Turbulence in fluids and PDEs

January 27^{th}-31^{st} **2020**

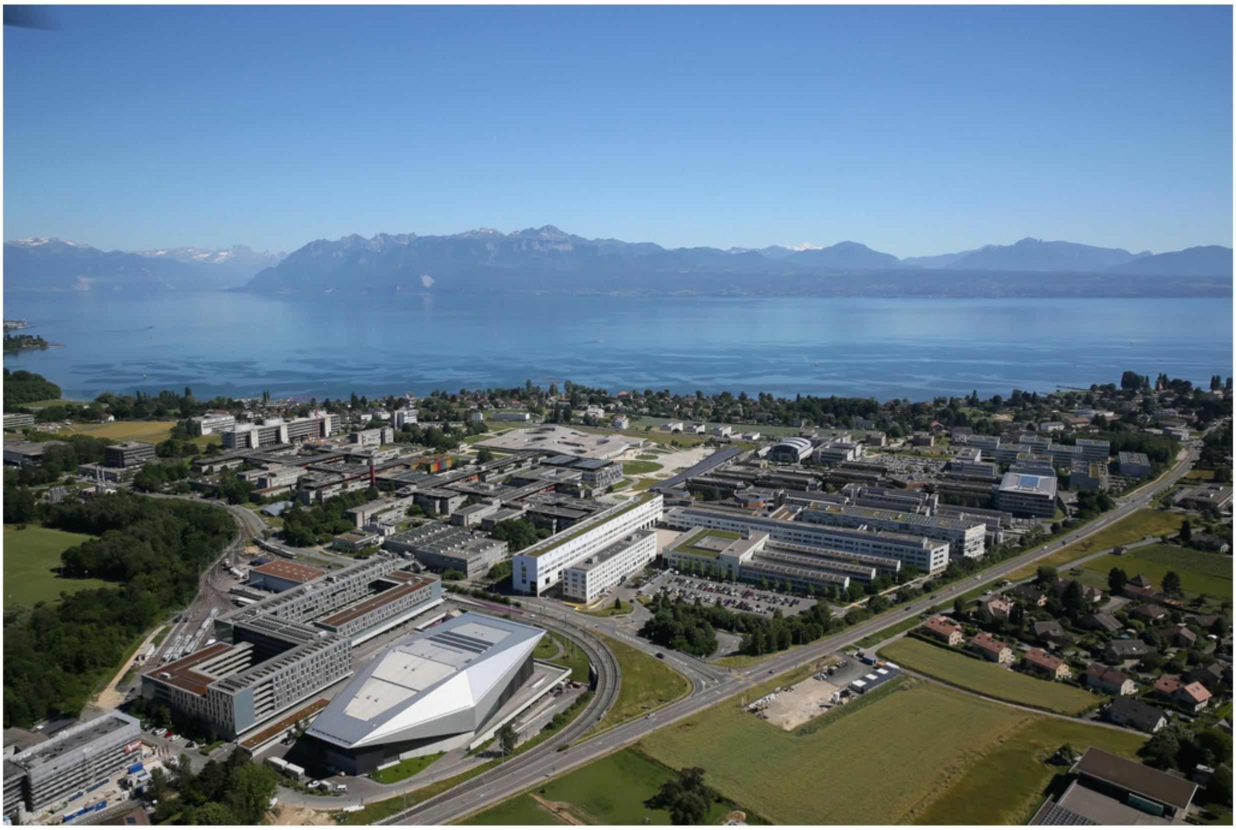

Figure 1: Courtesy of EPFL.

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Schedule

	Monday 27	Tuesday 28	Wednesday 29	Thursday 30	Friday 31
9:00 - 9:30	Registration	Saint-Raymond	Saint-Raymond	Vicol	Staffilani
9:30 - 10:30	Vicol	Saint-Raymond	Saint-Raymond	Vicol	Staffilani
10:30 - 11:00	Coffee break	Coffee break	Coffee break	Coffee break	Coffee break
11:00 - 11:30	Staffilani	Vicol	Staffilani	Saint-Raymond	Rodriguez
11:30 - 12:30	Staffilani	Vicol	Staffilani	Saint-Raymond	Sohinger
12:30 - 14:00	Lunch	Lunch	Lunch	Lunch	Lunch
14:00 - 15:00	Iacobelli	Saffirio		Coti Zelati	
15:00 - 15:30	Duerinckx	Novack		Beekie	
15:30 - 16:00	Coffee break	Coffee break		Coffee break	
16:00 - 16:30	Vicol	Grande Izquierdo		Staffilani	
16:30 - 17:00	Vicol	Saint-Raymond		Staffilani	
17:00 - 17:30	Vicol	Saint-Raymond			

Minicourses

Saint-Raymond, *Forcing and dissipation of internal waves.*

Stratification of the density in an incompressible fluid is responsible for the propagation of internal waves. In domains with topography, the interaction of these waves with the boundary produces a cascade towards small wavelengths. This phenomenology is reminiscent from turbulence.

Plan of the course:

- The WKB approximation
- Spectral analysis of forced internal waves
- Wave spectrum
- Stochastic forcing
- Viscous dissipation

Staffilani, *Deterministic and probabilistic methods in the study of the dynamics of periodic dispersive equations.*

In these lectures, using the periodic 2D cubic Schrödinger initial value problem I will present a variety of results and tools that have appeared in the study of dispersive equations. On one hand the periodicity of an initial value problem is a feature that often changes dramatically the behavior of its wave solutions, and as a consequence it affects the choice of mathematical tools used for their analysis. On the other hand periodicity represents the ideal set up in order to introduce a probabilistic approach that allows for precise description of generic properties for these wave solutions. More precisely in these lectures I will investigate the following questions concerning solutions $u(t,x)$ to the periodic 2D cubic Schrödinger initial value problem.

1. Long time dynamics: what do we know about $\lim_{t\to\infty} u(t,x)$? Here we will introduce the concept of forward cascade and polynomial bounds of Sobolev norms. We also introduce the very difficult problem of constructing solutions that exhibit a growth. Open problems will be listed along the way.
2. Dynamics near $t = 0$: here we analyze how one can make sense of the $\lim_{t\to 0} u(t,x)$ when the initial data of the problem are not very smooth. The question is a classical one in harmonic analysis, and for the linear solution in the Euclidean space only recently has been solved using beautiful harmonic analysis. I will consider both the linear and the nonlinear solution in the periodic case, I will construct counterexamples, and using probability I will show how generically these counterexamples are in fact rare. Open problems will be also listed.

Bibliography:

- Colliander, J.; Keel, M.; Staffilani, G.; Takaoka, H.; Tao, T. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. Invent. Math. 181 (2010), no. 1, 39–113.
- Carles, R.; Faou, E. Energy cascades for NLS on the torus. Discrete Contin. Dyn. Syst. 32 (2012), no. 6, 2063–2077.
- Bourgain, J. On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE. Internat. Math. Res. Notices 1996, no. 6, 277–304.
- Du, X.; Guth, L.; Li, X. A sharp Schrödinger maximal estimate in $\mathbb{R}^2$. Ann. of Math. (2) 186 (2017), no. 2, 607–640.
- Du, X.; Zhang. Sharp l2 estimates of the Schrödinger maximal function in higher dimensions. Annals of Mathematics, 189(3):837, 2019.

Vicol, *Convex integration and phenomenologies in turbulence.*

We will discuss a number of recent results concerning wild weak solutions of the incompressible Euler and Navier-Stokes equations. These results build on the groundbreaking works of De Lellis and Székelyhidi Jr., who extended Nash's fundamental ideas on flexible isometric embeddings, into the realm of fluid dynamics. These techniques, which go under the umbrella name convex "integration" have fundamental analogies with the phenomenological theories of hydrodynamic turbulence. Mathematical problems arising in turbulence (such as the Onsager conjecture) have not only sparked new interest in convex integration, but certain experimentally observed features of turbulent flows (such as intermittency) have also informed new convex integration constructions. Our goal is to introduce the audience to "Nash schemes" for constructing wild Hölder continuous weak solutions of the 3D Euler equations and to discuss the usage of intermittency in constructing finite energy weak solutions of 3D Navier-Stokes which attain any energy profile. Towards the end, we will discuss more recent developments in the field and will mention a number of open problems at the intersection of convex integration and hydrodynamic turbulence.

Bibliography:

Buckmaster, T.; Vicol, V. Convex integration and phenomenologies in turbulence. Available at https://arxiv.org/abs/1901.09023

Talks

Beekie, *Weak solutions of Ideal MHD which do not conserve magnetic helicity.*

We construct weak solutions to the ideal magneto-hydrodynamic (MHD) equations which have finite total energy, and whose magnetic helicity is not a constant function of time. In view of Taylor's conjecture, this proves that there exist finite energy weak solutions to ideal MHD which cannot be attained in the infinite conductivity and zero viscosity limit. Our proof is based on a Nash-type convex integration scheme with intermittent building blocks adapted to the geometry of the MHD system. Based on joint work with Tristan Buckmaster and Vlad Vicol.

Coti Zelati, *A stochastic approach to enhanced dissipation*

We provide examples of initial data which saturate the enhanced diffusion rates proved for general shear flows which are HÃ¶lder regular or Lipschitz continuous with critical points, and for regular circular flows, establishing the sharpness of those results. The proof makes use of a probabilistic interpretation of the dissipation of solutions to advection diffusion equations.

Duerinckx, *On the size of chaos via Glauber calculus in the classical mean-field dynamics.*

We consider a system of classical particles, interacting via a smooth, long-range potential, in the mean-field regime, and we optimally analyze the propagation of chaos in form of sharp estimates on the many-particle correlation functions. While approaches based on the BBGKY hierarchy are doomed by uncontrolled losses of derivatives, we propose a novel non-hierarchical approach that focusses on the empirical measure of the system and exploits a Glauber type calculus with respect to initial data. This main result allows to rigorously truncate the BBGKY hierarchy to an arbitrary precision on the mean-field timescale, thus justifying the Bogolyubov corrections to mean field. As a corollary, we discuss the justification of the Lenard-Balescu limit in the case of a spatially homogeneous system.

Iacobelli, *Well-posedness and singular limits for the VPME system.*

The Vlasov-Poisson system with massless electrons (VPME) is widely used in plasma physics to model the evolution of ions in a plasma. It differs from the classical Vlasov-Poisson system (VP) in that the Poisson coupling has an exponential nonlinearity that creates several mathematical difficulties. We will discuss a recent result proving uniqueness for VPME in the class of solutions with bounded density, and global existence of solutions with bounded density for a general class of initial data, generalising to this setting all the previous results known for VP. Moreover we will talk about a mean field derivation of the VPME and a rigorous quasi neutral limit for initial data that are close to analytic data deriving the Kinetic Isothermal Euler (KIE) system from the VPME in dimensions $d = 1, 2, 3$. Lastly, we combine these two singular limits in order to show how to obtain the KIE system from an underlying particle system.

Grande Izquierdo, *Continuum limit of discrete NLS-type equations.*

In this talk, we will discuss the continuum limit of discrete NLS-type equations and some issues with dispersive estimates in the discrete setting. We will compare the approach of Kirkpatrick, Lenzmann and Staffilani (2013) and that of Hong and Yang (2019). Finally, we will explain how to combine these methods to tackle more general dispersive equations.

Novack, *Non-uniqueness for the 3D Quasi-Geostrophic Equations.*

In this talk, I will present and describe some of the main ideas from a recent work which constructed dissipative weak solutions to the 3D quasi-geostrophic equations.

Rodriguez, *The Radiative Uniqueness Conjecture for Bubbling Wave Maps.*

We will discuss the finite time breakdown of solutions to a canonical example of a geometric wave equation; energy critical wave maps. Breakthrough works of Krieger-Schlag-Tataru, Rodnianski- Sterbenz and Raphaël-Rodnianski produced examples of wave maps that develop singularities in finite time. These solutions break down by concentrating energy at a point in space (via bubbling a harmonic map) but have a regular limit, away from the singular point, as time approaches the final time of existence. The regular limit is referred to as the radiation. This mechanism of breakdown occurs in many other PDE including energy critical wave equations, Schrödinger maps and Yang-Mills equations. A basic question is the following:

- Can we give a precise description of all bubbling singularities for wave maps with the goal of finding the natural unique continuation of such solutions past the singularity?

In this talk, we will discuss recent work (joint with J. Jendrej and A. Lawrie) which is the first to directly and explicitly connect the radiative component to the bubbling dynamics by constructing and classifying bubbling solutions with a simple form of prescribed radiation. Our results serve as an important first step in formulating and proving the following Radiative Uniqueness Conjecture for a large class of wave maps: every bubbling solution is uniquely characterized by its radiation, and thus, every bubbling solution can be uniquely continued past blow-up time while conserving energy.

Saffirio, *From the many-body quantum dynamics to the Vlasov-Poisson equation.*

We review some results on the joint mean-field and semiclassical limit of the fermionic N-body Schrödinger dynamics leading to the Vlasov equation, which is a model in kinetic theory for charged or gravitating particles. The results we present include the case of singular interactions and provide explicit estimates on the convergence rate, using the Hartree-Fock theory for interacting fermions as a bridge between many-body and Vlasov dynamics.

Sohinger, *Gibbs measures of nonlinear Schrödinger equations as limits of many-body quantum states.*

Gibbs measures of nonlinear Schrödinger equations are a fundamental object used to study low-regularity solutions with random initial data. In the dispersive PDE community, this point of view was pioneered by Bourgain in the 1990s. We study the problem of the derivation of Gibbs measures as mean-field limits of thermal states in many-body quantum mechanics.

In our work, we apply a perturbative expansion in the interaction. This expansion is then analysed by means of Borel resummation techniques. In two and three dimensions, we need to apply a Wick-ordering renormalisation procedure. Moreover, in one dimension, our methods allow us to obtain a microscopic derivation of the time-dependent correlation functions for the cubic nonlinear Schrödinger equation. This is based partly on joint work with Jürg Fröhlich, Antti Knowles, and Benjamin Schlein.