Stability and Instability of Vortex Patches

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2D Incompressible Euler.

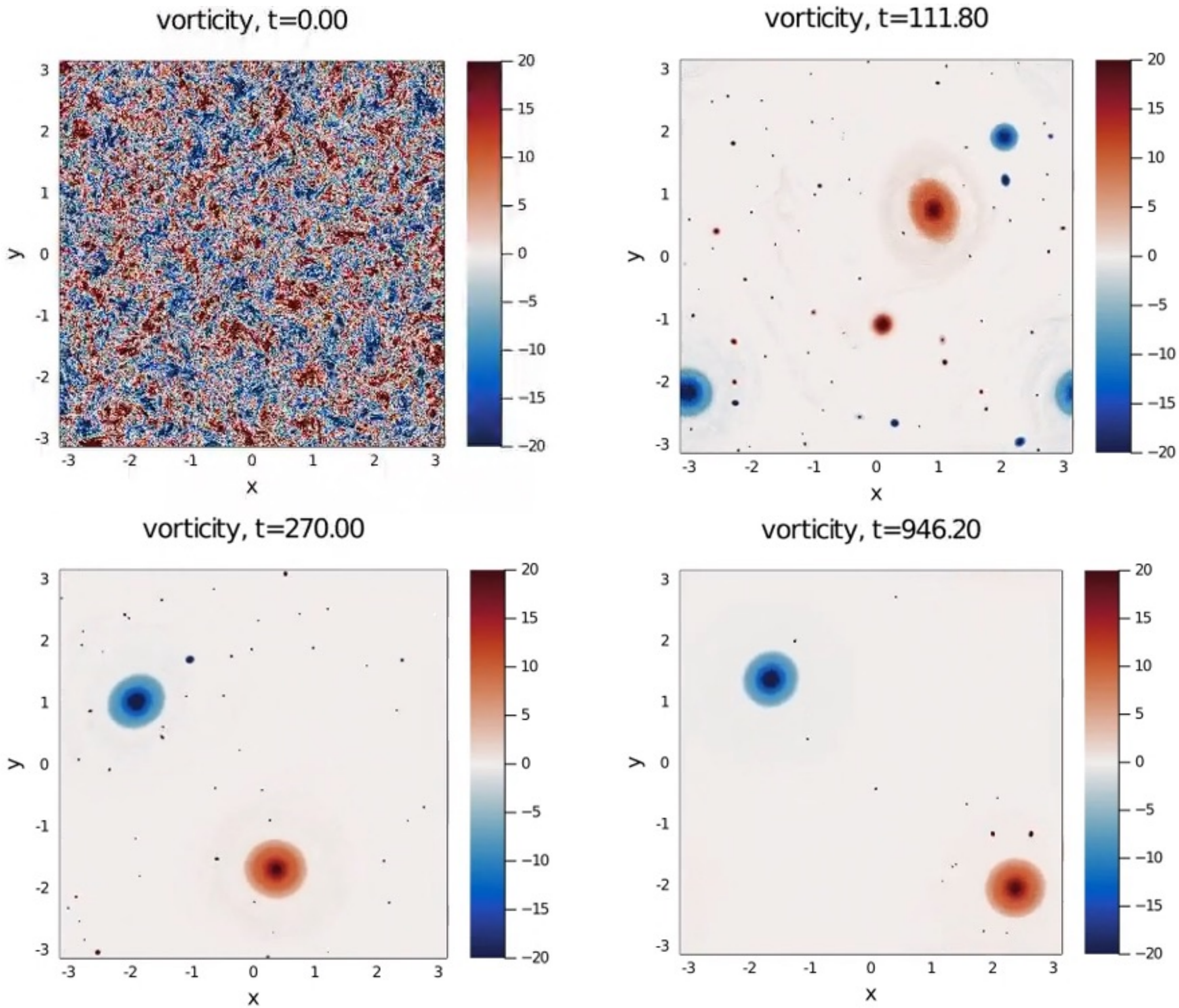
$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ u = \nabla^{\perp} \Delta^{-1} \omega = \nabla^{\perp} \psi \end{cases}$$

- Fluid domain: $\mathbb{R}^2$ or $\mathbb{T}^2$
- Velocity $u(t, x)$, $x = (x_1, x_2)$
- Vorticity $\omega(t, x)$
- Stream $\psi(t, x)$
- Relations $u = (u_1, u_2) = (-\partial_2 \psi, \partial_1 \psi)$

$\Delta \psi = \omega$, $\nabla \times u = \omega$, $\nabla \cdot u = 0$

Long-time dynamics

- Initial value problem: GWP in $L^1 \cap L^\infty$
  $\omega_0 \longmapsto \omega(t)$ for large t?
- Relation with Navier–Stokes? Turbulence?
- Vortex condensate.
- Difficulties:
  - Understanding $\omega \mapsto u$.
  - Too many steady/quasi-steady states
  - Conservation laws
  - Numerical simulations


vorticity, t=0.00
vorticity, t=111.80
vorticity, t=270.00
vorticity, t=946.20
x
y
-3 -2 -1 0 1 2 3
20 15 10 5 0 −5 −10 −15 −20

T. Drivas

Goal.

- Patches: partial understanding of long-time dynamics
- Stability and Instability of some patches for 2D Euler.
- Present a few explicit computations

Vortex Patches.

- Initial data $\omega_0 = 1_{A_0}$ $\longrightarrow$ $\exists!$ Solution $\omega(t) = 1_{A(t)}$

- Reduction to boundary behavior

  $\rightarrow$ Simpler analysis / simulations

- Energy maximizers under natural constraints.

  $\longrightarrow$ Stability / Instability

Energy (pseudoenergy)

- $E[\omega] = -\frac{1}{2}\int_{\mathbb{R}^2} \omega\psi \, dx.$ conserved

$\omega = \Delta\psi = \nabla\cdot\nabla\psi$, so that

$-\frac{1}{2}\int \omega\psi = -\frac{1}{2}\int \nabla\cdot\nabla\psi \;\psi = \frac{1}{2}\int |\nabla\psi|^2$

if ψ decays at infinity.

- Explicit $(-\Delta)^{-1}\omega(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln\frac{1}{|x-x'|}\,\omega(x')\,dx'$

$$\Rightarrow E[\omega] = \frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\frac{1}{|x-x'|}\,\omega(x')\,\omega(x)\,dx\,dx'$$

Energy & Dynamics

- Critical point of E $\iff$ Euler steady states

Given $\overline{\omega} \in L^1 \cap L^\infty$, consider variation by area-preserving diffeo

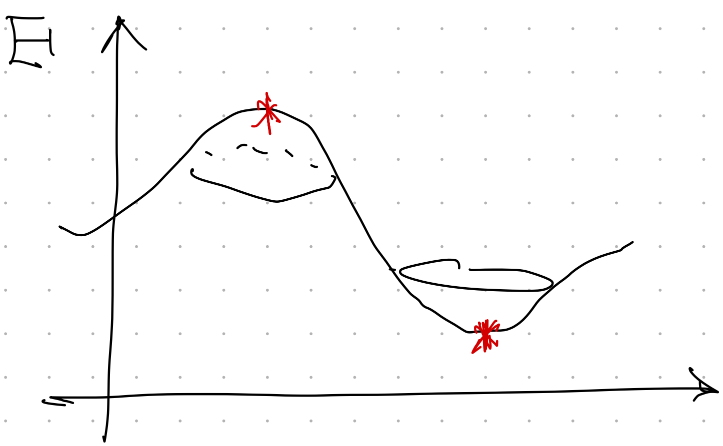
$$\frac{d}{dt}\Big|_{t=0} E[\overline{\omega} \circ \phi^t] = \int_{\mathbb{R}^2} \overline{\omega}\, \overline{u} \cdot \left(\frac{d}{dt}\Big|_{t=0} \phi^t \right)^{\perp}$$

Weak formulation of steady Euler:

for any φ, $\displaystyle\int_{\mathbb{R}^2} \overline{\omega}\, \overline{u} \cdot \nabla \varphi = 0$

Energy & Dynamics II

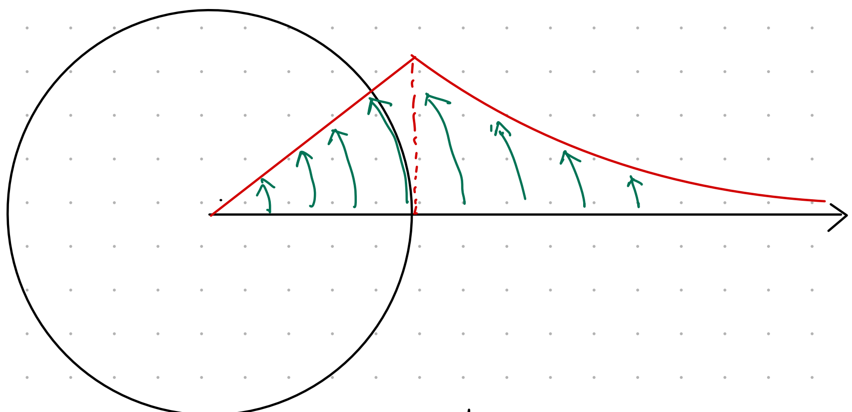
- (Arnol'd) Extremal points of E
  $\sim$ Stable Euler steady states



- Bounded domain case
  - minimum = constant vorticity $\rightarrow$ stable

Rankine Vortex

- $\overline{\omega}_D = 1_D$, $D =$ unit disc.



- $\overline{u}_D = \begin{cases} \frac{1}{2} x^{\perp}, & |x| \leq 1 \\ \frac{1}{2} \frac{x^{\perp}}{|x|^2}, & |x| > 1 \end{cases}$

$\Delta \overline{\psi} = \overline{\omega} \Rightarrow \partial_{rr} \overline{\psi} + \frac{1}{r} \partial_r \overline{\psi} = \overline{\omega}$

Stability and Energy for Rankine.

Pulvirenti, Wan, Tang, ...

- $\boxed{L^1 \text{ stability}}$. If $\omega_0 = 1_A$ is a patch

and $\|\omega_0 - \overline{\omega}_D\|_{L^1} < c_0 \varepsilon^8$ then for all t,

$$\|\omega(t) - \overline{\omega}_D\|_{L^1} < \varepsilon.$$

cf. stability by impulse minimization (Sideris–Vega)

- $\boxed{\text{Admissible class}}$ $\max_{\mathcal{A}} E[\omega] = E[\overline{\omega}_D]$

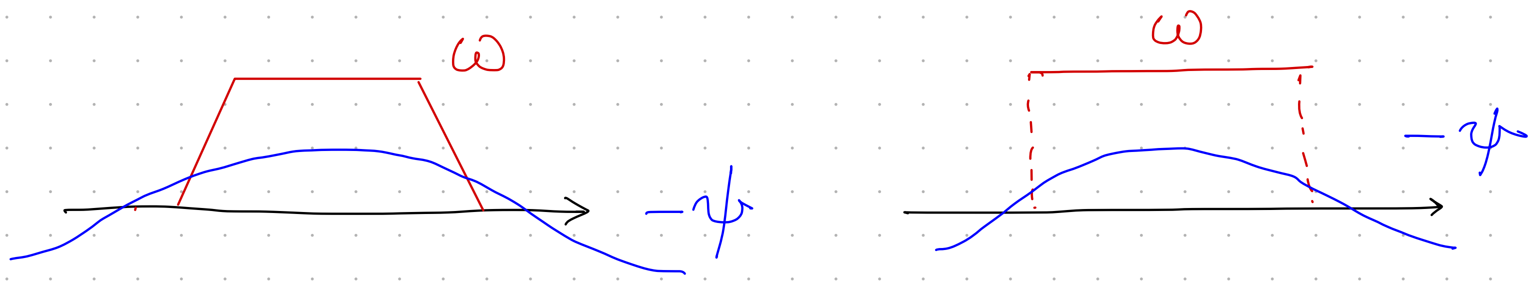
$$\mathcal{A} = \left\{ \omega = 1_A \text{ is a patch} \quad \int_{\mathbb{R}^2} \omega \, dx = \pi, \quad \int_{\mathbb{R}^2} x\omega \, dx = 0 \right\}$$

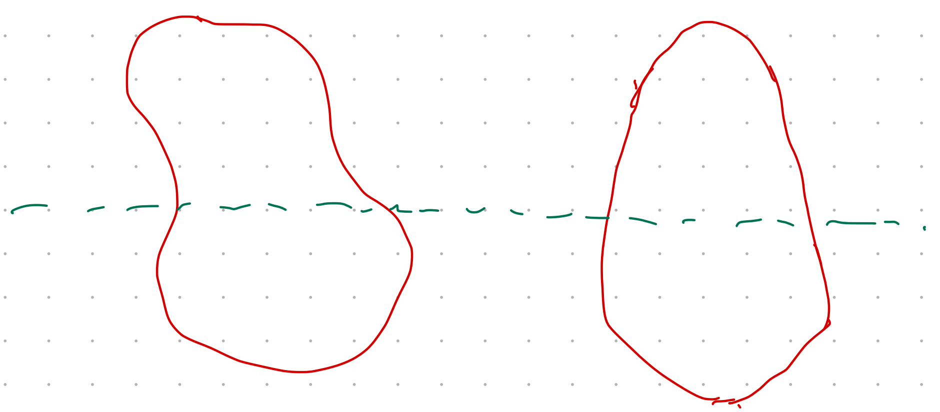
Can be relaxed to $\boxed{0 \le \omega(x) \le 1 \text{ for all } x}$

Idea for Maximization.

Turkington

$$\frac{1}{4\pi}\iint \omega(x)\,\omega(x')\,\ln\frac{1}{|x-x'|}\,dx\,dx' = -\frac{1}{2}\int \omega\psi\,dx$$





Cf. Without L^∞ in $\mathcal{A}$, maximizer is "Dirac delta" (infinite energy), but one can penalize concent– by instead maximizing $E - \frac{1}{2}\|\omega\|^2_{L^2}$

Stability of strict local maximizer

- Assume: $\overline{\omega}$ is strict maximizer, in a class $\mathcal{A}$ which is preserved by Euler

$$\left| E[\overline{\omega}] - E[\omega_0] \right| \leqslant C_{\overline{\omega}} \| \overline{\omega} - \omega_0 \|_{L^1}$$

$$\| \qquad\qquad$$

$$E[\overline{\omega}] - E[\omega_0]$$

$$\| \qquad\qquad$$

$$\mathcal{F}(\| \overline{\omega} - \omega(t) \|_{L^1}) \leqslant E[\overline{\omega}] - E[\underbrace{\omega(t)}_{\in \mathcal{A}}]$$

Energy inequality

- If $\omega_0 \notin \mathcal{A}$: "projection" find $\tilde{\omega}_0 \in \mathcal{A}$,

$$\| \omega_0 - \tilde{\omega}_0 \|_{L^1}, \quad \| \tilde{\omega}_0 - \overline{\omega} \|_{L^1} \ll 1.$$

Energy inequality for perturb. of $\overline{\omega}_D$

$\boxed{\text{Part I}}$. Reduction to graph perturbation.

$\boxed{\text{Part II}}$ Fourier analysis for graph perturbation

$$E[\overline{\omega}_D] - E[\omega] = \underset{\text{graph}}{(\exists \tilde{\omega})} \overbrace{\left(E[\overline{\omega}_D] - E[\tilde{\omega}] \right)}^{\boxed{\text{Part II}}} + \overbrace{\left(E[\tilde{\omega}] - E[\omega] \right)}^{\boxed{\text{Part I}}}$$

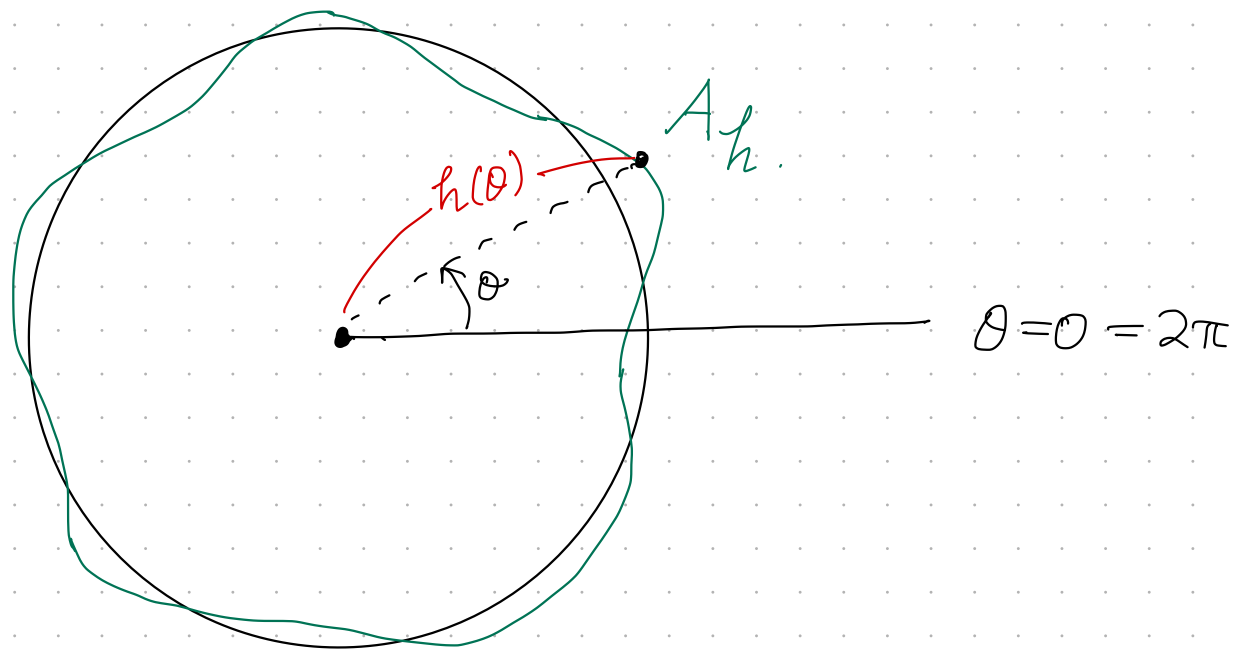
$$\left.\begin{aligned} &\mathrm{I}: \quad \gtrsim \|\tilde{\omega} - \omega\|_{L^1}^2 \\ &\mathrm{II}: \quad \gtrsim \|\overline{\omega}_D - \tilde{\omega}^*\|_{L^1}^2 \gtrsim \|\overline{\omega}_D - \tilde{\omega}\|_{L^1}^4 \end{aligned}\right\} \gtrsim \|\overline{\omega}_D - \omega\|_{L^1}^4$$

Reduction to graph.

∘ Graph perturbation ω_h

$$h : S^1 \longrightarrow \mathbb{R}, \qquad \|h\|_{C^1} \ll 1.$$

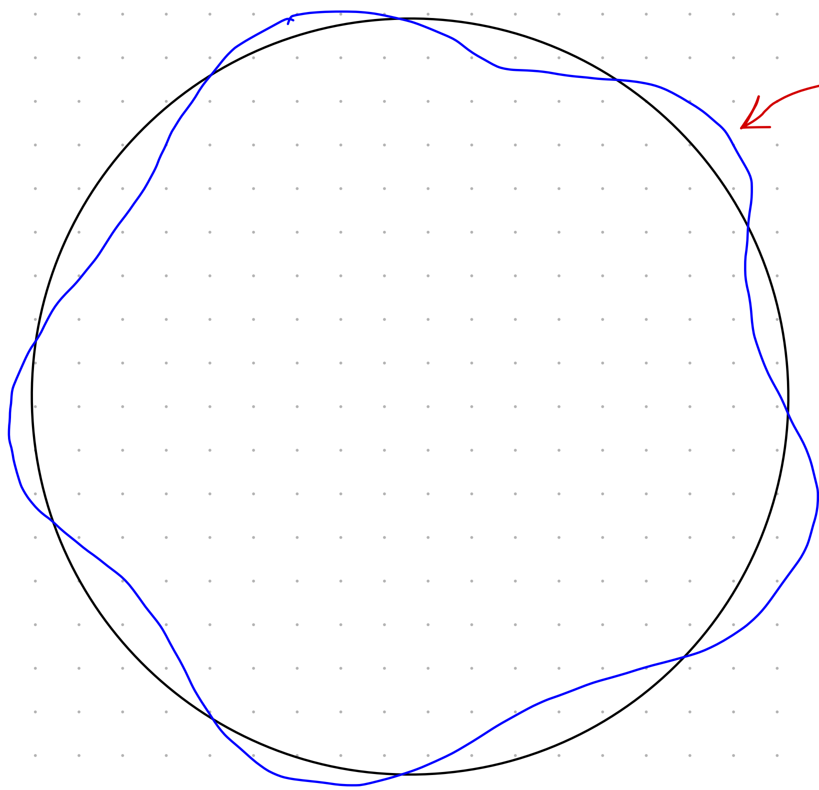
$$\omega_h = \{ (r, \theta) : \quad r < 1 + h(\theta) \} = 1_{A_h}.$$



Idea for the reduction.

$\omega \in \mathcal{A}$, $\quad \|\omega - \overline{\omega}_D\|_{L^1} \ll 1$

$\Rightarrow$ Nice Structure of $\psi = \Delta^{-1}\omega$ near ∂D



a level set of ψ

$\boxed{\text{uniquely}}$ chosen s.t.

area inside $= \pi$

Then define

$$\tilde{\omega} = 1_A$$

$$E[\tilde{\omega}] - E[\omega] \gtrsim \|\tilde{\omega} - \omega\|_{L^1}^2$$

Energy favors concentration

Graph Part

$$\boxed{\omega_h = \tilde{\omega}}$$

$$E[\overline{\omega}_D] - E[\omega_h] = \underbrace{\int (\overline{\omega}_D - \omega_h)(-\Delta)^{-1}\overline{\omega_D}}_{\mathrm{I}} \underbrace{- \frac{1}{2}\int (\overline{\omega}_D - \omega_h)\,(-\Delta)^{-1}(\overline{\omega}_D - \omega_h)}_{\mathrm{II}}.$$

$$\mathrm{I} = \int_0^{2\pi}\int_{1-h(\theta)}^{1} \overbrace{(-\Delta)^{-1}\overline{\omega}_D}^{\text{vanishes } r=1.}\, \underbrace{r}_{\simeq 1 \text{ for } \|h\| \ll 1.}\, dr\, d\theta.$$

$$\simeq \int_0^{2\pi}\int_{1-h(\theta)}^{1} \underbrace{\frac{\partial}{\partial r}(-\Delta)^{-1}\overline{\omega}_D\Big|_{r=1}}_{-\frac{1}{2}.}(r-1)\, dr\, d\theta.$$

$$= \frac{1}{4}\int_0^{2\pi} h^2(\theta)\, d\theta$$

$$\boxed{\mathrm{I} = \left\langle \frac{1}{4}h, h\right\rangle + o(\|h\|^2)}$$

$$\mathrm{II} = -\frac{1}{4\pi}\int_0^{2\pi}\int_{1-h(\theta)}^{1}\int_0^{2\pi}\int_{1-h(\theta')}^{1} \ln\frac{1}{|re^{i\theta}-r'e^{i\theta'}|}\, rr'\,dr'\,d\theta'\,dr\,d\theta.$$

$$\simeq -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} h(\theta)\,h(\theta')\,\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}\,d\theta'\,d\theta.$$

$$=: -\langle Kh, h\rangle$$

K = Convolution with kernel $\frac{1}{4\pi}\ln\frac{1}{|1-e^{i\theta}|}$

$\Rightarrow$ Given by a Fourier multiplier !

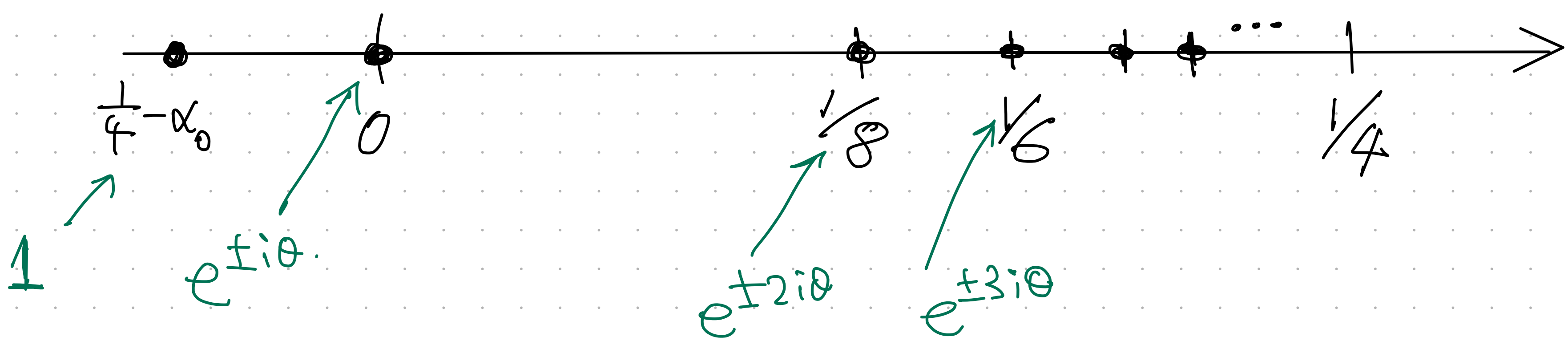
$$K[e^{in\theta}] \overset{(n\geq 1)}{=} \frac{1}{4n}e^{in\theta} \quad , \quad K[1] = \alpha_0 .$$

$$\boxed{\mathrm{II} = \langle -Kh, h\rangle + o(\|h\|^2)}$$

Summary

$$E[\overline{\omega}_D] - E[\omega_{f_h}] = \overbrace{\langle \mathcal{L} h , h \rangle}^{\text{want: positive definite.}} + o(\|h\|^2)$$

$$\mathcal{L} = \frac{1}{4} - K$$



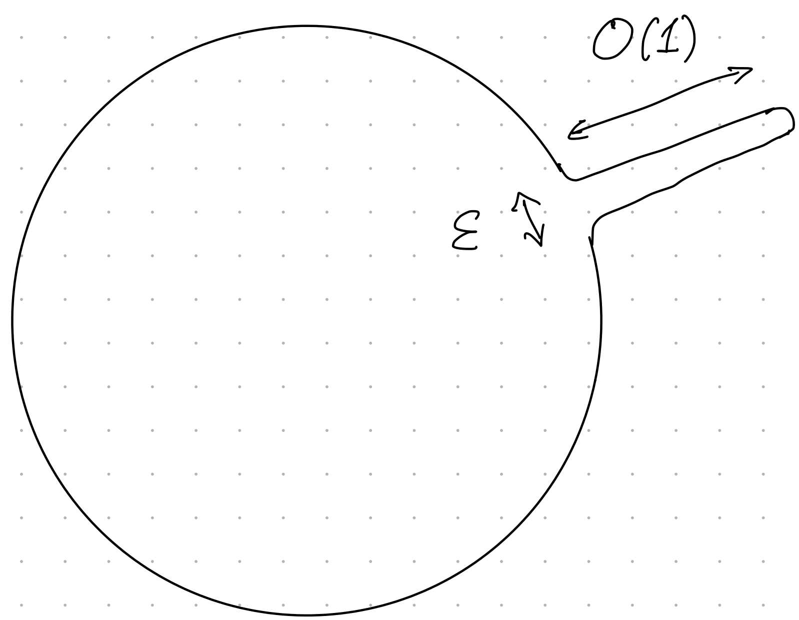
Three problematic eigenvectors:

$$\int \overline{\omega}_D - \omega_{f_h} = 0 , \qquad \int x(\overline{\omega}_D - \omega_{f_h}) = 0$$

Application of Stability.

- Filamentation based on

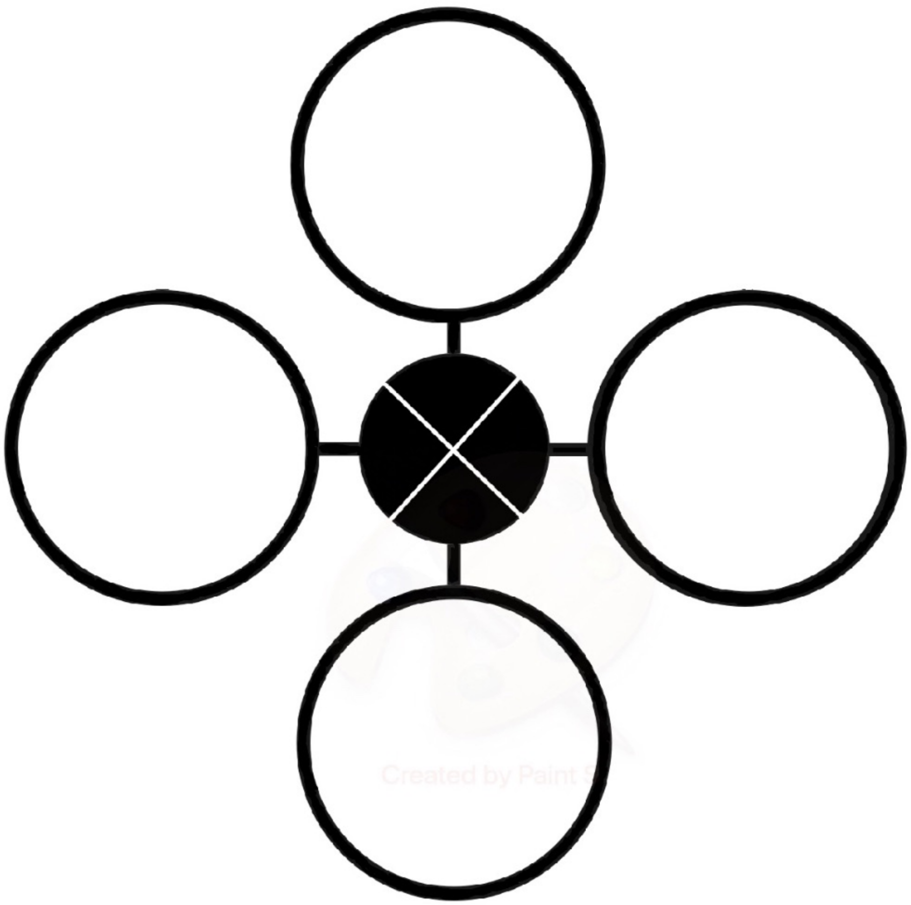
$$\| u(t) - \bar{u}_D \|_{L^\infty} \lesssim \| \omega(t) - \bar{\omega}_D \|_{L^1}^{1/2}$$



Choi-J. perimeter growth at least up to $C\varepsilon^{-\alpha}$

Infinite Perimeter Growth.

Elgindi – Drivas – J.



Stability of shape

$\Downarrow$

Stability of winding in averaged sense

$\Downarrow$

Twisting of boundary

Q. Infinite perimeter growth for simply connected vortex patch?

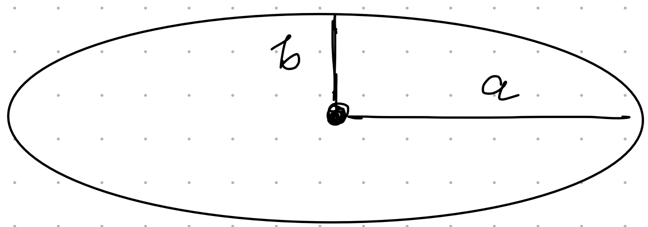
Kelvin Waves

- Rotating solution:

$\exists\, \Omega$ s.t. $\omega(t) = R_{\Omega t}(\omega_0)$ for all t.

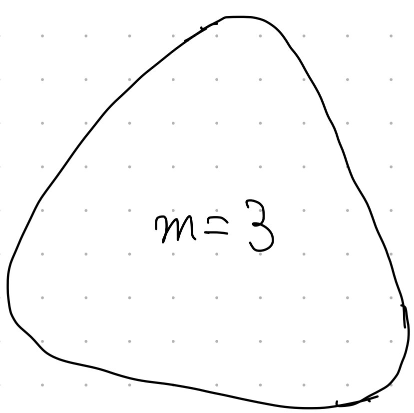
"angular speed"

- Kirchhoff ellipse



$$\Omega = \frac{ab}{(a+b)^2} \longrightarrow \frac{1}{4} \neq \frac{1}{2}$$

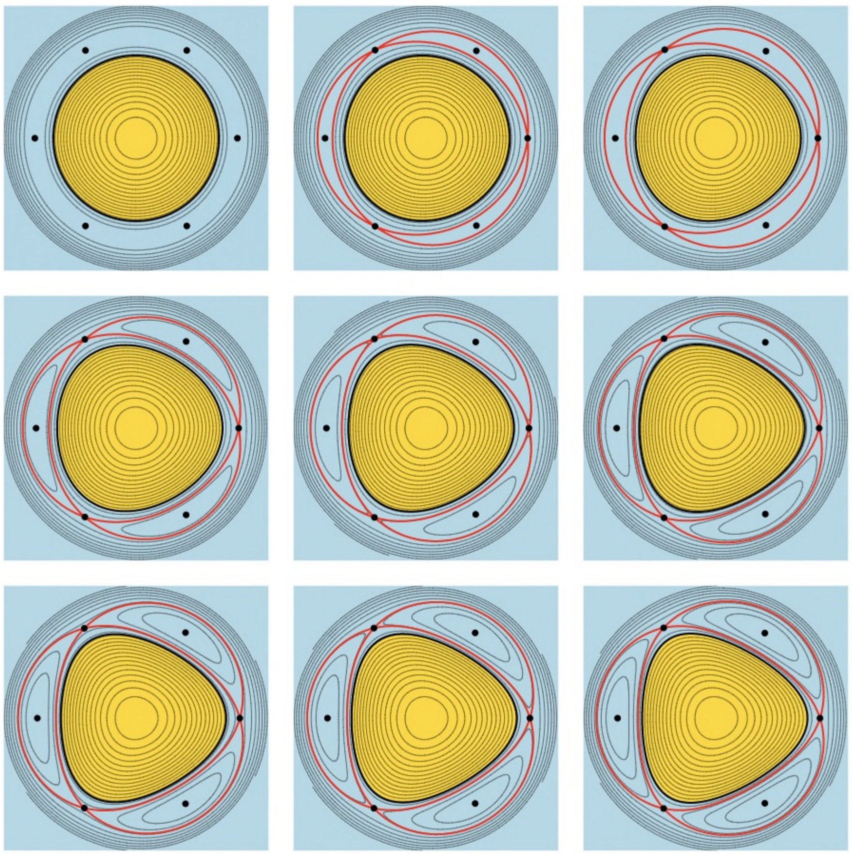
- Kelvin m-waves for any $m \geq 2$. (Burbea, HMV)



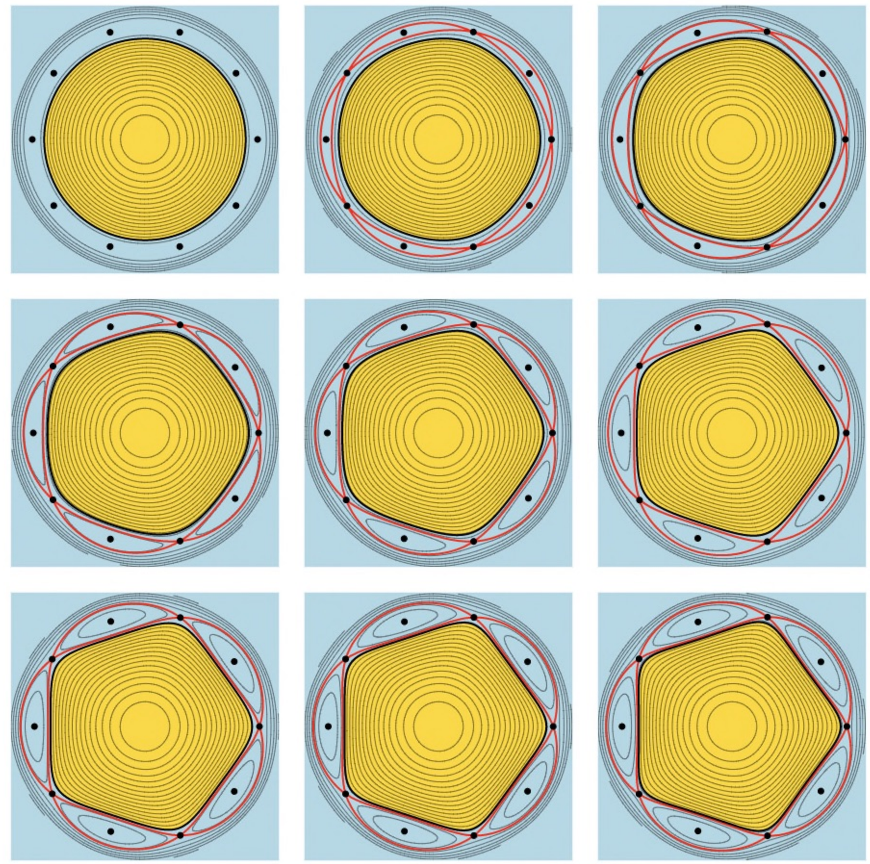
Limiting angular velocity.

$$\Omega_m = \frac{1}{2} - \frac{1}{2m}$$

eigenvalues of $2L$



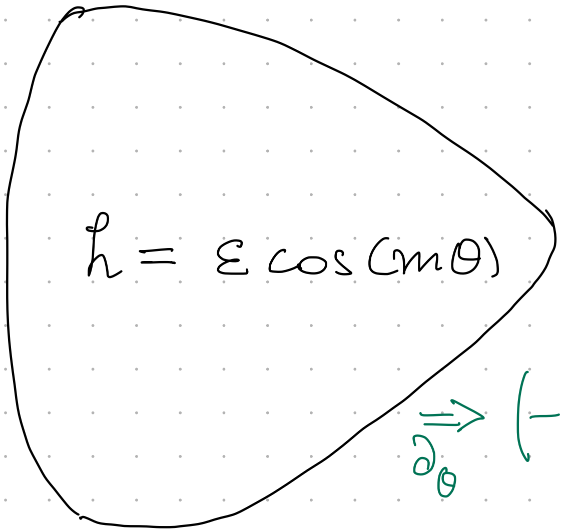
$m = 3$



$m = 5$

M. Wheeler

Kelvin Computation



Relative Stream

$$\Delta^{-1}\omega^{m,\varepsilon} - \frac{1}{2}\Omega^{m,\varepsilon}|x|^2 \equiv C$$

on $\{(1+\varepsilon\cos(m\theta), \theta)\}$

$$\underset{\partial_\theta}{\Rightarrow} \left(-\frac{1}{2}+\Omega^{m,\varepsilon} + \partial_r \Delta^{-1}(\omega^{m,\varepsilon} - \bar{\omega}_D)\right)(-\varepsilon m \sin(m\theta)) + \partial_\theta \Delta^{-1}(\omega^{m,\varepsilon} - \bar{\omega}_D) = 0$$

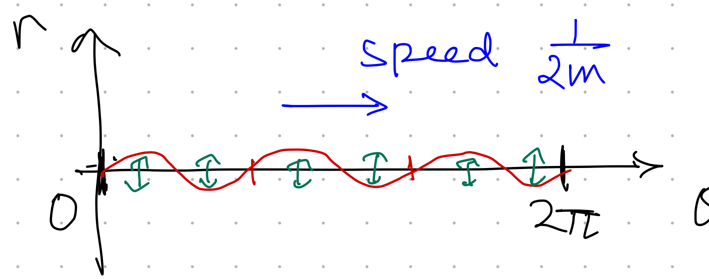
$$\Delta^{-1}(\omega^{m,\varepsilon} - \bar{\omega}_D) = \frac{1}{2\pi}\int_0^{2\pi}\int_1^{1+\varepsilon\cos(m\theta)} \ln\frac{1}{|re^{i\theta} - r'e^{i\theta'}|}\, r'\,dr'\,d\theta'$$

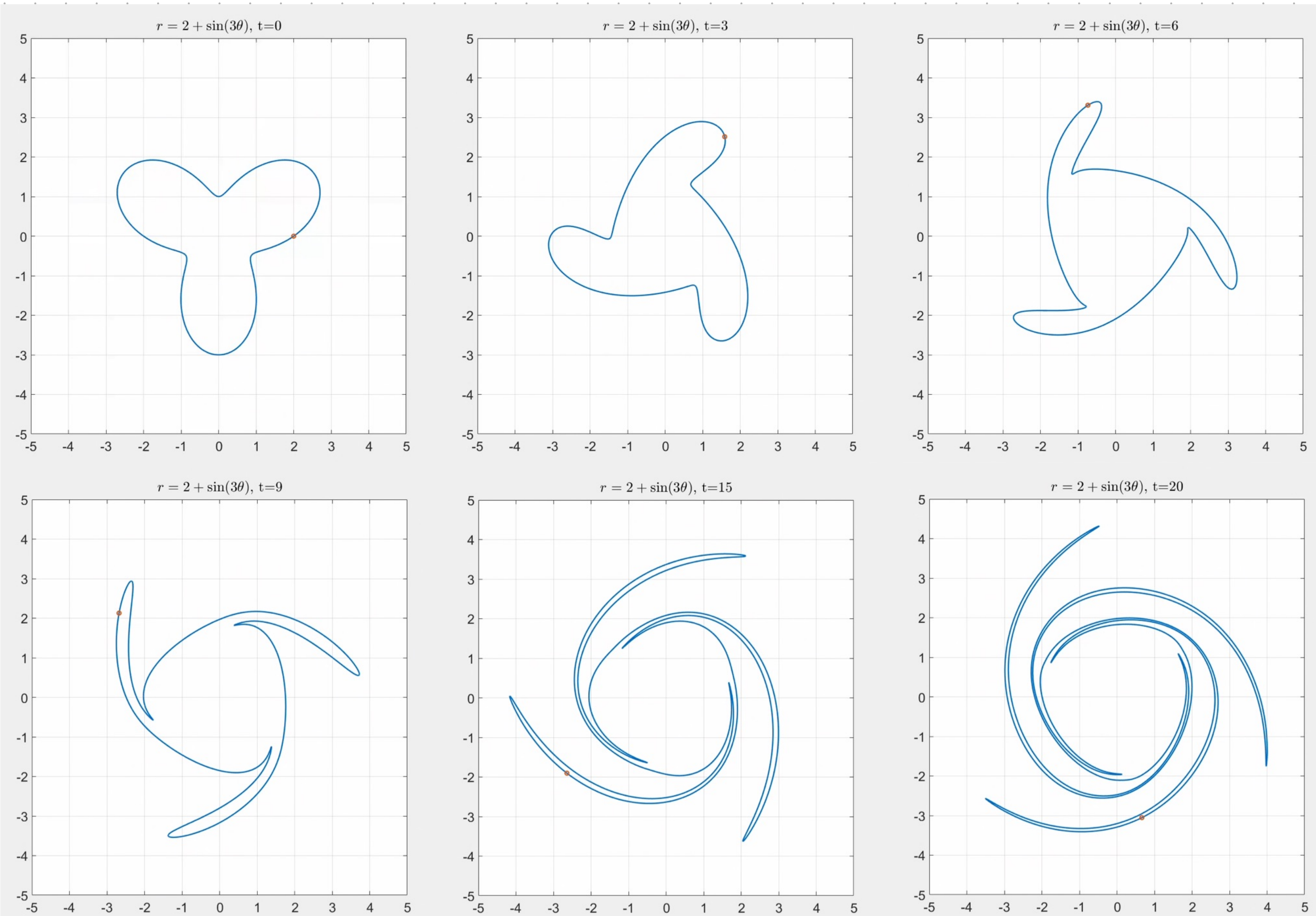
Expand the log, depending on $r > r'$ or $r' > r$.

$$\mathrm{Re}\left[\sum_{n\geq 1}\frac{1}{n}\left(\frac{r'}{r}\right)^n e^{in(\theta'-\theta)}\right]$$

$\Rightarrow$ then integrate term-by-term

$$\Rightarrow \begin{cases} \Omega^{m,\varepsilon} = \frac{1}{2} - \frac{1}{2m} + o(\varepsilon). \\ u^r = -\frac{1}{2}\varepsilon\sin(m\theta) + o(\varepsilon) \end{cases}$$




$r = 2 + \sin(3\theta)$, t=0
$r = 2 + \sin(3\theta)$, t=3
$r = 2 + \sin(3\theta)$, t=6
$r = 2 + \sin(3\theta)$, t=9
$r = 2 + \sin(3\theta)$, t=15
$r = 2 + \sin(3\theta)$, t=20

J. Choi

Constrained Maximization.

Tang

Ellipse Stability If $1/3 < a/b < 3$ then the Kirchhoff ellipse is L^1 stable, under an assumption on the support.

Admissible Class concentration up to constraint

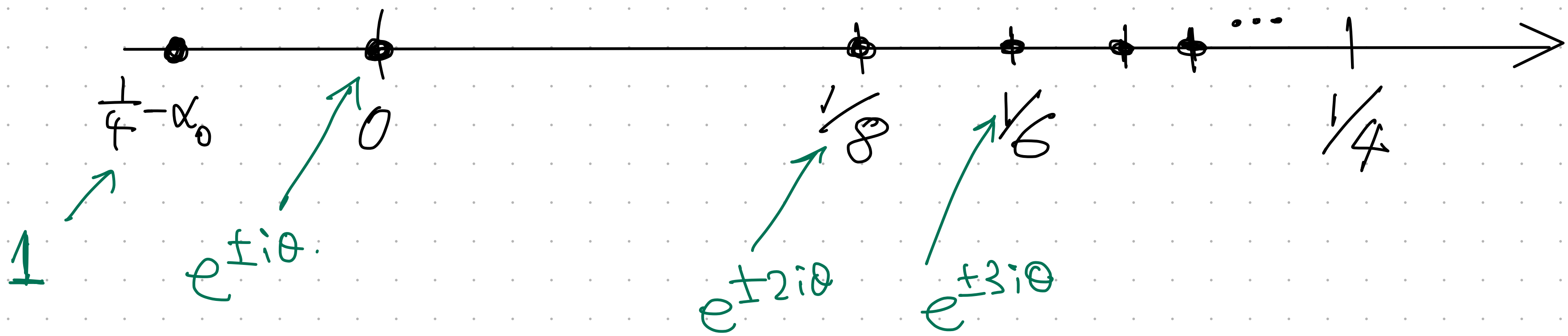
$$\mathcal{A}_\mu = \left\{ \omega = 1_A, \quad \int \omega = \pi, \quad \int x\omega = 0, \quad \int |x|^2 \omega = \int |x|^2 \bar{\omega}_D + \mu \right\}$$

Energy expansion

$\overline{\omega}_\varepsilon$: ellipse close to disc

$$E[\overline{\omega}_\varepsilon] - E[\omega_h] = \langle \mathcal{L} h, h \rangle + o(\|h\|^2)$$

$$\mathcal{L} = \frac{1}{8} - K + o(1)$$



Five problematic eigenvectors :

$$\int \overline{\omega}_\varepsilon - \omega_h = 0, \qquad \int x(\overline{\omega}_\varepsilon - \omega_h) = 0$$

$$\int |x|^2(\overline{\omega}_\varepsilon - \omega_h) = 0, \qquad \int x_1 x_2 (\overline{\omega}_\varepsilon - \omega_h) = 0.$$

Kelvin wave case.

Wan, Choi–J.

- Stability of m–waves sufficiently close to the disc for m-fold symmetric perturbations

- Filamentation as an application of stability.

- Questions
  - Instability for non m-fold symmetric perturbations
  - Secondary bifurcations
  - "Convergence"