Construction of quasi-periodic solutions to active scalar equations

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Deterministic and random features of fluids, EPFL

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- Quasi-periodic solutions
- Hamiltonian system and KAM theory
- Application to active scalar equations (Generalized surface quasi-geostrophic (gSQG) equations)

Quasi-periodic functions

- Periodic function: $f : \mathbb{R} \mapsto X$ s.t.

$$f(t+T) = f(t), \text{ for all } t \in \mathbb{R}$$

- Frequency $\omega := \frac{2\pi}{T}$
- In other words,

$$\exists\ u : \mathbb{T} \mapsto X, \text{ s.t. } f(t) = u(\omega t).$$

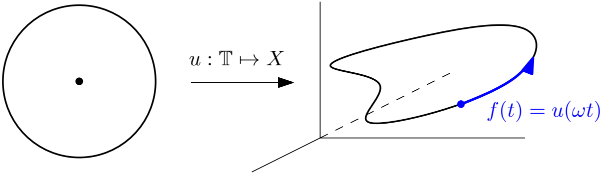


Figure: Orbit of f = image of u="embedded torus"

- Quasi-periodic motion e.g. $f(t) = \begin{pmatrix} e^{i\omega_1 t} \\ e^{i\omega_2 t} \end{pmatrix}$. What if $\omega_1 = 1.\ \omega_2 = \sqrt{2}$?
- Given $\nu \in \mathbb{N}$, $\vec{\omega} \in \mathbb{R}^\nu$, $f : \mathbb{R} \mapsto X$ is quasi-periodic with frequency $\vec{\omega}$, if

$$\exists\ u : \mathbb{T}^\nu \mapsto X \text{ s.t. } f(t) = u(\vec{\omega} t).$$

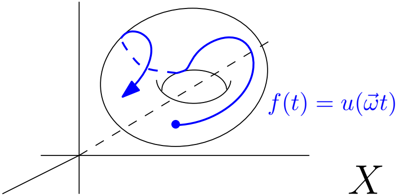


Figure: Orbit of $f \subset$ image of u="embedded (higher dim) torus"

- Common in **Hamiltonian systems**

Hamiltonian system

- (X,H,J): **Phase space** X, **Hamiltonian** $H : X \mapsto \mathbb{R} \cup \{\infty\}$ and **anti-symmetric operator** $J : TX \mapsto TX$.

$$f_t(t) = J\nabla H(f(t)), \ f(0) = f_0.$$

- $f \mapsto J\nabla H(f)$: Hamiltonian vector field
- e.g. undamped pendulum, harmonic osillator

$f_t(t) = J\nabla H(f(t))$

- e.g. **Airy equation** on $\mathbb{T}$: $f_t = -\partial_{\theta\theta\theta} f$.

$$X := \left\{ f \in L^2(\mathbb{T}) : \int_{\mathbb{T}} f(\theta) d\theta = 0 \right\}, \ H(f) := \frac{1}{2} \int_{\mathbb{T}} |\partial_\theta f(\theta)|^2 d\theta, \ J = \partial_\theta,$$
$$\implies J\nabla H(f) = -f_{\theta\theta\theta}.$$

  - $\frac{d}{dt}\hat{f}(n) = -(\mathrm{i}n)^3 \hat{f}(n) \implies f(t,\theta) = \sum_n \hat{f}_0(n) e^{\mathrm{i}n^3 t} e^{\mathrm{i}n\theta} \implies$ Quasi-periodic.

- e.g. **KdV equation**: $f_t = -f_{\theta\theta\theta} + 6f\partial_\theta f$

$$H(f) := \frac{1}{2} \int_{\mathbb{T}} |\partial_\theta f|^2 d\theta + \int_{\mathbb{T}} f^3 d\theta \implies J\nabla H = -f_{\theta\theta\theta} + 6f\partial_\theta f.$$

- If initial data f_0 is small,

Can KdV possess quasi-periodic solutions?

Main question in the KAM theory

Given X (phase space), H (Hamiltonian) and J (anti-sym. operator), suppose

$$f_t = J\nabla H(f)$$

possesses a quasi-periodic solution.
Q: Can a perturbed system

$$f_t = J\nabla H_p(f), \quad H_P = H + P, \text{ for some small } P : X \mapsto \mathbb{R} \cup \{\infty\}$$

possess a quasi-periodic solution?

- Nonlinear system: Suppose $\nabla H(0) = 0$.

$$f_t = J\nabla H(f) = J(\nabla^2 H(0)[f] + nonlinear(\sim O(f^2)))$$
$$\implies f_t = J\nabla^2 H(0)[f] + perturbation$$

- Imaginary eigenvalues of $J\nabla^2 H(0) \implies$ oscillations (quasi-periodic)
- Difficulty in PDE: semilinear/quasilinear nature of perturbation.

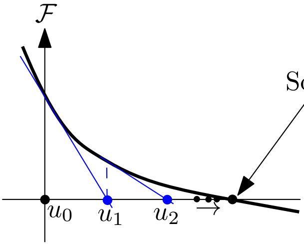
- Semillinear perturbation: Bourgain, Craig, Kappeler, Kuksin, Pöschel, Wayne and many others.
- Quasiliear perturbation: P. Baldi, M. Berti, R. Montalto, R. Feola, F. Giuliani...
- Recent applications:
  - Time-Q.P. solutions for gSQG and 2D Euler: N. Masmoudi, T. Hmidi, M. Berti, Z. Hassainia, E. Roulley
  - Benjamin-Feir instability (water waves): P. Ventura, M. Berti, A. Maspero
  - Space-Q.P. stationary solutions near Couette flow (2D Euler): L. Franzoi, N. Masmoudi, R. Montalto

General idea

- Toy model: $\underbrace{f_t = \partial_{\theta\theta\theta} f}_{\text{Airy}} + \epsilon P(f)$.
  - Ansatz: $f(t) = u(\vec{\omega} t)$ (for some $\vec{\omega} \in \mathbb{R}^\nu$ and $u : \underset{\varphi \mapsto u(\varphi) = u(\varphi,\theta)}{\mathbb{T}^\nu \mapsto L^2(\mathbb{T})}$). ν is free.

$$\mathcal{F}(u) := \vec{\omega} \cdot \partial_\varphi u - \partial_{\theta\theta\theta} u + \epsilon P(u) = 0.$$

  - Newton's method



$$u_{n+1} = u_n - D\mathcal{F}(u_n)^{-1}[\mathcal{F}(u_n)]$$

Requirement:
Invertibility of $h \mapsto D\mathcal{F}(u)[h]$.

  - $D\mathcal{F}(u)[h] = (\vec{\omega} \cdot \partial_\varphi - \partial_{\theta\theta\theta})[h] + \underbrace{\epsilon DP(u)[h]}_{(\textit{perturbative})}$.
  - In a general model,

$$D\mathcal{F}(u) = \vec{\omega} \cdot \partial_\varphi - J\nabla^2 H(0) + pert.$$

Generalized surface quasi-geostrophic equations

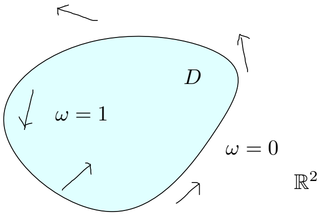
- gSQG equations: For $\alpha \in [0, 2)$,

$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ u = \nabla^\perp \Delta^{-(1-\frac{\alpha}{2})} \omega, \quad \nabla^\perp := (\partial_{x_1}, \partial_{x_2})^\perp := (-\partial_{x_2}, \partial_{x_1}) \\ \omega(0, \cdot) = \omega_0(\cdot). \end{cases}$$

- $\alpha = 0$: 2D Euler equation, $\alpha = 1$: SQG equation (Toy model for the 3D Euler (Constantin, Majda, Tabak ('94))).
- Global well-posedness?
  - For $\alpha > 0$, existence of global smooth solutions unknown.
  - Local (in time) existence: Chae, Constantin, Wu, Córdoba, Gancedo...
  - In a half-plane: local well-posedness (Jeong, Kim, Yao ('23), A. Zlatoš ('23)), Finite time singularity (A. Zlatoš ('23))

Vortex patch problem

- $\omega_0 = 1_D$ for some $D \subset \mathbb{R}^2$.



$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0, \\ u = \nabla^{\perp} \Delta^{-1+\frac{\alpha}{2}} \omega, \\ \omega(0, \cdot) = 1_D \end{cases}$$

Figure: Vortex patch

- Transport nature: The shape of D evolves in time $\implies \omega(t) = 1_{D_t}$.
- Initially smooth boundary remain smooth? Not known but finite time singularity in a half-plane (Kiselev, Ryzhik, Yao, Zlatos ('16), Gancedo, Patel ('21)).

Hamiltonian Structure for star-shaped patch problem

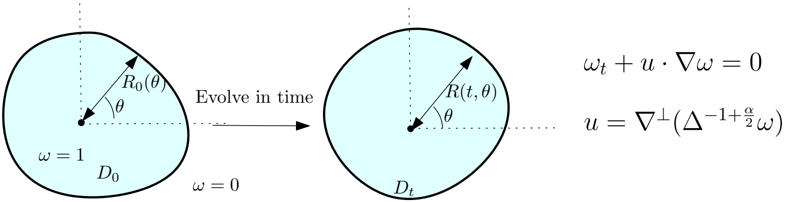
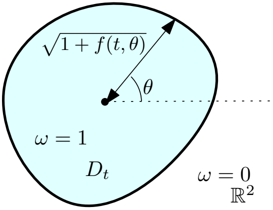


Figure: Motion of star shaped patches

- gSQG equation tells the dynamics of $R(t, \theta)$
- $R(t, \theta)$ not a good variable.

$$R(t, \theta) =: \sqrt{1 + f(t, \theta)}.$$

- Conserved quantity: $|D| = \int_{\mathbb{T}} |R|^2 d\theta \implies \int_{\mathbb{T}} f(t, \theta) d\theta$ is conserved
- WLOG $\int_{\mathbb{T}} f(t, \theta) d\theta = 0$, (as in KdV, Airy)



$$\omega_t + u\cdot\nabla\omega = 0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2$$
$$u = (-\Delta)^{-1+\alpha/2}\omega$$

Figure: Motion of star shaped patches

- gSQG equation tells us

$$\partial_t f = \text{complicated..}$$

- Hamiltonian structure (Marsden, Weinstein ('83) for Euler):

$$X := \left\{ f\in L^2(\mathbb{T}) : \int_{\mathbb{T}} f(\theta)d\theta = 0 \right\},\ H(f) := \frac{1}{2}\int_D \Delta^{-1+\frac{\alpha}{2}}(1_D)dx, \quad J = \partial_\theta.$$

- *gSQG* for a star-shaped patch:

$$\partial_t f = J\nabla H(f).$$

- $f = 0$ (radial patch) is stationary: Quasi-periodic solutions for small f?

Recent works on KAM for gSQG

- Existence of quasi periodic (patch) solutions for "almost all" $\alpha \in (0, \frac{1}{2})$
  Hassainia, Hmidi, Masmoudi ('21)
- Existence of quasi periodic (patch) solutions for Euler $(\alpha = 0)$ near ellipses
  Berti, Hassainia, Masmoudi ('22)

Theorem (Gómez-Serrano, Ionescu, P ('23))

Fix $\alpha \in (1, 2)$. There exists quasi-periodic patch type solutions to the gSQG equations.

- Non-trivial global solutions: ∂D_t is regular in H^s for $s \gg 1$.
- "Generic" initial data (patch) near radial patch give rise to Q.P. solutions (KAM)
- Every $\alpha \in (1, 2)$: Extraction of parameter from the nonlinearity of the Hamiltonian structure instead of α.

Thank You for Your Attention!