The Strong Onsager Conjecture

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Navier-Stokes and Euler Equations

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p - \nu \Delta u &= f \\ \operatorname{div} u &= 0 \end{aligned}$$

- $u(t,\cdot) : \mathbb{T}^3 \to \mathbb{R}^3$, $p(t,\cdot) : \mathbb{T}^3 \to \mathbb{R}$, $f(t,\cdot) : \mathbb{T}^3 \to \mathbb{R}^3$
- NSE - $\nu > 0$, Euler - $\nu = 0$

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- **Facts:** (1) Anomalous dissipation of energy, (2) $4/5$-law, (3) intermittency

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- **Focus:** Turbulent regime $\nu \to 0$
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- **Onsager program:** Build solutions to the PDEs consistent with experiments and numerics!

Main Theorem

Theorem (Giri-Kwon-N., '23)

For any fixed $\beta < 1/3$, there exist weak solutions to the 3D Euler equations

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}$$

which, in addition, belong to $C_t^0 B_{3,\infty}^{\beta}(\mathbb{T}^3)$ and satisfy the local energy inequality

$$\partial_t \left(\frac{1}{2}|u|^2 \right) + \operatorname{div} \left(u \left(\frac{1}{2}|u|^2 + p \right) \right) \leq 0$$

in the sense of distributions.

Turbulence Basics

- Navier-Stokes equations for an incompressible fluid of constant density

$$\partial_t u + \operatorname{div}(u \otimes u) = \frac{1}{\mathrm{Re}} \Delta u - \nabla p + f$$
$$\operatorname{div} u = 0$$

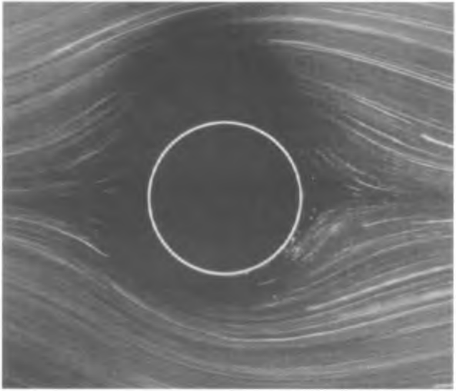
u is velocity, p is pressure, f is an external force

- The *Reynolds number*

$$\mathrm{Re} = \frac{UL}{\nu} = \frac{\text{(characteristic velocity)} \cdot \text{(characteristic length)}}{\text{kinematic viscosity}}$$

- Euler equations correspond to $\mathrm{Re} = \infty$, or $\nu = 0$

What happens as the Reynolds number increases?



Flow behind a cylinder at $\mathrm{Re} = 1.54$

Turbulence Basics

What happens as the Reynolds number increases?

Flow behind a cylinder at $\mathrm{Re} = 140$

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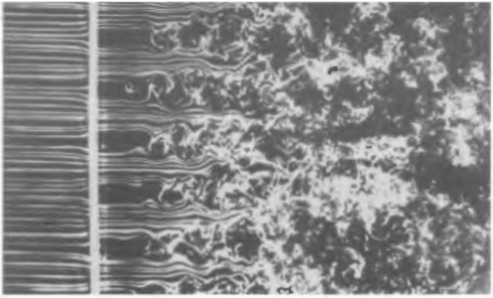
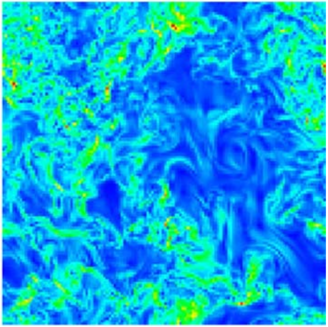


Flow behind a grid at Re $= 1800$

Turbulence Basics

- *Homogeneous isotropic turbulence* arises at large Reynolds numbers (or small ν)
- What about anomalous dissipation, the $4/5$ law, and intermittency?



Contour plot of dissipation in a turbulent velocity field
Source: Kaneda-Ishihara '05

Fact #1: Anomalous Dissipation

$$\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu = \nu \Delta u^\nu - \nabla p^\nu \,, \qquad \qquad \operatorname{div} u^\nu = 0$$

- Pointwise energy balance for smooth solutions

$$\partial_t \left(\frac{1}{2} |u^\nu|^2 \right) + \operatorname{div} \left(\left(\frac{1}{2} |u^\nu|^2 + p^\nu \right) u^\nu - \nu \nabla \frac{|u^\nu|^2}{2} \right) = -\nu |\nabla u^\nu|^2$$

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- Integrating in $\mathbb{T}^3$ and from 0 to T, we have

$$\frac{1}{2} \|u^\nu(T, \cdot)\|^2_{L^2(\mathbb{T}^3)} - \frac{1}{2} \|u^\nu(0, \cdot)\|^2_{L^2(\mathbb{T}^3)} = - \int_0^T \nu \|\nabla u^\nu(t, \cdot)\|^2_{L^2(\mathbb{T}^3)} \, dt$$

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- Thus smooth Euler solutions conserve energy, and dissipation in smooth Navier-Stokes solutions is caused by $\nu \Delta u^\nu$

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$$D[u^\nu](t,x) = \lim_{\ell \to 0} \frac{1}{4} \int_{\mathbb{T}^3} \nabla \phi_\ell(z) \cdot (u(t,x+z) - u(t,x)) \, |u(t,x+z) - u(t,x)|^2 \, dz$$

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- Zeroth law of turbulence (no proof exists!)

$$\varepsilon = \liminf_{\nu \to 0} \underbrace{\left\langle \nu |\nabla u^\nu|^2 + D[u^\nu] \right\rangle}_{\varepsilon^\nu} > 0$$

Fact #1: Anomalous Dissipation

- Caffarelli-Kohn-Nirenberg's "suitable solutions" to Navier-Stokes satisfy

$$u^\nu \in L_t^\infty L_x^2 \cap L_t^2 W_x^{1,2}\,, \qquad D[u^\nu] \geq 0$$

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- Thus if $D[u] \neq 0$, u^ν *cannot* remain bounded in $L_t^3 B^\alpha_{3,\infty,x}$ for $\alpha > 1/3$ as $\nu \to 0$, where

$$\|f\|_{B^\alpha_{3,\infty}(\mathbb{T}^3)} = \sup_{|z|>0} \frac{1}{|z|^\alpha} \|f(\cdot + z) - f(\cdot)\|_{L^3(\mathbb{T}^3)}$$

Fact #2: Kolmogorov's 4/5 law

- **K41 Assumptions:** the zeroth law ($\varepsilon > 0$), translation, rotation, and scaling symmetries for law of $u^\nu(t, x + \ell\hat{z}) - u^\nu(t, x)$ (here $\ell > 0, \hat{z} \in \mathbb{S}^2$)

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- Local, deterministic 4/5 law (Eyink, '02)

$$\lim_{\ell \to 0} \frac{1}{\ell} S_3^{\|}(\ell) = \lim_{\ell \to 0} \frac{1}{\ell} \fint_{\mathbb{T}^3} \fint_{\mathbb{S}^2} [(u^\nu(t, x + \ell z) - u^\nu(t, x)) \cdot z]^3 \, dz \, dx$$
$$= -\frac{4}{5} D[u^\nu]$$

Onsager's Conjecture and the $L_t^\infty C_x^{1/3}$ Threshold

- "It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable!" – Onsager, '49

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- If $\alpha < 1/3$, the kinetic energy of 3D Euler solutions need not be conserved (Isett '18) and can dissipate (Buckmaster-De Lellis-Székelyhidi-Vicol '19)

Adding to the story: local energy inequality and intermittency

- The solutions of Isett and Buckmaster et. al. do not satisfy $D[u] \geq 0$, and so cannot arise as limits of suitable Navier-Stokes solutions

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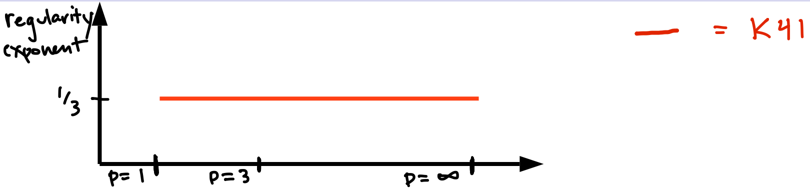
- Conservation of energy requires only $L^3_t B^\alpha_{3,\infty}$ for $\alpha > 1/3$, but dissipative solutions belong to $C^\alpha_{t,x}$ for $\alpha < 1/3$ ... is this merely a curiosity concerning function spaces?

Intermittency: Deviations from K41/Onsager



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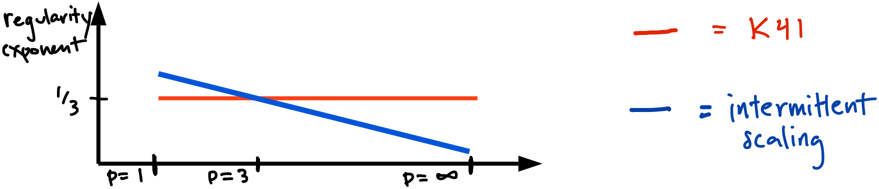
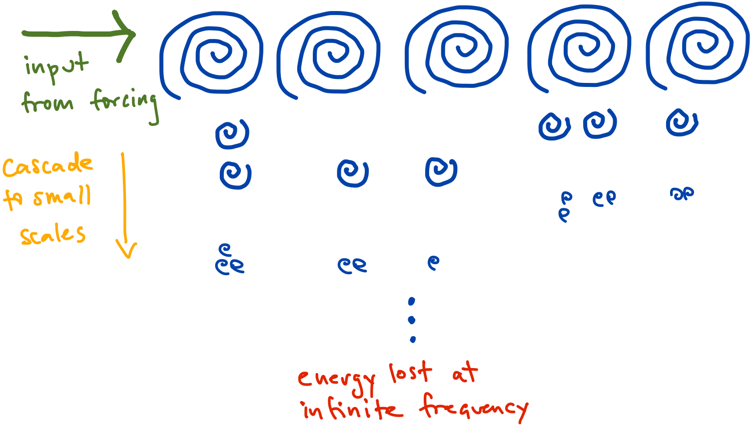


- **Onsager, unpublished work** - "[Anomalous scaling for ζ_2] would require a "spotty" distribution of the regions in which the velocity varies rapidly"
- **Kolmogorov '62** - "I have formulated appropriate modifications to the two similarity hypotheses that I put forward in 1941 ..."
- **Chen, Dhruva, Kurien, Sreenivasan, Taylor '05** - "It is now believed that the scaling exponents of moments of velocity increments are anomalous ... anomalous scaling is a genuine result worth of a serious theoretical effort."
- **Iyer, Sreenivasan, Yeung '20** - "The $4/5$-ths law holds in an intermediate range of scales and the second-order exponent over the same range of scales is *anomalous*, departing from the self-similar value of $2/3$."
- **See also** - Ishihara-Kaneda-Gotoh, Frisch, Anselmet-Gagne-Hopfinger-Antonia, ...

Takeaway: $B^{1/3}_{3,\infty} \cap L^\infty$ may be the correct space

Intermittency: Deviations from K41/Onsager

- **Symmetry assumptions:** Turbulence is *isotropic*, *homogeneous*, *but not purely self-similar ... fewer eddies of higher intensity!*



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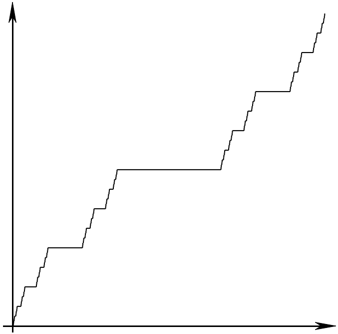
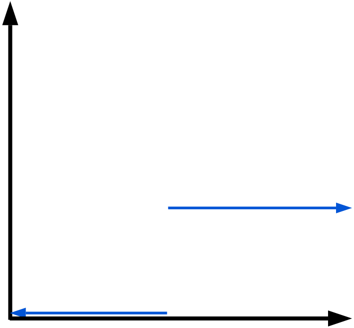
- **Symmetry assumptions:** Turbulence is *isotropic*, *homogeneous*, *but not purely self-similar ... fewer eddies of higher intensity!*

- **Dissipativity assumption:** Dissipation occurs even in the absence of viscosity

- **Implications for regularity:** Cantor function, Heaviside function ($B^{1/p}_{p,\infty}$)





Strong Onsager Conjectures

Consider weak solutions u to the Euler equations, with the local energy balance

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div}\, u = 0 \\ \partial_t \left(\frac{1}{2}|u|^2\right) + \operatorname{div}\, \left(\left(\frac{1}{2}|u|^2 + p\right) u\right) + D[u] = 0\,. \end{cases}$$

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Conjecture (Strong L^3 Onsager Conjecture)

For any $\alpha < 1/3$, there exist weak solutions $u \in L^\infty_t B^{\alpha}_{3,\infty}$ of 3D Euler for which $D[u] \geq 0$ and $\frac{d}{dt}\|u\|^2_{L^2} < 0$.

The Strong L^3 Onsager Theorem (and its antecedents)

Theorem (Buckmaster-Masmoudi-N.-Vicol, '21)

There exist solutions to 3D Euler which have decreasing kinetic energy and belong to $L^\infty_t H^{1/2-}_x$ (intermittent, but no $4/5$ law or local energy inequality)

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- Giri-Kwon-N., '23a - Reprove the results in NV '22 using a "wavelet-inspired" scheme, along with further improvements

Theorem (Giri-Kwon-N., '23b)

There exist solutions to 3D Euler which have decreasing kinetic energy, belong to $L_t^\infty \left(L_x^{\infty-} \cap B_{3,\infty}^{1/3-} \right)$*, and satisfy* $D[u] \geq 0$ *(intermittent,* $4/5$ *law, local energy inequality)*

A bird's-eye view of our construction

Basic Strategy: Construct a weak solution $u = \lim_{q\to\infty} u_q$ in $L^\infty_t\left(L^{\infty-} \cap B^{1/3-}_{3,\infty}\right)$ via Nash iteration

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Two questions: How to set up the induction, and how to propagate the inductive assumptions?

Wavelet-inspired, $B^{1/3-}_{3,\infty} \cap L^{\infty-}$ inductive set-up

- Assume the existence of $(u_q, p_q, R_q, \Phi_q, \pi_q)$ satisfying

$$\partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q = \underbrace{\operatorname{div}_x (R_q - \pi_q \mathrm{Id})}_{\rightharpoonup 0 \text{ as } q \to \infty}, \qquad \operatorname{div} u_q = 0$$

$$\partial_t \left(\frac{|u_q|^2}{2} \right) + \operatorname{div} \left(\left(\frac{|u_q|^2}{2} + p_q \right) u_q \right) \leq \underbrace{\operatorname{div}_{t,x} \Phi_q(\pi_q)}_{\rightharpoonup 0 \text{ as } q \to \infty}$$

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- **Heuristics:**
  - Let $\lambda_q = a^{(b^q)}$ quantify the inverse of the diameter of an oscillation, and $\lambda_q r_q$ for $r_q \ll 1$ quantify the inverse of the distance between oscillations

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  - $u_q = \sum_{q' \leq q} w_{q'}$, where $w_{q'}$ has oscillations of size λ_q^{-1}, distance $(\lambda_q r_q)^{-1}$ between oscillations, and frequency support $[\lambda_q r_q, \lambda_q]$ (not necessarily disjoint from $[\lambda_{q'} r_{q'}, \lambda_{q'}]$!)

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- **Heuristics:**
  - Let $\lambda_q \approx a^{(b^q)}$ quantify the inverse of the width of an oscillation, and $\lambda_q r_q$ for $r_q \ll 1$ describe the inverse of the distance between oscillations
  - $u_q = \sum_{q' \leq q} w_{q'}$, where $w_{q'}$ has oscillations of size λ_q^{-1}, distance $(\lambda_q r_q)^{-1}$ between oscillations, and frequency support $[\lambda_q r_q, \lambda_q]$
  - "Wavelet parameter" $\bar{n}$ quantifies the exchange between space-time and frequency support and decomposes R_q, Φ_q, and π_q by frequency:

$$\mathrm{supp}_{t,x} w_{q'} \cap w_{q''} = \emptyset \quad \text{if} \quad |q' - q''| < \bar{n}$$

$$\mathrm{supp}_{\xi} \hat{w}_{q'} \cap \hat{w}_{q''} = \emptyset \quad \text{if} \quad |q' - q''| \geq \bar{n}$$

Wavelet-inspired, $B^{1/3-}_{3,\infty} \cap L^{\infty-}$ inductive set-up

- Assume the existence of $(u_q, p_q, R_q, \mathit{\Phi}_q, \pi_q)$ satisfying

$$\partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q = \underbrace{\sum_{k=q-\bar{n}+1}^{q} \operatorname{div}_x \left(R_q^k - \pi_q^k \mathrm{Id} \right)}_{\rightharpoonup 0 \text{ as } q \to \infty}, \qquad \operatorname{div} u_q = 0$$

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- **Heuristics:**
  - Use parameters λ_q, r_q, and $\bar{n}$ to describe u_q as a "partial wavelet sum," decompose R_q, $\mathit{\Phi}_q$, π_q by frequency
- **Inductive bounds:**
  - Assume bounds for $\|\pi\|_{3/2}$, $\|\pi\|_\infty$ (which depend on their frequency)

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- **Heuristics:**
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- **Inductive bounds:**
  - Assume bounds for $\|\pi\|_{3/2}$, $\|\pi\|_\infty$ (which depend on their frequency)
  - All other bounds are "pointwise in terms of π," e.g.

$$\left|D^N(\partial_t + u\cdot\nabla)^M \pi\right| + \left|D^N(\partial_t + u\cdot\nabla)^M R\right| \lesssim \pi \lambda^N \left(\pi^{1/2}\lambda\right)^M$$

Wavelet-inspired, $B^{1/3-}_{3,\infty} \cap L^{\infty-}$ inductive set-up

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- **Heuristics:**
  - Use parameters λ_q, r_q, and $\bar{n}$ to decompose u_q into a "partial wavelet sum," decompose R_q, Φ_q, π_q by frequency
- **Inductive bounds:**
  - Assume bounds for $\|\pi\|_{3/2}$, $\|\pi\|_\infty$ (which depend on their frequency)
  - All other bounds are *bointwise bounds* in terms of π
- **Scaling law:** Mollifying high-freq. π's gives rescaled low-freq. π's

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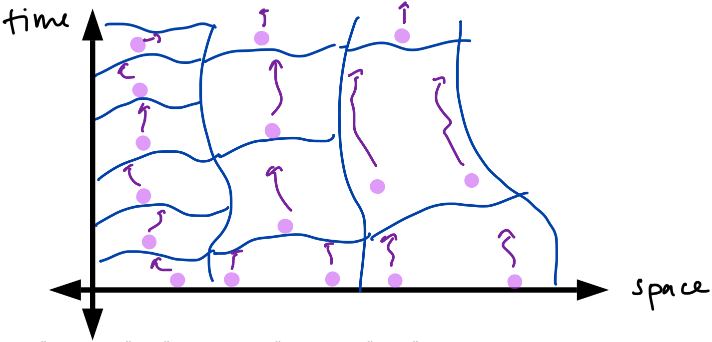
- **Scaling law:** Mollifying high-freq. π's gives rescaled low-freq. π's
  - $\phi_{\lambda^{-1}} * \pi_\Lambda = \left(\frac{\lambda}{\Lambda} \right)^{2/3} \pi_\lambda , \qquad \lambda < \Lambda$

How to propagate spatial support properties (multi-scale pipe flows)

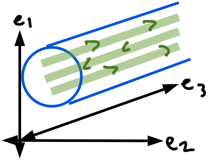
- $u_{q'} = \sum_{q \le q'} w_q = \sum_{q \le q'} \sum_k \underbrace{a_k^q(t,x)}_{\text{low-frequency amplitudes, partition spacetime}} \underbrace{\mathbb{B}_k^q(\Phi_k^q)}_{\text{Intermittent Mikado bundle } (\partial_t + u_{q-1}\cdot\nabla)\Phi_k^q = 0}$

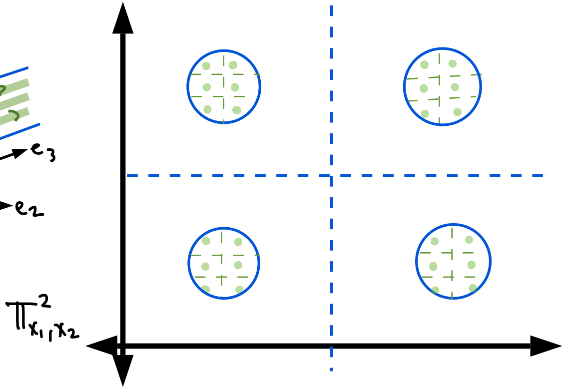


- $\operatorname{supp} a_k^q(t,x)\, \mathbb{B}_k^q(\Phi_k^q) \cap \operatorname{supp} a_{k'}^q(t,x)\, \mathbb{B}_{k'}^q(\Phi_{k'}^q) = \emptyset$
- Either $\operatorname{supp} w_q \cap \operatorname{supp} w_{\tilde{q}} \equiv 0$ or $\operatorname{supp} \hat{w}_q \cap \operatorname{supp} \hat{w}_{\tilde{q}}$

How to propagate spatial support properties (multi-scale pipe flows)

- $u_{q'} = \sum_{q \le q'} w_q = \sum_{q \le q'} \sum_k \underbrace{a_k^q(t,x)}_{\substack{\text{low-frequency partition}\\ \text{of unity in time and space}}} \underbrace{\mathbb{B}_k^q(\Phi_k^q)}_{\substack{\text{Intermittent Mikado bundle}\\ (\partial_t + u_{q-1}\cdot\nabla)\Phi_k^q = 0}}$
- Intermittent Mikado bundle $\mathbb{B}_k^q = \mathbb{B}_{k,\mathrm{high}}^q \mathbb{B}_{k,\mathrm{low}}^q$ is a multi-scale, intermittent shear flow





How to propagate spatial support properties (multi-scale pipe flows)

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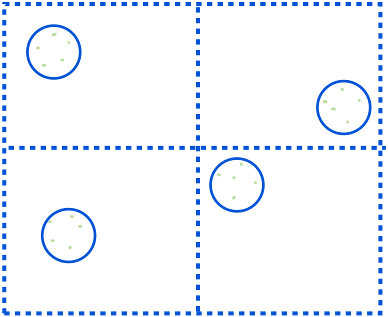
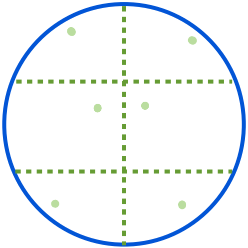
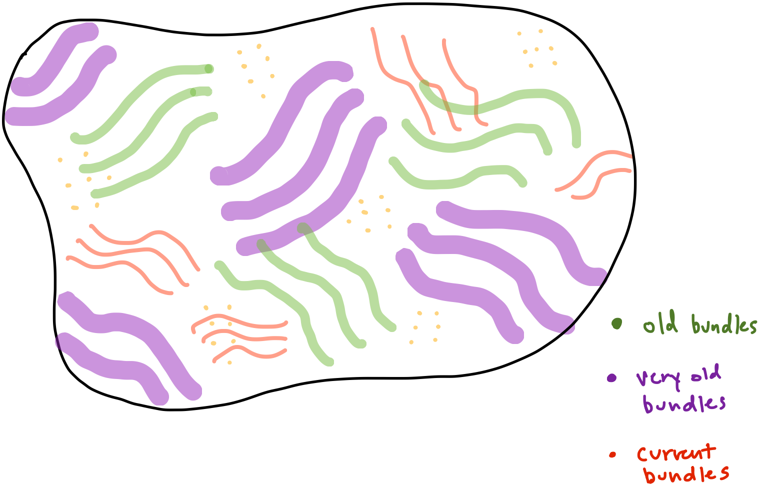
- Intermittent Mikado bundle $\mathbb{B}_k^q = \mathbb{B}_{k,\mathrm{high}}^q \mathbb{B}_{k,\mathrm{low}}^q$ is a multi-scale, intermittent shear flow with flexibility in the support





How to propagate spatial support properties (multi-scale pipe flows)

- Need to choose the support of $\mathbb{B}^q_k = \mathbb{B}^q_{k,\mathrm{high}} \mathbb{B}^q_{k,\mathrm{low}}$ to dodge any other bundles (for other k' or q') which have overlapping frequency support



Oscillation error and the choice of r_{q+1}

- Simple example, ignoring local energy inequality:

$$R_q^{q-\bar{n}+1} - \pi_q^{q-\bar{n}+1}\mathrm{Id} \to \begin{pmatrix} R-\pi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w_{q+1,R} = \vec{e}_1(-R+\pi)^{1/2}\underbrace{\mathbb{B}_{q+1,R}(x_2,x_3)}_{\substack{\text{freq's} \\ [\lambda_{q+1}r_{q+1},\lambda_{q+1}]}}$$

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- Set $\fint_{\mathbb{T}^3} \mathbb{B}^2_{q+1,R} = 1$, use stationarity of $\vec{e}_1\mathbb{B}_{q+1,R}$, and rewrite

$$\begin{aligned} \operatorname{div}\left((R-\pi)(e_1\otimes e_1) + w_{q+1,R}\otimes w_{q+1,R}\right) &= \vec{e}_1\partial_x\left((R-\pi)(1-\mathbb{B}^2_{q+1,R})\right) \\ &= \vec{e}_1\partial_x\,(R-\pi)\left(\fint_{\mathbb{T}^3} - \mathrm{Id}\right)(\mathbb{B}^2_{q+1,R}) \end{aligned}$$

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- Put this vector field in divergence form and estimate

Oscillation error and the choice of r_{q+1}

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- Estimate error in $L^{3/2}$ (since R,π scale like velocity squared)

$$\left\| \mathrm{div}^{-1}\left(\vec{e}_1\partial_x(R-\pi)\left(\mathrm{Id} - \fint_{\mathbb{T}^3}\right)(\mathbb{B}^2_{q+1,R})\right)\right\|_{3/2}$$

Oscillation error and the choice of r_{q+1}

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$$\left\| \mathrm{div}^{-1}\left(\vec{e}_1\partial_x(R-\pi)\left(\mathrm{Id} - \fint_{\mathbb{T}^3}\right)(\mathbb{B}_{q+1,R}^2)\right)\right\|_{3/2}$$

- Requires knowledge of $\|\mathbb{B}_{q+1,R}\|_3$, which *grows* as $r_{q+1}\to 0$ from Bernstein's inequality and $\fint \mathbb{B}_{q+1,R}^2 = 1$

Oscillation error and the choice of r_{q+1}

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- Requires knowledge of $\|\mathbb{B}_{q+1,R}\|_3$, which *grows* as $r_{q+1} \to 0$ from Bernstein's inequality and $\fint \mathbb{B}^2_{q+1,R} = 1$

- **Conclusion:** Nonlinear error prefers $r_{q+1} = 1$, i.e. *less* intermittency

Scaling law: $\pi_\Lambda * \phi_{\lambda^{-1}} = \left(\frac{\lambda}{\Lambda}\right)^{2/3} \pi_\lambda,\ \lambda < \Lambda$

- Simple example (requires $R \leq \pi$!!):

$$R_q^{q-\bar{n}+1} - \pi_q^{q-\bar{n}+1}\mathrm{Id} \to \begin{pmatrix} R-\pi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w_{q+1,R} = \vec{e}_1(-R+\pi)^{1/2} \underbrace{\mathbb{B}_{q+1,R}(x_2, x_3)}_{\substack{\text{freq's} \\ [\lambda_{q+1} r_{q+1}, \lambda_{q+1}]}}$$

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- We need to build π_Λ such that

$$\pi_\Lambda \geq \left| \mathrm{div}^{-1} \left(\vec{e}_1 \partial_x \left(R - \pi\right) \left(\mathrm{Id} - \fint_{\mathbb{T}^3} \right) \left(\mathbb{B}_{q+1,R}^2\right) \right) \right|$$

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- Assuming div^{-1} gains a power of high frequency $\Lambda \approx \lambda_{q+1} r_{q+1}$, and using the inductive assumption $|\nabla_x R| + |\nabla_x \pi| \lesssim \lambda \pi$, we set

$$\begin{aligned} \pi_\Lambda &= \frac{\pi\lambda}{\Lambda} \left[\fint + \left(\mathrm{Id} - \fint \right) \right] \left| \left(\mathrm{Id} - \fint \right) (\mathbb{B}_{q+1,R}^2) \right| \\ &= \frac{\pi\lambda}{\Lambda} + \text{high frequency} \end{aligned}$$

Linear errors and the choice of r_{q+1}

- Recall that

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- Construct wild solutions to 3D Euler with well-defined helicity $\int_{\mathbb{T}^3} u \cdot (\nabla \times u)$

Thanks for your attention!