $\sigma_1 \circ \bar{R}_1$ translates each subsquare of the grid $\mathbb{N} \times \mathbb{N}\frac{1}{M}$ into a subsquare of the same grid, i.e. it is a permutation of squares.



Approximation by cyclic permutations

Theorem (Second approximation result, Bianchini S., Z., 2021)

Let $b \in L^\infty_t BV_x$ be a divergence-free vector field. Then for every $\epsilon > 0$ there exist $\tilde{C}_1, \tilde{C}_2 > 0$ positive constants, $D \in \mathbb{N}$ arbitrarily large and a divergence-free vector field $b^c \in L^\infty_t BV_x$ such that

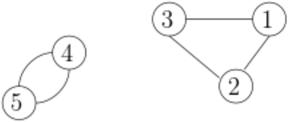
1. *it holds*

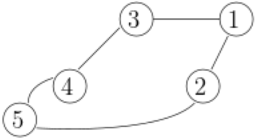
$$\|b - b^c\|_{L^1(L^1)} \leq \epsilon, \quad ||TV(b^c)(K)||_\infty \leq \tilde{C}_1 ||TV(b)(K)||_\infty + \tilde{C}_2, \quad (4)$$

2. *the map $X^c \llcorner_{t=1}$ generated by b^c at time $t = 1$ is a cyclic permutation of squares of size $\frac{1}{D}$.*

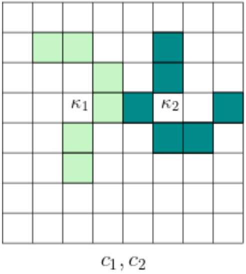
Adding transpositions to connect cycles

$$(45)(123) \Longrightarrow (45)(34)(123)$$





Transpositions as rotations



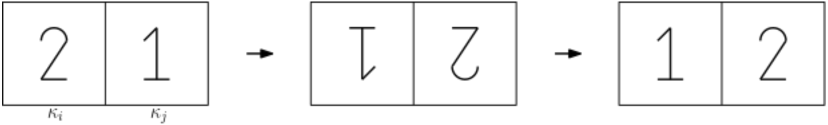


Figure: A transposition between two adjacent squares.

Rotation Flow

Call

$$V(x) = \max\left\{\left|x_1 - \frac{1}{2}\right|, \left|x_2 - \frac{1}{2}\right|\right\}^2, \quad (x_1, x_2) \in K.$$

Then the *rotation field* is $r : K \to \mathbb{R}^2$

$$r(x) = \nabla V^{\perp}(x), \tag{5}$$

where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ is the orthogonal gradient.

The rotation flow R_t is the flow of the vector field r, i.e. the unique solution to the following ODE system:

$$\begin{cases} \dot{R}_t(x) = r(R_t(x)), \\ R_0(x) = x. \end{cases} \tag{6}$$

This flow rotates the cube counterclockwise of an angle $\frac{\pi}{2}$ in a unit interval of time.

Folded Baker's map - Universal Mixer [Elgindi-Zlatoš, '18]

The universal mixer is $b^U \in L^\infty_t \mathrm{BV}_x$ s.t. its RLF $X^U_{t=1} = U$ is the **Folded Baker's map**.

$$U = \begin{cases} \left(-2x+1, -\frac{y}{2}+\frac{1}{2}\right) & x \in \left[0, \frac{1}{2}\right), \\ \left(2x-1, \frac{y}{2}+\frac{1}{2}\right) & x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

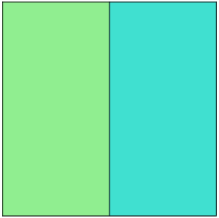


Figure: Starting set

Folded Baker's map

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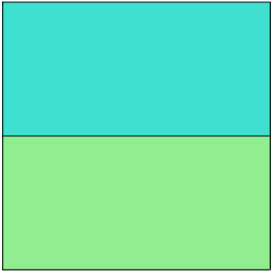

Figure: Action of the Folded Baker's map

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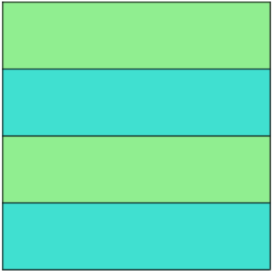

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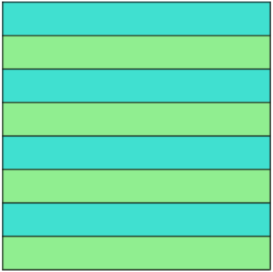

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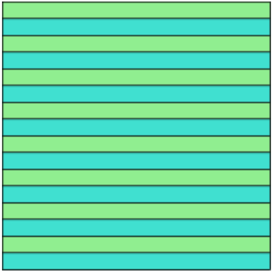

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A simple case study

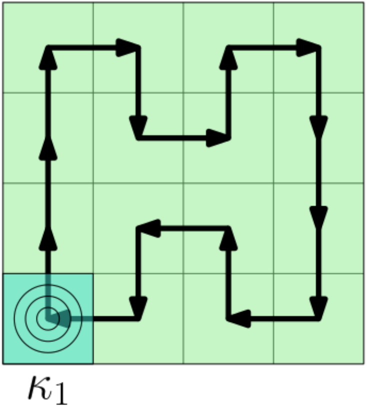

Figure: Construction of an ergodic vector field from a cyclic permutation.

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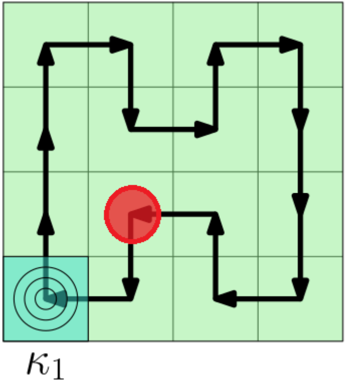

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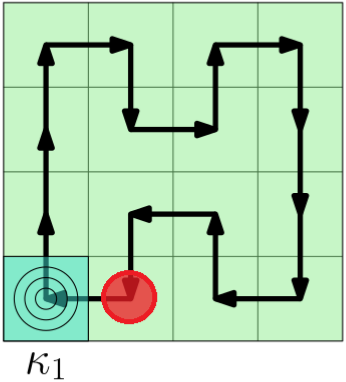

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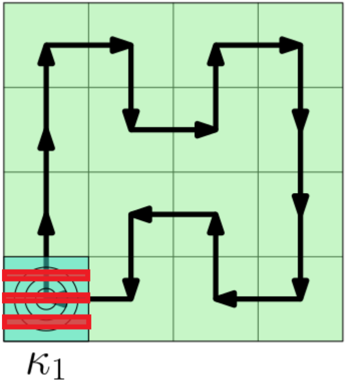

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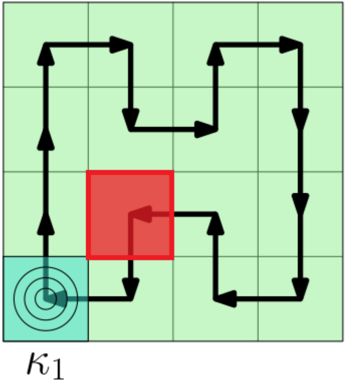

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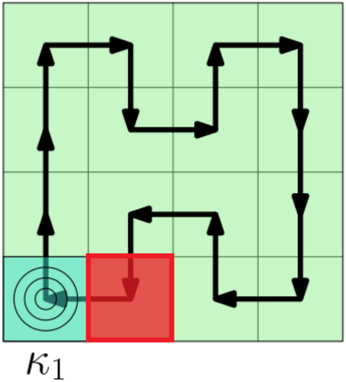

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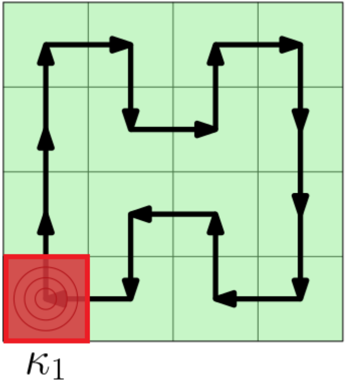

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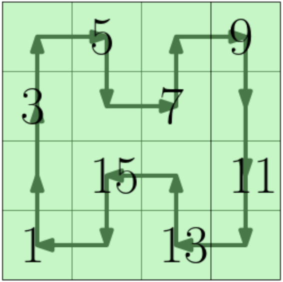
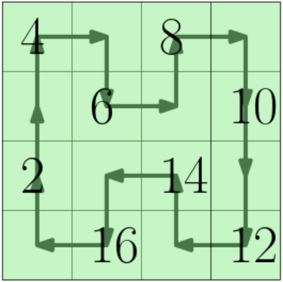
How to generate mixing vector fields





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The proof is obtained by using the theory of Markov processes.

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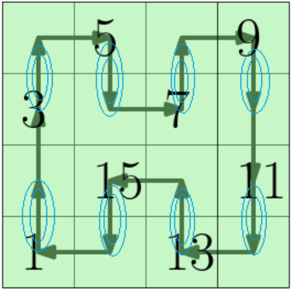

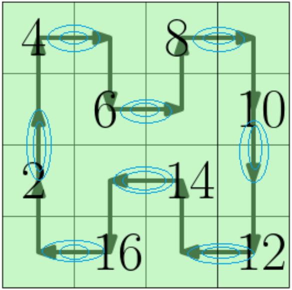

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Idea of the proof

One defines

$$X_t^s(x) = \begin{cases} M_t(x) & t \in [0, 2\delta], \\ X_t \circ M_{2\delta}(x) & t \in [2\delta, 1], \end{cases}$$

where the map M_t, $t \in [0, 2\delta]$, is defined as follows:

$$M_t(x) = \begin{cases} U_t^{\ell,\ell+1}(x) & t \in [0, \delta], \ell \text{ odd}, \\ U_t^{\ell,\ell+1}(x) \circ M_\delta(x) & t \in [\delta, 2\delta], \ell \text{ even}. \end{cases}$$

The proof follows observing that the matrix associated to the process is **aperiodic**.

The weakly mixing behaviour

Weakly mixing vector fields are typical, but it is hard to obtain weakly mixing vector fields that are not strongly mixing.

Theorem (Chacon '69)

There exists $T \in G(K)$ which is weakly mixing but not strongly mixing.

Theorem (Z., '22)

There exists $b \in L^\infty_t \mathrm{BV}_x$ divergence-free which is weakly mixing but not strongly mixing.

Canonical Chacon's map

We look for those automorphisms which are weakly mixing but not strongly mixing.

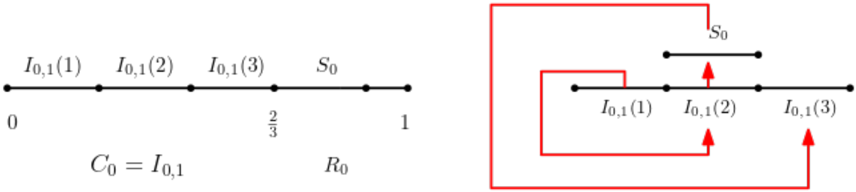
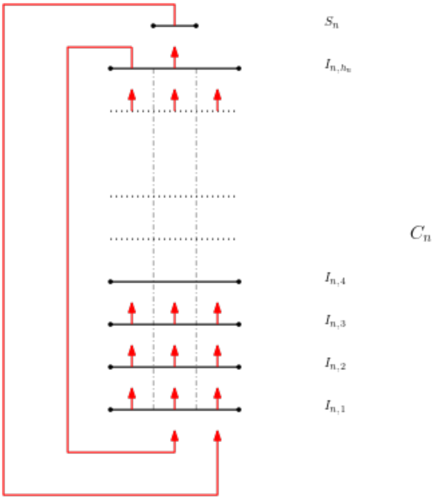


Figure: In the left figure the Column C_0, in the right figure the geometric representation of the action of the automorphism T_1.


S_n
I_{n,h_n}
C_n
$I_{n,4}$
$I_{n,3}$
$I_{n,2}$
$I_{n,1}$

Thanks for your attention