Instability and non-uniqueness in the partial differential equations of fluid dynamics

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... describe the motion of a viscous, incompressible fluid.

Conservation of Momentum

$$\underbrace{\partial_t u + (u \cdot \nabla) u}_{\textit{acceleration}} - \nu \Delta u + \frac{1}{\varrho} \nabla p = 0$$

- $u(x,t)$ is the *fluid velocity field*
- $p(x,t)$ is the (scalar) *pressure*
- $\nu > 0$ is the *viscosity*
- $\varrho = \text{constant} > 0$ is the *density*

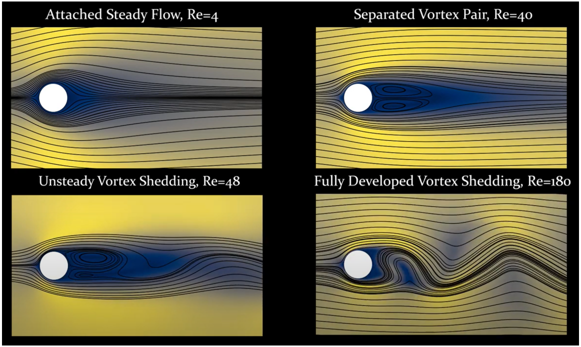
Incompressibility

$$\operatorname{div} u = 0$$

The *Reynolds number*

$$\mathrm{Re} = \frac{LU}{\nu} \qquad \approx \qquad \frac{|(u \cdot \nabla)u|}{|\nu \Delta u|}$$

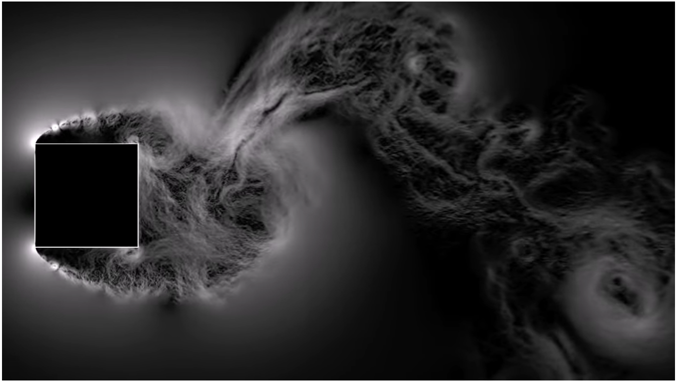
is a dimensionless number which roughly characterizes the flow regime.



The onset of vortex shedding in flow around a cylinder, from *Dynamics and Control of Flow around Circular Cylinder*, by Ramsay, Sellier, and Ho, in the 2019 APS DFD Gallery of Fluid Motion. Available on YouTube.

Vortex shedding at $Re = 22000$

Three-dimensional turbulent flows develop features at length scales $L \times \mathrm{Re}^{-3/4}$ and, hence, are extremely difficult to resolve.



Turbulent flow around a square cylinder at Reynolds number 22000: a DNS study, by Triasa, Gorobetsa, and Oliva (Computers & Fluids, 2015). Available on YouTube.

A rapid tour of the PDE theory of the Navier-Stokes equations

The Navier-Stokes equations are expected to be a self-consistent deterministic model capable of *predicting* the motion of a fluid.

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p = f \\ \operatorname{div} u = 0 \end{cases} \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+ \qquad \text{(NS)}$$

(Q) Are the Navier-Stokes equations well-posed *globally in time*?

- Local-in-time well-posedness in L^p, $p > 3$

  Heuristic: $u \sim \frac{\sigma}{|x|^\kappa} \quad \Longrightarrow \quad u \cdot \nabla u \sim \frac{\sigma^2}{|x|^{2\kappa+1}}, \quad \Delta u \sim \frac{\sigma}{|x|^{\kappa+2}}$

  When $\kappa < 1$ ($p > 3$), the viscous term wins at small scales.

- The energy balance ($f = 0$, for simplicity)

  $$\frac{1}{2}\int |u(x,t)|^2\,dx + \int_0^t \int |\nabla u|^2\,dx\,ds = \frac{1}{2}\int |u_0(x)|^2\,dx\,.$$

  is not strong enough to prove global well-posedness.

For each $u_0 \in L^2$ ($\operatorname{div} u_0 = 0$) and $f \in L^1_t L^2_x$, there exists a global-in-time *Leray-Hopf solution* (Leray, Acta 1934), (Hopf, Math. Nachr. 1951):

$$u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(\mathbb{R}^3 \times \mathbb{R}_+)$$

- solves (NS) for some pressure p,
- attains the initial data u_0, and
- satisfies the energy inequality for all $t > 0$:

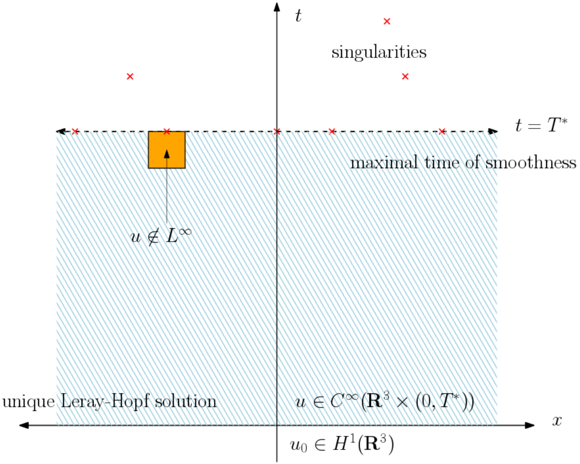
$$\frac{1}{2}\int |u(x,t)|^2\,dx + \int_0^t \int |\nabla u|^2\,dx\,ds \leq \frac{1}{2}\int |u_0(x)|^2\,dx + \int_0^t \int f \cdot u\,dx\,ds\,.$$

Suitable weak solutions further satisfy the local energy inequality:

$$(\partial_t - \Delta)\frac{1}{2}|u|^2 + |\nabla u|^2 + \operatorname{div}\left[\left(\frac{1}{2}|u|^2 + p\right)u\right] \leq f \cdot u$$

and *partial regularity* (CKN, CPAM 1982) when $f \in L^{5/2+}$:

Singularities cannot fill a curve in spacetime.



Weak solutions become relevant if the strong solutions break down.

Weak solutions: a success story

- (Kruzhkov 1970) In spite of shock discontinuities, for each $u_0 \in L^\infty$, there exists a **unique** *entropy solution* of Burgers equation

$$\partial_t u + u\partial_x u = 0\,. \tag{B}$$

- Viscous regularization converges to the unique entropy solution:

$$\partial_t u + u\partial_x u = \varepsilon\Delta u\,. \tag{B_ε}$$

- A dispersive regularization need not (Lax–Levermore, CPAM 1983):

$$\partial_t u + u\partial_x u + \varepsilon^2\partial_x^3 u = 0\,. \tag{KdV_ε}$$

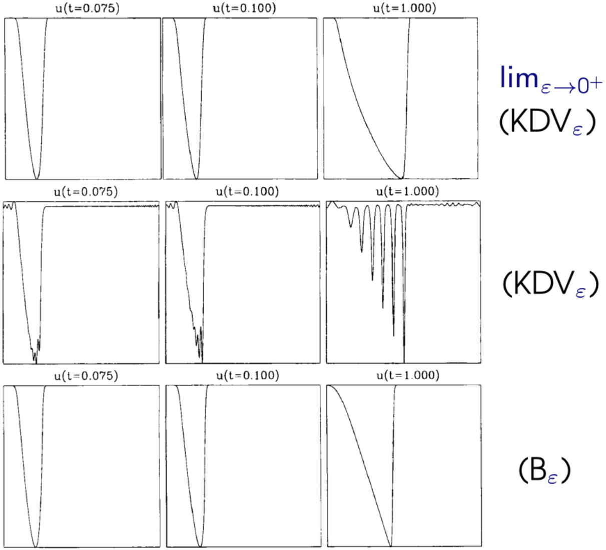


Figure 1. Weak limit, KdV solution and Burgers profile with $\varepsilon = 0.08$. The interval is $[-2, 10]$ and each plot is scaled to fit the solution. The break time for this solution is $t_b = 1/12 = 0.08666\ldots$.

(McLaughlin & Strain, Computing the Weak Limit of KdV, CPAM 1994).

NB: The PDE convention here is $\partial_t u - 6u\partial_x u$ rather than $\partial_t u + u\partial_x u$.

In 1969, Ladyzhenskaya constructed an example of non-uniqueness for (NS) within a Leray–Hopf-type class with caveats.

> The example described here can provoke "displeasure" for only one reason. It has been constructed for boundary conditions of type (18) but not for adhesion conditions...
>
> The examples presented here are interesting to me in that they refute the entrenched opinion on the "naturalness" for nonstationary problems of physics and mechanics of the class of solutions which have finite energy norm.

Recent progress.

- (Jia-Šverák, Inventiones 2014, JFA 2015) and (Guillod-Šverák, arXiv 2017) produced compelling numerical evidence of non-unique Leray solutions but no "proof."
- Convex integration constructions of (Buckmaster-Vicol, Ann. Math. 2019) in $C_t H_x^\beta$ (small β) *without energy inequality*. Proof, but no Leray solutions.

Theorem (A.-Brué-Colombo, Ann. Math. 2022)

There exists a one-parameter family of distinct suitable Leray–Hopf solutions to the Navier–Stokes equations with identical body force $f \in L^1_t L^2_x$ *and identical initial velocity* $u_0 \equiv 0$.

Self-similarity

- (NS) has a one-parameter *scaling symmetry* :

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)\,, \quad f_\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t)\,, \quad \lambda > 0\,.$$

- *Self-similar solutions* are invariant under the scaling symmetry:

$$u(x,t) = \frac{1}{\sqrt{t}} U(\xi), \quad f(x,t) = \frac{1}{t^{\frac{3}{2}}} F(\xi)\,, \quad \xi = \frac{x}{\sqrt{t}}\,.$$

(Leray 1934) proposed self-similar singularity formation

- The Navier-Stokes equations for the *similarity profile* U are

$$\underbrace{-\frac{1}{2}\left(1 + \xi \cdot \nabla_\xi\right) U}_{\text{additional terms}} - \Delta U + U \cdot \nabla U + \nabla P = F, \quad \operatorname{div} U = 0 \qquad \text{(NS-SS)}$$

with *boundary condition*

$$|U - u_0| = o\left(\frac{1}{|\xi|}\right)\,,$$

where u_0 is -1-homogeneous.

- Fix a_0 divergence-free and -1-homogeneous and $f = 0$.
- Increase size σ of initial data $u_{0,\sigma}$,

$$u_{0,\sigma} = \sigma a_0 , \quad \sigma \approx \text{ Reynolds number} \geq 0 .$$

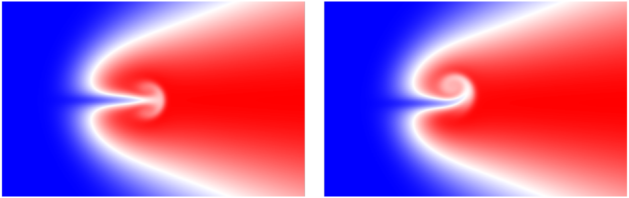
- Extract curve-in-σ of self-similar solutions (Jia-Šverák, Invent. 2014).

Theorem (Jia-Šverák, JFA 2015)

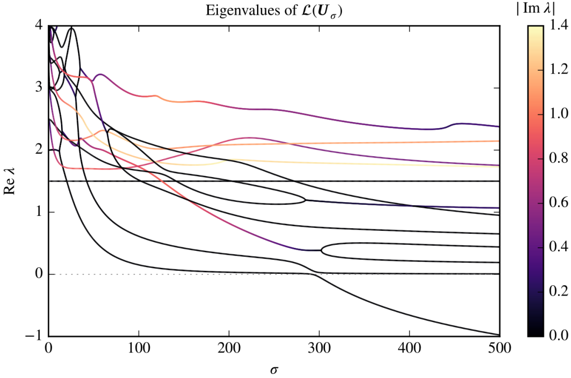
Suppose *bifurcation (saddle-node, Hopf...). Then, upon truncating properly, there exist two distinct Leray-Hopf solutions with identical compactly supported data* u_0*, and* $|u_0| = O(1/|x|)$ *at* $x = 0$*.*

- *Linearized operator* around self-similar solution U_σ:

$$-\boldsymbol{L}_\sigma U = -\frac{1}{2}(1 + \xi \cdot \nabla_\xi) U - \Delta U + \mathbb{P}(U_\sigma \cdot \nabla U + U \cdot \nabla U_\sigma) .$$



$|x|U_\sigma \cdot e_\theta$ for two solutions when $\sigma \approx 300$ (Guillod-Šverák, arXiv 2017). See video on Guillod's website.



Low-lying eigenvalues of the linearized operator L_σ (Guillod-Šverák, arXiv 2017)

- Self-similar solutions are *steady states* of the evolutionary PDE

$$\partial_\tau U \underbrace{-\frac{1}{2}(1+\xi\cdot\nabla_\xi)\,U}_{\text{additional terms}} - \Delta U + U\cdot\nabla U + \nabla P = F, \quad \operatorname{div} U = 0\,. \qquad \text{(NS-SS)}$$

- This is (NS) in *self-similarity variables*

$$\xi = \frac{x}{\sqrt{t}}, \quad \tau = \log t \in \mathbb{R} \quad (\tau \to -\infty \iff t \to 0^+)$$

$$u(x,t) = \frac{1}{\sqrt{t}} U(\xi,\tau), \quad f(x,t) = \frac{1}{t^{\frac{3}{2}}} F(\xi,\tau)$$

(Giga-Kohn 1980s) semilinear heat equation

- $U = O(1)$, smooth, decaying $\implies$ u in critical spaces, e.g., $L^\infty_t L^{3,\infty}_x$

The Jia-Šverák-Guillod picture is difficult to verify analytically.

We incorporate a *body force* to add flexibility to the problem.

Theorem (A.-Brué-Colombo, Ann. Math. 2022)

Suppose there exist $\bar{U} \in C_0^\infty(\mathbb{R}^3)$ $(\operatorname{div} \bar{U} = 0)$ *satisfying*

- *(Linear instability)* $\boldsymbol{L}_{\mathrm{ss}}$ *has an unstable eigenvalue.*

Then there exists a non-trivial solution U *on the* ***unstable manifold*** *of* $\bar{U}$*, that is,* $U \overset{\tau \to -\infty}{\longrightarrow} \bar{U}$ *exponentially backward-in-*$\log$ *time, and*

$$\bar{u} = \frac{1}{\sqrt{t}} \bar{U}(\xi), \quad u = \frac{1}{\sqrt{t}} U(\xi, \tau)$$

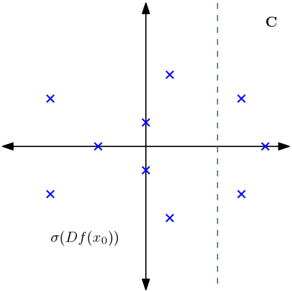
are the desired distinct suitable Leray-Hopf solutions with zero initial velocity and identical force $\bar{f}$*, whose similarity profile is*

$$\bar{F} := -\frac{1}{2}(1 + \xi \cdot \nabla_\xi)\bar{U} - \Delta \bar{U} + \bar{U} \cdot \nabla \bar{U} .$$

“Non-uniqueness in backward time”

$\dot{x} = f(x) \quad \ldots \quad$ equilibrium $f(x_0) = 0$

Linearized equation : $\quad \dot{y} = (Df)(x_0)y$



Unstable manifold M_u

- All trajectories $x(\tau) \overset{\tau \to -\infty}{\longrightarrow} x_0$ with certain exponential rate
- $\dim M_u = \dim E_u$
- Generalization to semilinear parabolic PDEs: (Henry 1981)

Finding unstable solutions

Claim. To complete the argument, it is enough to find a smooth, decaying steady state to the Euler equations in three dimensions,

$$\partial_t u + u \cdot \nabla u + \nabla p = f \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+ \,, \tag{E_{3d}}$$

unstable due to an unstable eigenvalue of the linearized operator.

Heuristic.

$$-\boldsymbol{L}_{\mathrm{ss}}^{(\beta)} U = \underbrace{-\frac{1}{2}\left(1 + \xi \cdot \nabla_\xi\right) U - \Delta U}_{\text{perturbative when } \beta \gg 1} + \beta \mathbb{P}\left(\bar{U} \cdot \nabla U + U \cdot \nabla \bar{U}\right)$$

Remarkably, no suitable steady state was known, and a key component and major difficulty of our proof is its construction!

The *two-dimensional* Euler equations

$$\partial_t \omega + u \cdot \nabla \omega = 0\,, \quad \Delta \psi = \omega\,, \quad u = \nabla^\perp \psi \qquad (E_{2d})$$

have a family of steady states known as *vortices*,

$$\bar{u}(x) = \zeta(r) x^\perp, \quad \bar{\omega}(x) = \bar{\omega}(r), \quad x \in \mathbb{R}^2,\ r = |x|\,.$$

Theorem (Vishik, arXiv ’18)

There exists a smooth unstable vortex $\bar{\omega}$. Its velocity profile $\bar{u}$ can be chosen to be compactly supported (ABC 2022).

Remarks.

- Instability of shear flows explored in (Tollmien 1935), (Lin, SIMA 2002)
- Vishik’s mechanism is the same and, to our knowledge, one of the *only known mechanisms* for generating unstable eigenvalues.

The unstable two-dimensional vortex is a key ingredient in

Theorem (Sharpness of the Yudovich class, Vishik, arXiv '18)

For every $p \in (2, +\infty)$, there exist two distinct finite-energy weak solutions $\bar{u}, u$ of the Euler equations

$$\partial_t \omega + u \cdot \nabla \omega = \operatorname{curl} f\,, \quad \Delta\psi = \omega\,, \quad u = \nabla^{\perp}\psi \quad \text{on } \mathbb{R}^2 \times (0,1) \qquad (E_{2d})$$

satisfying

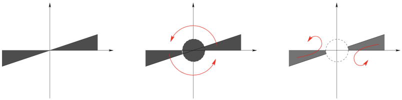
$$\bar{\omega}, \omega \in L^\infty_t (L^1 \cap L^p)_x,$$

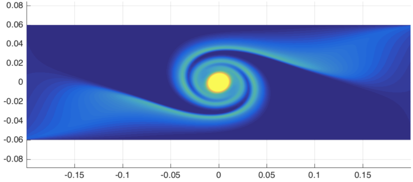
with zero initial velocity (ABCDGJK, arXiv '21) and identical body force

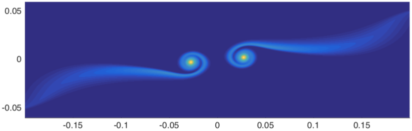
$$f \in L^1_t L^2_x, \quad \operatorname{curl} f \in L^1_t (L^1 \cap L^p)_x\,.$$

(Bressan-Shen '21), (Bressan-Murray '20), inspired by (Elling), (Pullin)

$$\omega_0(x) = \frac{1}{r^{\frac{1}{\alpha}}}\phi\left(\frac{x}{|x|}\right) \quad (-\alpha)\text{-homogeneous}$$

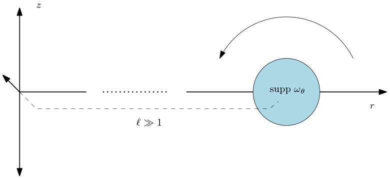






Steam vortex ring above Mount Etna, Italy

Our unstable three-dimensional object will be a *vortex ring*.



Vortex ring in axisymmetric coordinates

Key observation

As $r \to +\infty$, the axisymmetric Euler equations without swirl formally converge to the two-dimensional Euler equations.

I. Axisymmetric vorticity equation ($\omega = -\omega^\theta(r,z)e_\theta, \psi = \psi^\theta(r,z)e_\theta$):

$$\partial_t \omega^\theta + u \cdot \nabla \omega^\theta - \frac{u^r}{r}\omega^\theta = 0$$

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} + \partial_z^2\right)\psi^\theta = \omega^\theta$$

$$u = -\partial_z \psi^\theta e_r + \left(\partial_r + \frac{1}{r}\right)\psi^\theta e_z$$

II. Two-dimensional vorticity equation

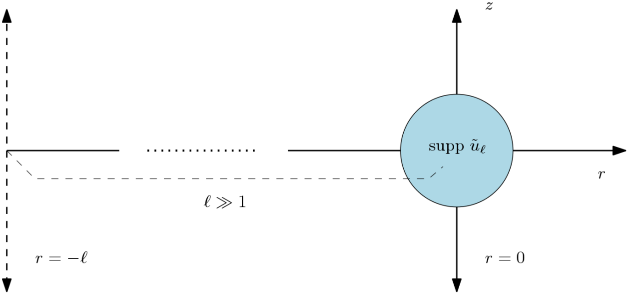
$$\partial_t \omega + u \cdot \nabla \omega = 0$$

$$\Delta_{x,y}\psi = \omega, \quad u = \nabla^\perp \psi$$

- Recenter coordinates on the vortex, so $r > -\ell$.
- Let $\tilde{u}$ be a truncation of Vishik's unstable vortex.

  $\rightsquigarrow$ This is the vortex core ($x \to r$, $y \to z$).
- Correction v_ℓ keeps $\tilde{u}_\ell$ divergence-free in the physical variables:

$$\tilde{u}_\ell = \tilde{u} + v_\ell, \quad \tilde{\omega}_\ell = \mathrm{curl}_\ell\, \tilde{u}_\ell := -\partial_z \tilde{u}^r_\ell + \partial_r \tilde{u}^z_\ell\,.$$



Linearized operators

- Linearization about the 3d vortex ring at distance ℓ:

$$-\boldsymbol{L}_\ell\omega := \underbrace{\tilde{u}_\ell \cdot \nabla\omega}_{=:-\boldsymbol{M}_\ell\omega} + \underbrace{\mathrm{BS}_\ell[\omega]\cdot\nabla\tilde{\omega}_\ell}_{=:-\boldsymbol{K}_\ell\omega}$$
$$\underbrace{-(r+\ell)^{-1}\mathrm{BS}_\ell[\omega]^r\tilde{\omega}_\ell - (r+\ell)^{-1}\tilde{u}_\ell^r\omega}_{=:-\boldsymbol{S}_\ell\omega}$$

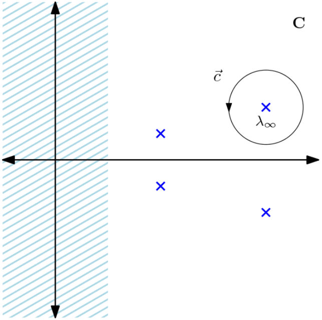
- Linearization about the 2d vortex:

$$-\boldsymbol{L}_\infty\omega := \underbrace{\tilde{u}\cdot\nabla\omega}_{=:-\boldsymbol{M}_\infty\omega} + \underbrace{\mathrm{BS}_{2d}[\omega]\cdot\nabla\tilde{\omega}}_{=:-\boldsymbol{K}_\infty\omega}$$

Proposition (Axisymmetric instability)

Let λ_∞ *be an unstable eigenvalue of* $\boldsymbol{L}_\infty$. *For all* $\varepsilon \in (0, \operatorname{Re}\lambda_\infty)$ *and* $\ell \gg_{\tilde{u},\varepsilon,\lambda_\infty} 1$, $\boldsymbol{L}_\ell$ *has an unstable eigenvalue* λ_ℓ *with* $|\lambda_\ell - \lambda_\infty| < \varepsilon$.

Spectral projection $\mathrm{Pr}_\ell \omega := \frac{1}{2\pi i} \int_{\vec{c}} R(\lambda, \boldsymbol{L}_\ell)\omega \, d\lambda$



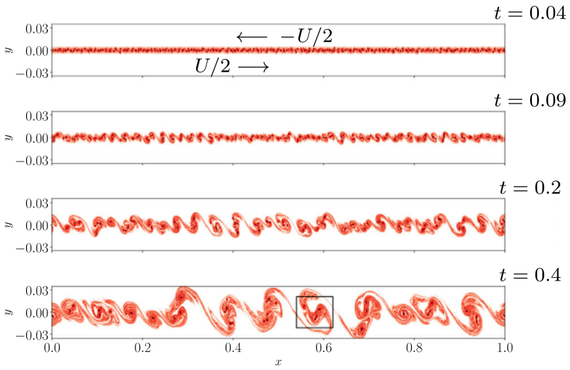
Claim: $R(\lambda, \boldsymbol{L}_\ell P_\ell)\omega \to R(\lambda, \boldsymbol{L}_\infty)\omega$ in L^2_γ, $\forall \omega \in C_0^\infty(\mathbb{R}^2)$, uniformly on $\vec{c}$.

Conclusion

Toward a more perfect picture

- Is it possible to remove the force?
  - Interactions with computer-assisted techniques
- Non-unique continuations of blow-ups?
  - Minimal mass blow-up in focusing nonlinear Schrödinger equation (Merle, CPAM 1992)
  - Bubbling and reverse bubbling in harmonic map heat flow (Davila–Del Pino–Wei, Invent. 21)
  - Generalized self-similar blow-up and continuation in complex Ginzburg-Landau equation (Lessons from CGL, Online Lecture by Šverák, 2020)
- Non-existence of a selection principle via regularization?
- What would the implications of non-uniqueness be for physics?
  - Case study: *Non-uniqueness in inviscid limit to vortex sheet*

Kelvin-Helmholtz instability



(Thalabard-Bec-Mailybaev, Communications Physics 2020)

- *Regularize*: Hyperviscous Navier-Stokes with viscosity ν
- *Randomize*: (vortex sheet) $+$ ($\varepsilon \times$ random perturbation)

As $\nu \sim \varepsilon \to 0^+$, a non-trivial *measure* of solutions emerges with (maybe) universal macroscopic properties $\rightsquigarrow$ spontaneous stochasticity

Conclusion

Lorenz, "The predictability of a flow which possesses many scales of motion", 1969:

> *It is proposed that certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error.*

- He refers to a more extreme version of chaos than the "standard butterfly effect", i.e., sensitive dependence on initial conditions (in which the solution map is still well-defined and continuous).
- This is termed the "real butterfly effect" in (Palmer-Döring-Seregin, Nonlinearity 2014).
- Our understanding of this concept from the perspective of rigorous PDEs and dynamics is in its early stages!

Thank you!