Stability–instability of vortex solutions in incompressible Euler equations

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Deterministic and random features in fluids, EPFL, Laussane

The vorticity form in $\mathbb{R}^2$

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^2, \qquad \omega|_{t=0} = \omega_0 \quad \text{on} \quad \mathbb{R}^2. \tag{1}$$

The velocity field u is determined by the scalar vorticity $\omega = -\partial_2 u^1 + \partial_1 u^2$ following the Biot-Savart law:

$$u = \nabla^{\perp}(-\Delta)^{-1}\omega.$$

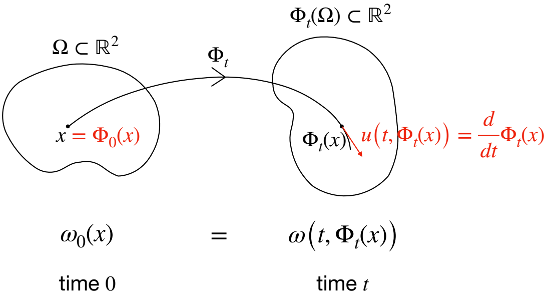


Figure: The dynamics of an ideal incompressible fluid with the flow map Φ_t

Conserved quantities in time

- The measure of any level set $|\{x \in \mathbb{R}^2 : \omega(t,x) > c\}|, \ c > 0$.
- L^p-norms $\|\omega(t)\|_{L^p(\mathbb{R}^2)}$ for any $p \in [1, \infty]$.
- The energy

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^2} |u(t,x)|^2 dx,$$

- The angular impulse

$$\mathcal{J}(\omega(t)) := \int_{\mathbb{R}^2} |x|^2 \omega(t,x) dx,$$

  which represents the rotational inertia of $\omega(t)$, i.e., the angular mass.

Goal

- To prove stabilities of the unit disc patch 1_{B_1} and, more generally, a radial, non-negative, and monotone vorticities in $\mathbb{R}^2$.
- To construct a vortex patch in the half cylinder having instability (infinite perimeter growth for all time.)

Steady state

- A solution $\overline{\omega}$ of (1) is called a steady state if it satisfies

$$\overline{u} \cdot \nabla \overline{\omega} = 0,$$

where $\overline{u} = \nabla^{\perp}(-\Delta)^{-1}\overline{\omega}$.

- For instance, any radial vorticity $\overline{\omega} = \overline{\omega}(|x|)$ is a steady state of (1) because its stream function $\overline{\psi} = (-\Delta)^{-1}\overline{\omega}$ is also radial, so the velocity field $\overline{u} = \nabla^{\perp}\overline{\psi}$ is in the tangential direction while $\nabla\overline{\omega}$ is in the radial direction.
- During the first section of this talk, we will discuss about stability of radial vorticities.

Lyapunov stability

- A steady state $\overline{\omega}$ is said to be (Lyapunov) stable if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t. if any perturbed vorticity ω_0 satisfies

$$\|\omega_0 - \overline{\omega}\|_X \leq \delta,$$

then for any $t \geq 0$, the corresponding solution $\omega(t)$ of (1) with initial data ω_0 satisfies

$$\|\omega(t) - \overline{\omega}\|_X \leq \varepsilon.$$

Related works

- L^1-stability : Wan–Pulvirenti '85, Marchioro–Pulvirenti '85, Sideris–Vega '09, etc.
- Other stability : Tang '87, Bedrossian–Masmoudi '14, Beichman–Denisov '17, Choi–Jeong '22, etc.

Vortex patches

- If a vorticity ω is given as a characteristic function;

$$\omega = 1_{\Omega},$$

  of some measurable set $\Omega \subset \mathbb{R}^2$, then we call it as a vortex patch.

- For example, the unit disc patch 1_{B_1} has its corresponding velocity field

$$u_{B_1}(x) = \begin{cases} \frac{x^{\perp}}{2} & \text{if } |x| \leq 1, \\ \frac{x^{\perp}}{2|x|^2} & \text{if } |x| > 1. \end{cases}$$

  Vortex patches are helpful in distinguishing regions where the local tendency to have rotation is strong or weak.

- If an initial data of (1) is $\omega_0 = 1_{\Omega_0}$, then $\forall\ t \geq 0$, its corresponding solution of (1) is

$$\omega(t) = 1_{\Omega_t}, \quad \Omega_t = \Phi_t(\Omega_0).$$

- If the boundary $\partial\Omega_0$ of 1_{Ω_0} is C^k-smooth, then the boundary $\partial\Omega_t$ of 1_{Ω_t} is also C^k-smooth $\forall\ t \geq 0$.

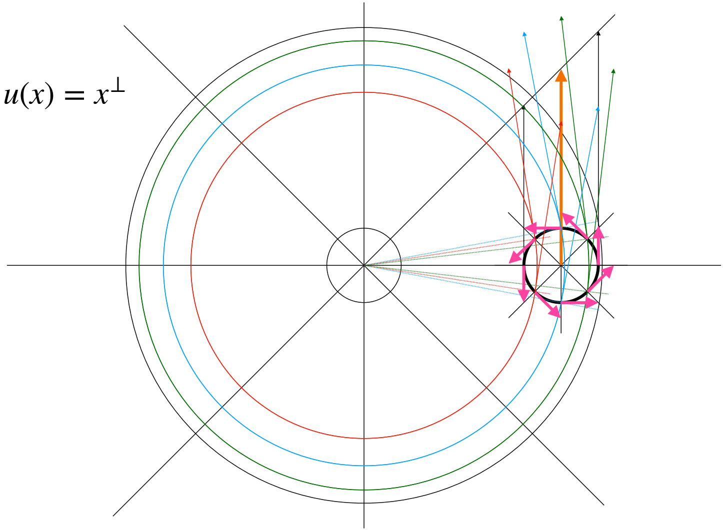


Figure: Relative velocity field around a certain point on the velocity field $u(x) = x^{\perp}$

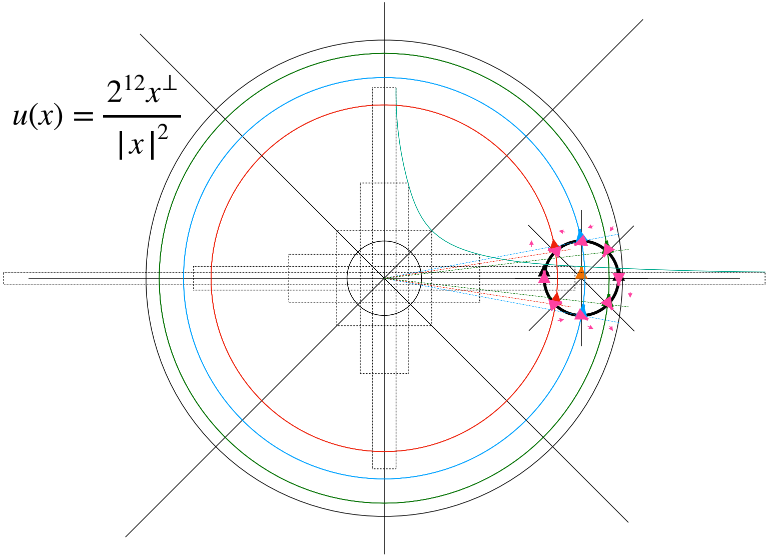


Figure: Relative velocity field around a certain point on the velocity field $u(x) = \frac{2^{12} x^{\perp}}{|x|^2}$

Main results

$\forall$ measurable set $\Omega \subset \mathbb{R}^2$ with finite measure and $\mathcal{J}(1_\Omega) < \infty$, let us denote

$$\|\Omega\|_J := \|1_\Omega\|_{L^1(\mathbb{R}^2)} + \mathcal{J}(1_\Omega) = \int_\Omega (1+|x|^2)dx.$$

Also, let us denote 1_{Ω_0} as the perturbed vortex patch with $\mathcal{J}(1_{\Omega_0}) < \infty$, and 1_{Ω_t} as its corresponding vortex patch solution of (1).

Theorem 1 (Choi–L. '22, $\theta = 1_{B_1}$, $\omega_0 = 1_{\Omega_0}$)

We have

$$\sup_{t\geq 0} \|\Omega_t \triangle B_1\|_J \lesssim \|\Omega_0 \triangle B_1\|_J^{1/2} + \|\Omega_0 \triangle B_1\|_J.$$

Remark

- This means that the unit disc patch 1_{B_1} is stable in J-norm. This tells us that if the initial perturbation is small in J-norm, then the perturbation stays small in the same norm for all time.
- However, it does not give us any information on the time evolution of the form of the perturbation.
- More generally, we proved that the same type of stability holds for a radial, non-negative, and decreasing vorticity, such as a Gaussian $e^{-|x|^2}$, as well.

Arnold-type stability

- Find a functional H of ω (or u) that satisfies

$$H(\omega(t)) = H(\omega_0) \quad \forall t \geq 0.$$

e.g. the energy $E(u) = \frac{1}{2}\int_{\mathbb{R}^2} |u|^2 dx$, the angular impulse $\mathcal{J}(\omega) = \int_{\mathbb{R}^2} |x|^2 \omega dx$, etc.

- For a steady state $\overline{\omega}$, and any $\hat{\omega}$ in some admissible class of functions, find constants $0 < c_1 \leq c_2 < \infty$ that satisfy

$$c_1 \|\hat{\omega} - \overline{\omega}\|_X \leq H(\hat{\omega}) - H(\overline{\omega}) \leq c_2 \|\hat{\omega} - \overline{\omega}\|_X.$$

The upper bound is easier to obtain. The lower bound is the non-trivial part.

- Once the above two conditions are satisfied, then we have

$$c_1 \|\omega(t) - \overline{\omega}\|_X \leq \underbrace{H(\omega(t))}_{=H(\omega_0)} - H(\overline{\omega}) \leq c_2 \|\omega_0 - \overline{\omega}\|_X \quad \forall t \geq 0.$$

Key ideas

Conservation of the angular impulse:

$$\mathcal{J}(1_{\Omega_t}) = \mathcal{J}(1_{\Omega_0}) \quad \forall t \geq 0.$$

Symmetric rearrangement Ω^* of $\Omega \subset \mathbb{R}^2$

- Ω^* is defined as the disc s.t. $|\Omega^*| = |\Omega| < \infty$.
- Basic property : $\mathcal{J}(1_{\Omega^*}) \leq \mathcal{J}(1_{\Omega})$, $(\Omega_t)^* = (\Omega_0)^*$.

Properties of Ω^*

- Nonexpansivity:

$$|\Omega^* \triangle B_1| \leq |\Omega \triangle B_1|.$$

- Estimates of the difference of angular impulse between 1_Ω and 1_{Ω^*} (adaptation of Marchioro–Pulvirenti '85 for patch); if $\Omega \subset B_R$ for some $R > 0$, then

$$|\Omega \triangle \Omega^*| \lesssim [\mathcal{J}(1_\Omega) - \mathcal{J}(1_{\Omega^*})]^{1/2} \lesssim_R |\Omega \triangle \Omega^*|^{1/2}.$$

  This means that 1_{Ω^*} is the unique minimizer of $\mathcal{J}$ among every vortex patch of a set having the same measure as Ω.

Instability

Goal

- ~~To prove stabilities of the unit disc patch 1_{B_1} and, more generally, a radial, non-negative, and monotone vorticities in $\mathbb{R}^2$.~~
- To construct a vortex patch in the half cylinder having instability (infinite perimeter growth for all time.)

Instability

- $\overline{\omega}$ is not stable : $\exists\ \varepsilon_0 > 0$ s.t. $\forall\ \delta > 0$, $\exists\ \omega_0$ and $\exists\ T_0 > 0$ s.t. we have

$$\|\omega_0 - \overline{\omega}\|_X \leq \delta,$$

  but

$$\|\omega(T_0) - \overline{\omega}\|_X > \varepsilon_0.$$

- For a vortex patch 1_Ω with smooth boundary, we can consider its perimeter length$(\partial\Omega)$ as a one way to describe instability, although it is not a norm.
- Does there exist a vortex patch 1_{Ω_0} satisfying

$$\text{length}(\partial\Omega_0) \leq C,$$

  and its corresponding solution 1_{Ω_t} of (1) showing

$$\text{length}(\partial\Omega_t) \gtrsim t, \quad \forall t \geq 0?$$

Related works on various instabilities of vortex solutions : Nadirashvili '91, Choi–Jeong '22, Choi–Jeong '22, etc.

Half cylinder $S_+ := \mathbb{R}_+ \times \mathbb{T}$, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ $(= [-\pi, \pi))$

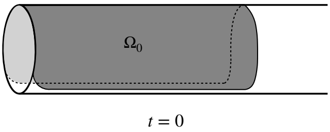


Figure: A diagram of some vortex patch on S_+

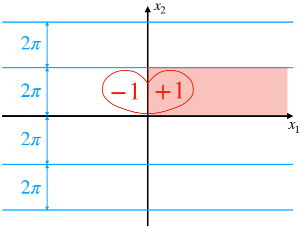


Figure: Considering the half cylinder as an infinite strip with boundary

Global well-posedness of a weak solution $\omega \in L^\infty(S_+)$ **with compact support** : Beichman–Denisov '17.

Conserved quantities of $\omega(t)$ **in time**

- The measure of any level set $|\{x \in S_+ : \omega(t,x) > c\}|$, $c > 0$.
- L^p-norms $\|\omega(t)\|_{L^p(S_+)}$ for any $p \in [1, \infty]$.
- The horizontal impulse

$$h(\omega(t)) := \int_{S_+} x_1 \omega(t,x)dx.$$

Stability of $1_{\overline{\Omega}}$

Let us denote $\overline{\Omega} := \{x_1 < 1\}$, and $\forall$ measurable set $\Omega \subset S_+$ with finite measure and $h(1_\Omega) < \infty$, let us denote

$$\|\Omega\|_Z := \|1_\Omega\|_{L^1(S_+)} + h(1_\Omega) = \int_\Omega (1 + x_1)dx.$$

Then we can produce the stability result of $1_{\overline{\Omega}}$ analogous to Theorem 1.

Theorem 2 (Choi–Jeong–L. '22)

We have

$$\sup_{t\geq 0} \|\Omega_t \triangle \overline{\Omega}\|_Z \lesssim \|\Omega_0 \triangle \overline{\Omega}\|_Z^{1/2} + \|\Omega_0 \triangle \overline{\Omega}\|_Z.$$

Note

We do not present key ideas of Theorem 2, since they are analogies of ideas from Theorem 1. We use its result to prove our second goal. In particular, we use the following; if $\Omega_0 \subset \{x_1 < 3\}$ and $|\Omega_0 \triangle \overline{\Omega}| \leq 1$, then we have

$$\sup_{t\geq 0} |\Omega_t \triangle \overline{\Omega}| \lesssim |\Omega_0 \triangle \overline{\Omega}|^{1/2}.$$

Theorem 3 (Choi–Jeong–L. '22)

$\exists$ *an open, bounded set* $\Omega_0 \in S_+$ *with smooth, connected boundary* $\partial\Omega_0 \in \overline{S_+} := \{x_1 \geq 0\}$ *that satisfies*

$$\text{length}(\partial\Omega_0) \leq 20, \quad \partial\Omega_0 \cap \partial S_+ \neq \emptyset,$$

and

$$\text{length}(\partial\Omega_t) \gtrsim t \quad \forall t \geq 0.$$

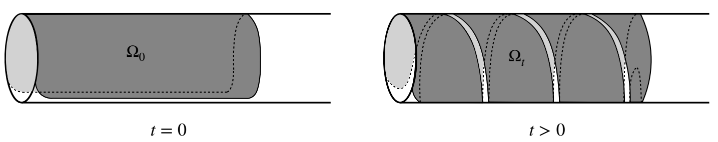


Figure: A schematic diagram of the patch 1_{Ω_t} from Theorem 3 on S_+

Remark

- This says that $1_{\overline{\Omega}}$ can be viewed as having instability in the sense of perimeter.
- This tells that Ω_t twists around on S_+ and length$(\partial\Omega_t)$ grows $\forall\ t \geq 0$, but this does not tell us what the precise form of 1_{Ω_t} is going to be throughout the time evolution.

Considering patches $1_{\overline{\Omega}}$ and 1_{Ω_0} in $\mathbb{R}^2_+$

- $\overline{u}$: velocity field in S_+ from $1_{\overline{\Omega}}$.
- $\overline{\Phi}_t$: flow map in $\mathbb{R}^2_+$ from periodic extension of $\overline{u}$.

- $u(t)$: velocity field in S_+ from 1_{Ω_t}.
- Φ_t : flow map in $\mathbb{R}^2_+$ from periodic extension of $u(t)$.

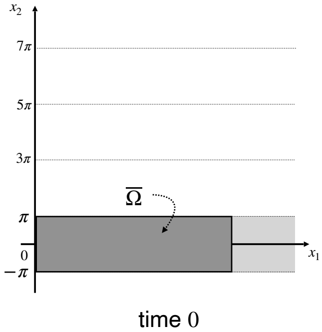


Figure: The patch $1_{\overline{\Omega}}$ in $\mathbb{R}^2_+$

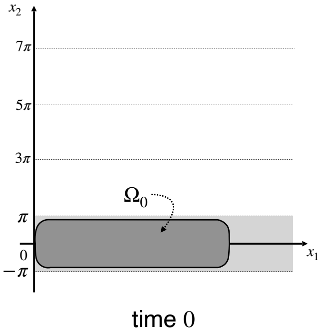


Figure: The patch 1_{Ω_0} in $\mathbb{R}^2_+$

Notable features of $\overline{u}^2$ on ∂S_+ and the patch $1_{\overline{\Phi}_t(\overline{\Omega})}$ in $\mathbb{R}^2_+$

- The vertical speed $\overline{u}^2$ of any point on the boundary ∂S_+ of S_+

$$\overline{u}^2\big|_{x_1=0} = 1.$$

- The growth rate of the vertical center of mass of the patch $1_{\overline{\Phi}_t(\overline{\Omega})}$ in $\mathbb{R}^2_+$

$$\frac{d}{dt}\frac{1}{2\pi}\int_{\overline{\Phi}_t(\overline{\Omega})} x_2 dx = \cdots = \frac{1}{2}.$$

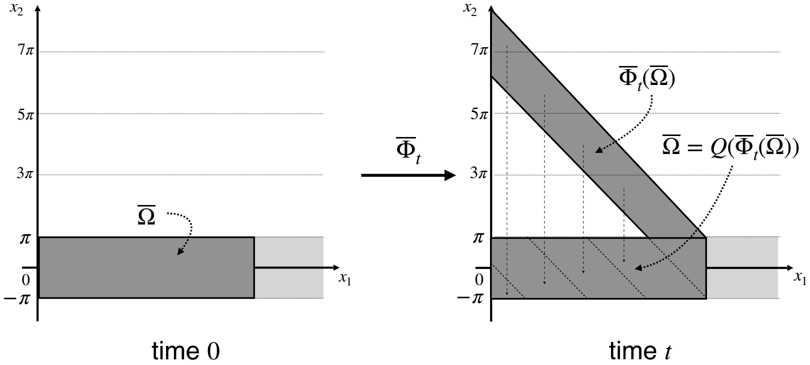


Figure: A schematic diagram describing the dynamics of $1_{\overline{\Phi}_t(\overline{\Omega})}$.

Controlling the vertical speed and the growth rate

Lemma 4 (Choi–Jeong–L. '22)

Let $\Omega_0 \subset \{x_1 < 3\}$ *satisfy* $|\Omega_0| = |\overline{\Omega}|$ *and* $|\Omega_0 \triangle \overline{\Omega}| \leq 1$. *Then we have*

$$\left| u^2(t,x)\big|_{x_1=0} - \overline{u}^2(x)\big|_{x_1=0} \right| \lesssim |\Omega_t \triangle \overline{\Omega}|^{1/2} \lesssim |\Omega_0 \triangle \overline{\Omega}|^{1/4} \quad \forall t \geq 0,\ x_2 \in \mathbb{T},$$

$$\left| \frac{d}{dt} \frac{1}{2\pi} \int_{\Phi_t(\Omega_0)} x_2 \, dx - \frac{d}{dt} \frac{1}{2\pi} \int_{\overline{\Phi}_t(\overline{\Omega})} x_2 \, dx \right| \lesssim |\Omega_0 \triangle \overline{\Omega}|^{1/4} \quad \forall t \geq 0.$$

Key ideas

The dynamics of $1_{\Phi_t(\Omega_t)}$

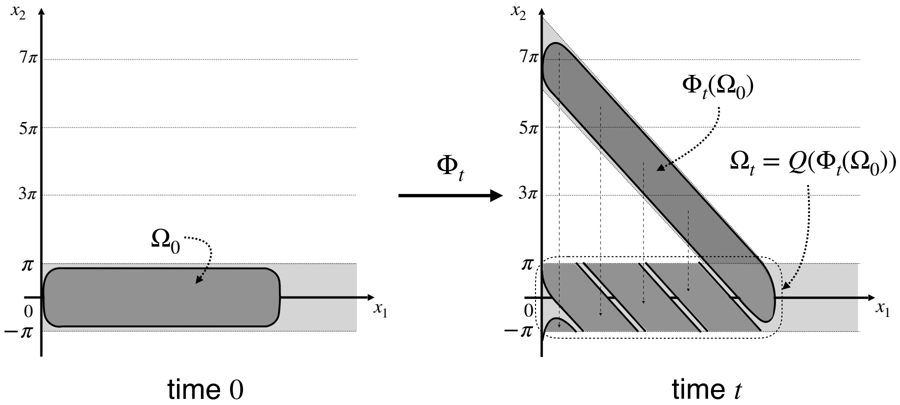


Figure: A schematic diagram describing the dynamics of $1_{\Phi_t(\Omega_t)}$

Open problems

- For 1_{B_1}, if we allow the perturbed vorticity ω_0 to have a negative part, i.e. a negative perturbation, then would it be stable? If not, then is there any counterexample which shows instability in some sense?
- Very recently, Drivas–Elgindi–Jeong (arXiv:2305.09582) proved the existence of a vortex patch of some multiply connected set in $\mathbb{R}^2$ showing infinite perimeter growth for infinite time, which is a surprising result. Then would there be a vortex patch of a simply connected set in an unbounded domain which shows infinite perimeter growth for infinite time, without using the boundary of the domain (if it exists)?

Thank you!

Please feel free to ask any questions.

Remark

- We enhanced the work of Wan–Pulvirenti '85 by expanding the domain to $\mathbb{R}^2$, and the work of Sideris–Vega '09 by allowing perturbations which are not necessarily compactly supported.
- More generally, we proved that a non-negative and monotone $\theta = \theta(|x|)$ in $L^\infty(\mathbb{R}^2)$ with some decay at infinity, such as a Gaussian $e^{-|x|^2}$, is stable in the norm $\|\cdot\|_{L^1} + \mathcal{J}(|\cdot|)$ w.r.t. nonpatch-type and non-negative perturbations not necessarily compactly supported in $\mathbb{R}^2$.

Related works on stability of vortex solutions

Related works on stability of radial vorticities

- Wan–Pulvirenti '85 : L^1-stability of the unit disc patch 1_{B_1} in $L^\infty(B_R)$ w.r.t. patch-type perturbations.
- Sideris–Vega '09 : L^1-stability of the unit disc patch 1_{B_1} in $L^\infty(\mathbb{R}^2)$ w.r.t. patch-type perturbations with compact support in $\mathbb{R}^2$.

$$\sup_{t\geq 0} |\Omega_t \triangle B_1|^2 \lesssim \sup_{\Omega_0 \triangle B_1} \big||x|^2 - 1\big| \cdot |\Omega_0 \triangle B_1|.$$

- Marchioro–Pulvirenti '85 : L^1-stability of a monotone vorticity $\theta = \theta(|x|)$ in $L^\infty(B_R)$ w.r.t. nonpatch-type perturbations.

Related works on stability of other vorticities in 2D domains

- Marchioro–Pulvirenti '85 : L^1-stability of a monotone vorticity $\zeta = \zeta(x_1)$ in $L^\infty([0,\alpha] \times \mathbb{T})$ w.r.t. nonpatch-type perturbations.
- Bedrossian–Masmoudi '14 : Asymptotic stability of a planar shear flow $\overline{u}(x) = (0, x_1)$ in the full cylinder $S := \mathbb{R} \times \mathbb{T}$.
- Beichman–Denisov '17 : Stability of a vortex patch $1_{\{|x_1|<L\}}$ in S for large enough L.
- Tang '87 : Stability of elliptic vortex patches in $\mathbb{R}^2$.
- Choi–Jeong '22 : Stability and instability of Kelvin waves in $\mathbb{R}^2$.

Main results

Let us denote the weighted L^1-norm involving the angular impulse as

$$\|f\|_J := \|f\|_{L^1(\mathbb{R}^2)} + \mathcal{J}(|f|).$$

Also, let us denote $\omega_0 \in (L^1 \cap L^\infty)(\mathbb{R}^2)$ as the non-negative perturbed vorticity and $\omega(t)$ as its corresponding solution of (1).

Theorem 5 (Choi–L. '22, $\theta = 1_{B_1}$)

We have

$$\sup_{t\geq 0} \|\omega(t) - 1_{B_1}\|_J \lesssim \|\omega_0 - 1_{B_1}\|_J^{1/2} + \|\omega_0 - 1_{B_1}\|_J.$$

Remark

The previous stability results had dependences on either the size of the domain (or the support of ω_0) or the supremum of ω_0, or both. Our result is independent on any information of the perturbed initial data ω_0. The L^∞-condition of ω_0 is only to guarantee the uniqueness of $\omega(t)$.

Proposition 6 (Adaptation of Marchioro–Pulvirenti '85 to patches on $\mathbb{R}^2$)

Let $R > 0$. Then $\forall$ compact set $\Omega_0 \subset B_R$, we have

$$\sup_{t\geq 0} \|1_{\Omega_t} - 1_{B_1}\|_{L^1(\mathbb{R}^2)} \lesssim_R \|1_{\Omega_0} - 1_{B_1}\|^{1/2}_{L^1(\mathbb{R}^2)} + \|1_{\Omega_0} - 1_{B_1}\|_{L^1(\mathbb{R}^2)}.$$

Symmetric rearrangement Ω^* of a finite-measured measurable set Ω in $\mathbb{R}^2$

- $\Omega^* := B_{\sqrt{\frac{|\Omega|}{\pi}}} = \{|x| < \sqrt{\frac{|\Omega|}{\pi}}\}$, i.e., $|\Omega^*| = |\Omega|$.
- $\mathcal{J}(1_{\Omega^*}) \leq \mathcal{J}(1_\Omega)$.

$$\because \mathcal{J}(1_\Omega) - \mathcal{J}(1_{\Omega^*}) = \int_{\Omega\setminus\Omega^*} |x|^2 dx - \int_{\Omega^*\setminus\Omega} |x|^2 dx \geq \frac{|\Omega|}{\pi} \cdot |\Omega \setminus \Omega^*| - \frac{|\Omega|}{\pi} \cdot |\Omega^* \setminus \Omega| = 0.$$

Properties of rearrangements

- Nonexpansivity:
$$\|1_{\Omega^*} - 1_{B_1}\|_{L^1(\mathbb{R}^2)} \leq \|1_{\Omega} - 1_{B_1}\|_{L^1(\mathbb{R}^2)}. \tag{2}$$
- Estimates of the difference of angular impulse between 1_Ω and 1_{Ω^*} (adaptation of Marchioro–Pulvirenti '85 for patch):
$$\|1_\Omega - 1_{\Omega^*}\|_{L^1(\mathbb{R}^2)} \lesssim [\mathcal{J}(1_\Omega) - \mathcal{J}(1_{\Omega^*})]^{1/2} \lesssim_R \|1_\Omega - 1_{\Omega^*}\|^{1/2}_{L^1(\mathbb{R}^2)}. \tag{3}$$

Indeed, if we let $B_b = \Omega^*$ and $\beta := |\Omega \setminus \Omega^*| = |\Omega^* \setminus \Omega| = \frac{1}{2}\|1_\Omega - 1_{\Omega^*}\|_{L^1}$, then

$$\int_{\Omega\setminus\Omega^*} |x|^2 dx = \int_{\Omega\cup\Omega^*} |x|^2 dx - \int_{\Omega^*} |x|^2 dx \geq \int_{(\Omega\cup\Omega^*)^*} |x|^2 dx - \int_{\Omega^*} |x|^2 dx = \frac{2\pi}{4}(a_1^4 - b^4),$$
$$\int_{\Omega^*\setminus\Omega} |x|^2 dx = \int_{\Omega^*} |x|^2 dx - \int_{\Omega\cap\Omega^*} |x|^2 dx \leq \int_{\Omega^*} |x|^2 dx - \int_{(\Omega\cap\Omega^*)^*} |x|^2 dx = \frac{2\pi}{4}(b^4 - a_2^4),$$

where $\pi(a_1^2 - b^2) = |\Omega \setminus \Omega^*| = \beta = |\Omega^* \setminus \Omega| = \pi(b^2 - a_2^2)$. Thus,

$$\mathcal{J}(1_\Omega) - \mathcal{J}(1_{\Omega^*}) = \int_{\Omega\setminus\Omega^*} |x|^2 dx - \int_{\Omega^*\setminus\Omega} |x|^2 dx \geq \frac{\pi}{2}(a_1^4 + a_2^4 - b^4)$$
$$= \frac{\beta^2}{\pi} = \frac{1}{4\pi}\|1_\Omega - 1_{\Omega^*}\|^2_{L^1}.$$

Key ideas

- Symmetric rearrangement Ω^* of $\Omega \subset \mathbb{R}^2$ is the disc s.t. $|\Omega^*| = |\Omega| < \infty$.
- Basic property : $\mathcal{J}(1_{\Omega^*}) \leq \mathcal{J}(1_\Omega)$.
- Properties of the solution 1_{Ω_t} of (1) with initial data 1_{Ω_0}:

$$\mathcal{J}(1_{\Omega_t}) = \mathcal{J}(1_{\Omega_0}), \quad (\Omega_t)^* = (\Omega_0)^*, \quad \forall t \geq 0.$$

- Nonexpansivity:

$$|\Omega^* \triangle B_1| \leq |\Omega \triangle B_1|.$$

- Estimates of the difference of angular impulse between 1_Ω and 1_{Ω^*} (adaptation of Marchioro–Pulvirenti '85 for patch); if $\Omega \subset B_R$ for some $R > 0$, then

$$|\Omega \triangle \Omega^*| \lesssim [\mathcal{J}(1_\Omega) - \mathcal{J}(1_{\Omega^*})]^{1/2} \lesssim_R |\Omega \triangle \Omega^*|^{1/2}.$$

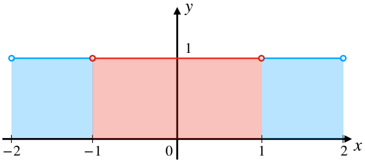
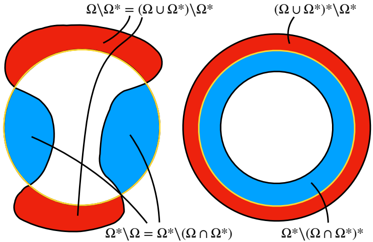


Figure: Nonexpansivity in 1D



$$\mathcal{J}(1_{(\Omega\cup\Omega^*)^*}) \leq \mathcal{J}(1_{\Omega\cup\Omega^*}), \; \mathcal{J}(1_{(\Omega\cap\Omega^*)^*}) \leq \mathcal{J}(1_{\Omega\cap\Omega^*}).$$

Figure: Estimating the difference of $\mathcal{J}$ between 1_Ω and 1_{Ω^*}

Sketch of the proof of Proposition 6

- We use the conservation $\mathcal{J}(1_{\Omega_t}) = \mathcal{J}(1_{\Omega_0})$ and the property $\Omega_t^* = \Omega_0^*$:

$$\| \underbrace{1_{\Omega_t^*}}_{=1_{\Omega_0^*}} - 1_{B_1} \|_{L^1} \leq \|1_{\Omega_0} - 1_{B_1}\|_{L^1}, \quad (\because (2))$$

$$\begin{aligned} \|1_{\Omega_t} - \underbrace{1_{\Omega_t^*}}_{=1_{\Omega_0^*}} \|_{L^1} &\lesssim [\underbrace{\mathcal{J}(1_{\Omega_t})}_{=\mathcal{J}(1_{\Omega_0})} - \mathcal{J}(1_{\Omega_0^*})]^{1/2} \lesssim_R \|1_{\Omega_0} - 1_{\Omega_0^*}\|_{L^1}^{1/2} \quad (\because (3)) \\ &\leq \|1_{\Omega_0} - 1_{B_1}\|_{L^1}^{1/2} + \|1_{B_1} - 1_{\Omega_0^*}\|_{L^1}^{1/2} \leq 2\|1_{\Omega_0} - 1_{B_1}\|_{L^1}^{1/2}. \end{aligned}$$

- Hence, we have

$$\|1_{\Omega_t} - 1_{B_1}\|_{L^1} \leq \|1_{\Omega_t} - 1_{\Omega_t^*}\|_{L^1} + \|1_{\Omega_t^*} - 1_{B_1}\|_{L^1} \lesssim_R \|1_{\Omega_0} - 1_{B_1}\|_{L^1}^{1/2} + \|1_{\Omega_0} - 1_{B_1}\|_{L^1}.$$

□

Properties of rearrangements

Symmetric-decreasing rearrangement f^* of a non-negative function $f \in L^1(\mathbb{R}^2)$:

$$\{x \in \mathbb{R}^2 : f^*(x) > c\} = \{x \in \mathbb{R}^2 : f(x) > c\}^* \quad \forall c > 0.$$

Note

- f^* is radial and non-increasing.
- $|\{f^* > c\}| = |\{f > c\}| \ \forall c > 0, \quad \|f^*\|_{L^1(\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2)}, \quad \mathcal{J}(f^*) \leq \mathcal{J}(f).$

Generalizations of key lemmas

Lemma 7 (Nonexpansivity)

Let $f, g \in L^1(\mathbb{R}^2)$ be non-negative with $g = g^*$. Then

$$\|f^* - g\|_{L^1(\mathbb{R}^2)} \leq \|f - g\|_{L^1(\mathbb{R}^2)}.$$

Lemma 8 (Adaptation of Marchioro–Pulvirenti '85)

Let $f \in L^\infty(\mathbb{R}^2)$ be non-negative with $\mathcal{J}(f) < \infty$. Then

$$\|f - f^*\|_{L^1(\mathbb{R}^2)}^2 \lesssim \|f\|_{L^\infty(\mathbb{R}^2)} \cdot [\mathcal{J}(f) - \mathcal{J}(f^*)].$$

Remark

To avoid the stability's dependence on $\|\omega_0\|_{L^\infty(\mathbb{R}^2)}$, we use the *cut-off operator* Γ:

$$(\Gamma f)(x) := \begin{cases} f(x) & \text{if } f(x) \leq 2, \\ 2 & \text{if } f(x) > 2. \end{cases}$$

Then the rearrangement and the cut-off operator commute with each other:

$$\Gamma(f^*) = (\Gamma f)^*.$$

In addition, we can estimate the measure of the region $\{f > 2\}$ as

$$|\{f > 2\}| = \int_{\{f>2\}} 1 dx \leq \int_{\{f>2\}} |f - 1| dx \leq \int_{\{f>2\}} |f - 1_{B_1}| dx \leq \|f - 1_{B_1}\|_{L^1(\mathbb{R}^2)}.$$

Sketch of the proof of Theorem 1

- We decompose the L^1-norm of the perturbation using the cut-off operator Γ as

$$\|\omega - 1_{B_1}\|_{L^1(\mathbb{R}^2)} = \|\omega - 1_{B_1}\|_{L^1(\{\omega>2\})} + \|\Gamma\omega - 1_{B_1}\|_{L^1(\{\omega\le 2\})}.$$

- For the upper part, we have

$$\begin{aligned}\|\omega - 1_{B_1}\|_{L^1(\{\omega>2\})} &\le \int_{\{\omega>2\}} \omega dx + \int_{\{\omega>2\}} 1_{B_1} dx = \int_{\{\omega_0>2\}} \omega_0 dx + \int_{\{\omega>2\}} 1_{B_1} dx \\ &\le \int_{\{\omega_0>2\}} |\omega_0 - 1_{B_1}| dx + \int_{\{\omega_0>2\}} 1_{B_1} dx + \int_{\{\omega>2\}} 1_{B_1} dx \\ &\le \|\omega_0 - 1_{B_1}\|_{L^1(\mathbb{R}^2)} + |\{\omega_0 > 2\}| + |\{\omega > 2\}| \lesssim \|\omega_0 - 1_{B_1}\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

- For the lower part, we get

$$\begin{aligned}\|\Gamma\omega - 1_{B_1}\|_{L^1(\{\omega\le 2\})} &\le \|\Gamma\omega - \underbrace{(\Gamma\omega)^*}_{=(\Gamma\omega_0)^*}\|_{L^1(\mathbb{R}^2)} + \underbrace{\|(\Gamma\omega)^* - 1_{B_1}\|_{L^1(\mathbb{R}^2)}}_{\le \|\Gamma\omega_0 - 1_{B_1}\|_{L^1(\mathbb{R}^2)}} \\ &\lesssim \left[\mathcal{J}(\Gamma\omega) - \mathcal{J}(\underbrace{(\Gamma\omega_0)^*}_{=\Gamma[(\omega_0)^*]})\right]^{1/2} + \|\omega_0 - 1_{B_1}\|_{L^1(\mathbb{R}^2)} \le \cdots \\ &\lesssim \mathcal{J}(|\omega_0 - 1_{B_1}|)^{1/2} + \|\omega_0 - 1_{B_1}\|_{L^1(\mathbb{R}^2)}^{1/2} + \|\omega_0 - 1_{B_1}\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

- Lastly, we have

$$\mathcal{J}(|\omega - 1_{B_1}|) \le \cdots \le 2\|\omega - 1_{B_1}\|_{L^1(\mathbb{R}^2)} + \mathcal{J}(|\omega_0 - 1_{B_1}|).$$

Related works on various instabilities of vortex solutions

- Nadirashvili '91 : Construction of a C^1-unstable smooth solution on an annulus domain, where two points on inner and outer circle have different tangential velocities for all time.
- Choi–Jeong '22 : Construction of a patch in $\mathbb{R}^2$, showing perimeter growth for any large finite time.

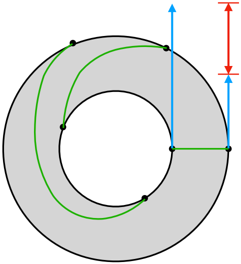


Figure: A schematic diagram of Nadirashvili '91

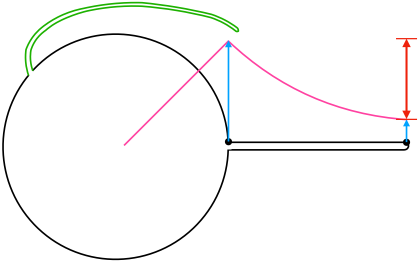


Figure: A schematic diagram of Choi–Jeong '22

Remark

These tell us about growth of a line or a boundary component, but these does not tell us about their precise form throughout time.

- Growth of support size : Choi–Denisov '19, Choi–Jeong '22, etc.

Instability

- Choi–Denisov '19 : Growth of support size of a non-negative $\omega \in L^\infty(S)$ in x_1-direction with rate $\mathcal{O}(t^{1/3} \ln^2 t)$.
- Choi–Jeong '22 : Filamentation for perturbation of Lamb dipole in $\mathbb{R}^2$ (growth of $\nabla\omega$ and support size for all time).

Key ideas

The dynamics of $1_{\overline{\Phi}_t(\overline{\Omega})}$ in $\mathbb{R}^2_+$

$$\overline{u}_{\text{ext}}: \text{ periodic extension of } \overline{u} \text{ on } \mathbb{R}^2_+, \quad \overline{\Phi}_t: \text{ flow map on } \mathbb{R}^2_+ \text{ induced from } \overline{u}_{\text{ext}}.$$

- The velocity field $\overline{u}$ determined by $1_{\overline{\Omega}}$.

$$\overline{u}^1(x) = -\partial_2\psi(x) = 0 \quad (\because \psi = \psi(x_1)),$$

$$\overline{u}^2(x) = \partial_1\psi(x) = -\psi'(s)\Big|_{s=x_1}^{s=\infty} = -\int_{x_1}^{\infty} \underbrace{\psi''(s)}_{=\Delta\psi(s)}\, ds = -\int_{x_1}^{\infty} 1_{(0,1)}(s)ds$$

$$= \begin{cases} 1 - x_1 & \text{if } 0 \le x_1 < 1, \\ 0 & \text{if } x_1 \ge 1. \end{cases} \quad (\because \psi|_{x_1=0} = 0,\ \partial_1\psi \to 0 \text{ as } x_1 \to \infty,\ |\psi(t,x)| \lesssim x_1 + 1.)$$

- The growth rate of the vertical center of mass of $1_{\overline{\Phi}_t(\overline{\Omega})}$ in $\mathbb{R}^2_+$.

$$k(f) := \left(\int_{\mathbb{R}^2_+} f(x)dx\right)^{-1} \cdot \int_{\mathbb{R}^2_+} x_2 f(x)dx.$$

$$\Rightarrow \frac{d}{dt}k(1_{\overline{\Phi}_t(\overline{\Omega})}) = \frac{d}{dt}\left[\left(\int_{\overline{\Phi}_t(\overline{\Omega})} 1dx\right)^{-1} \cdot \int_{\overline{\Phi}_t(\overline{\Omega})} x_2 dx\right] = \frac{1}{|\overline{\Omega}|}\int_{\overline{\Omega}} \frac{d}{dt}\overline{\Phi}^2_t(y)dy$$

$$= \frac{1}{2\pi}\int_{\overline{\Omega}} \overline{u}^2_{\text{ext}}(y)dy = \frac{1}{2\pi}\int_{\overline{\Omega}} \overline{u}^2(y)dy = \int_0^1 (1 - y_1)dy_1 = \frac{1}{2}.$$

Global regularity of some high-dimensional axisymmetric Euler flows

- Although $d \geq 4$ is not physical, there are several works suggesting possibility of a finite-time blow-up of smooth solutions of axisymmetric, swirl-free Euler equations in higher dimensions.
- (Choi–Jeong–L., to appear) In $d = 4$, if ω_0 vanishes at $r = 0$ and has some decay at infinity, then
$$\|\omega(t)\|_{L^\infty(\mathbb{R}^4)} \lesssim_{\omega_0} e^{Ct} \quad \forall t \geq 0.$$
Also, for $d \leq 7$, if ω_0 is single-signed and compactly supported, then
$$\|\omega(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim_{\omega_0} \begin{cases} (1+t)^{\frac{4(d-2)}{7-d}}, & d = 4, 5, 6, \\ e^{C_4 t}, & d = 7, \end{cases} \quad \forall t \geq 0.$$
- (L., preprint) $\forall$ $d \geq 3$, if ω_0 is single-signed and compactly supported, then
$$\|\omega(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim_{\omega_0} [(1+t)\ln(e+t)]^{(d-2)/(d+1)} \quad \forall t \geq 0.$$
- (Open question by Drivas–Elgindi) Can singularities form from smooth data for the axisymmetric no swirl Euler equations on $\mathbb{R}^d$ when $d \geq 4$?

Ongoing projects

Half cylinder

- Construction of a vortex solution which shows gradient growth for all time.
- Finding growth rate of the horizontal size of support of a vortex solution with compact support.

Bi-rotational flow in 4D coordinates (r, θ, s, ϕ)

- Obtaining upper or lower bound of radial impulse of the flow.