Mechanics of thin elastic rods

Engineering meets computer graphics

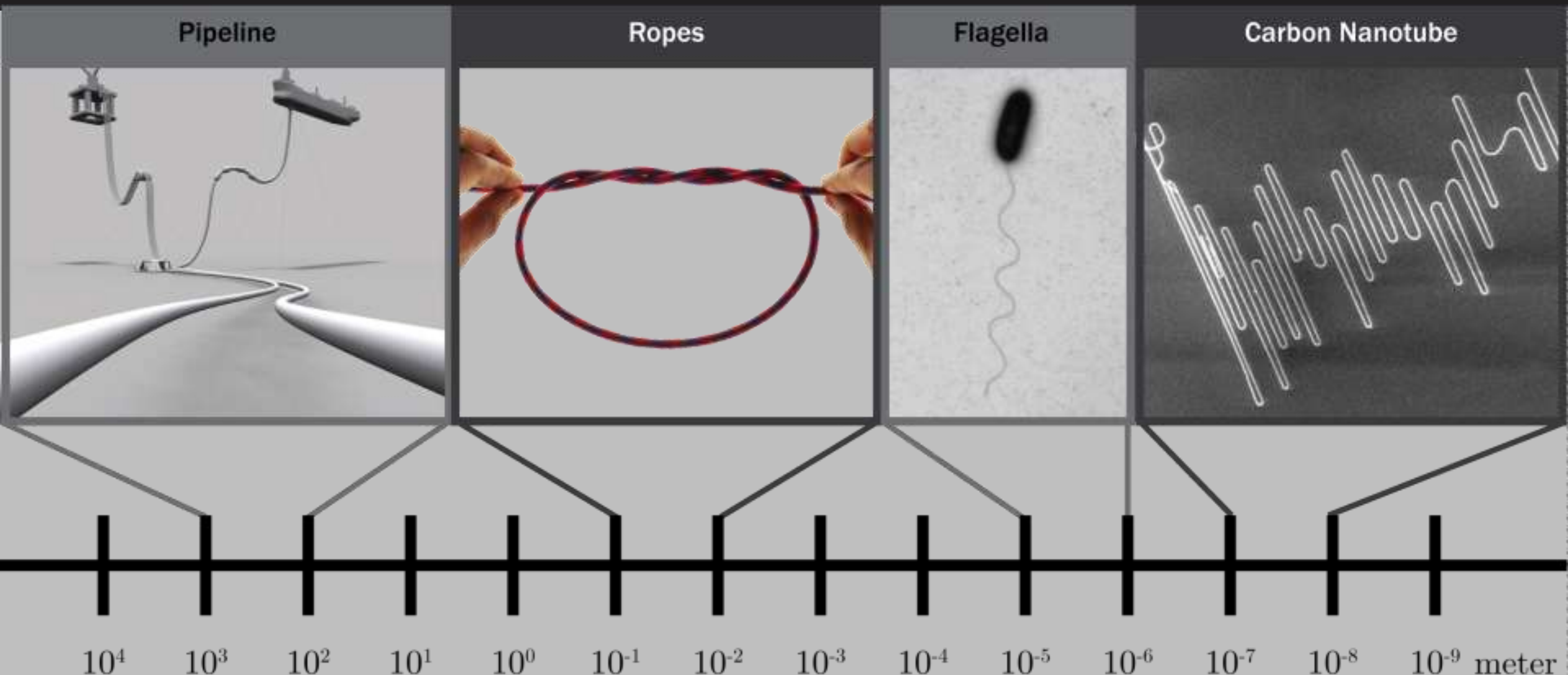

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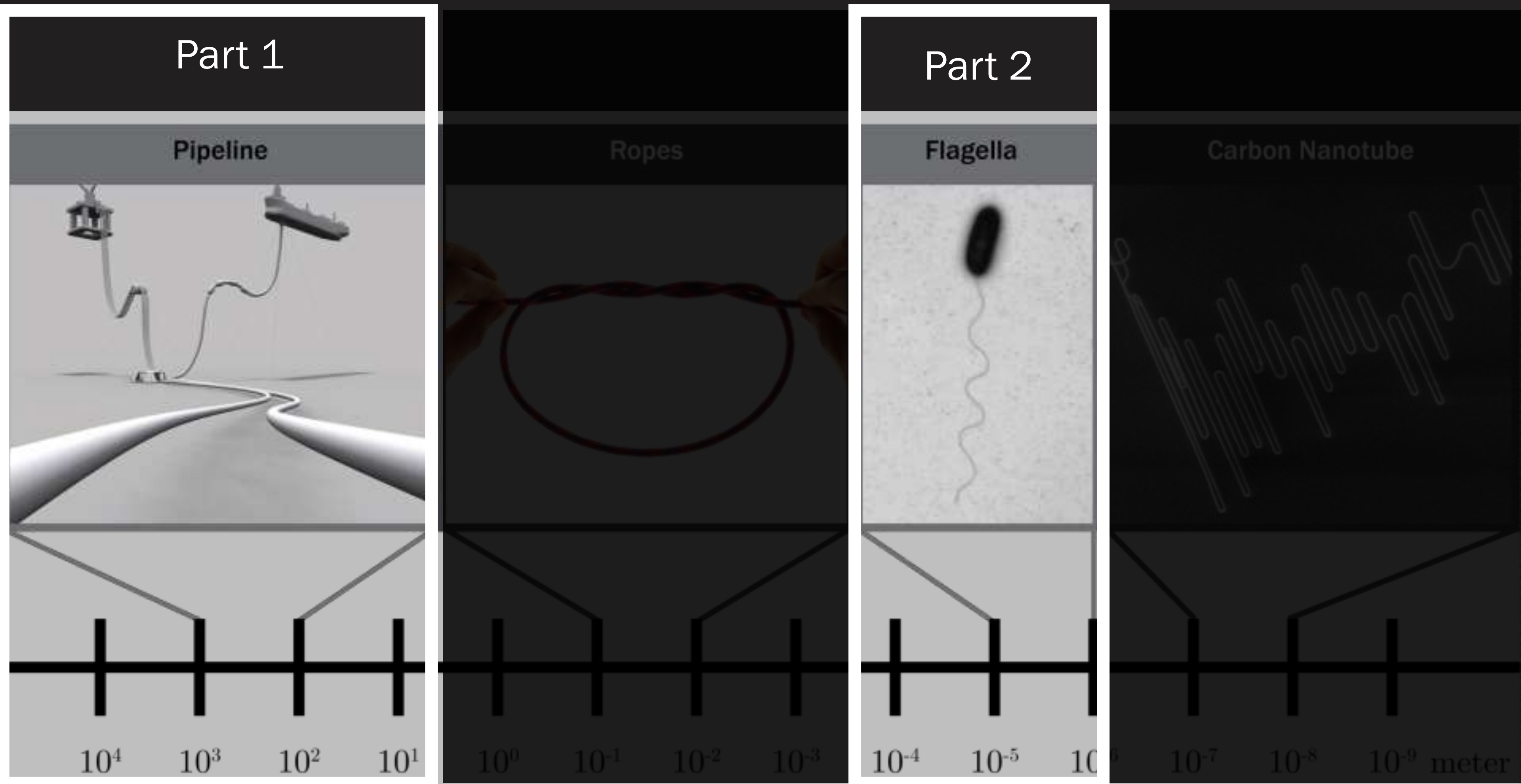
Image: Sketch-d.com, Roman Stocker (MIT), Geblinger (Weizmann)

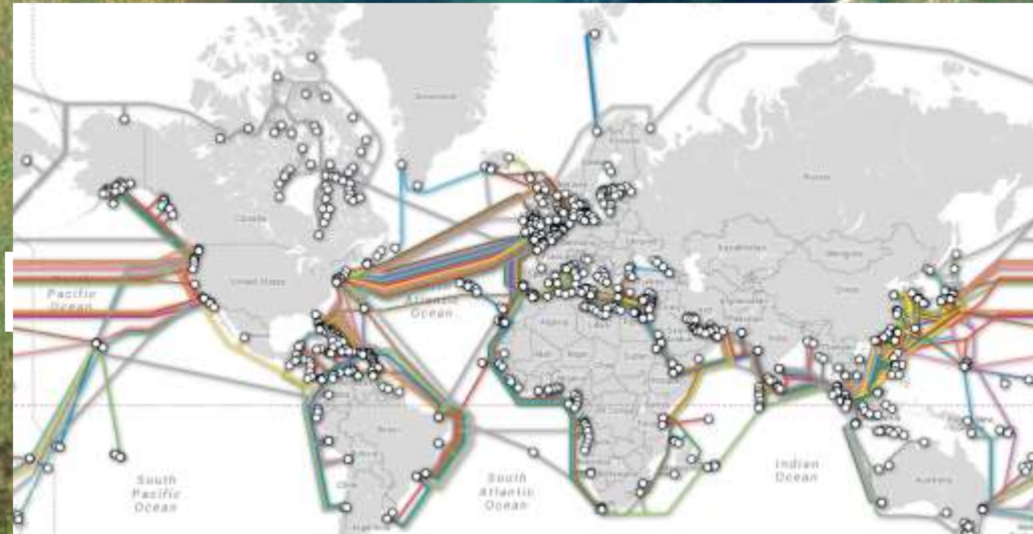
Agenda



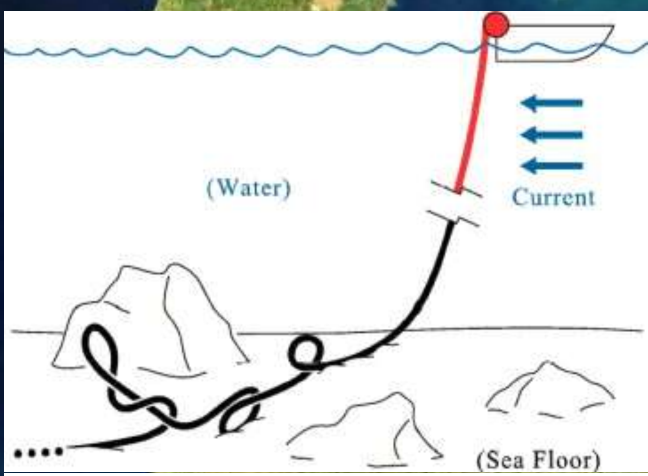
Coiling of rods

Propulsion of flagella

Submarine Cable Deployment
Global Marine Systems



Submarine Cable Map



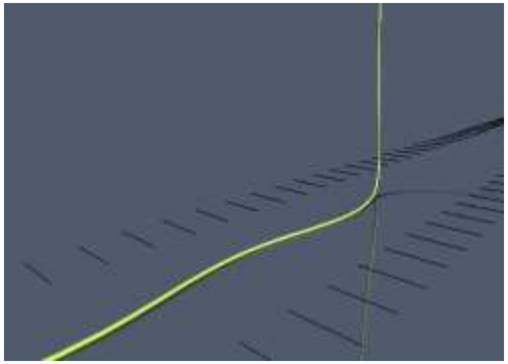
Tangles in cables
Goyal et al, *Int. J. Non Linear Mech.* 2008

With **Fang Da, J. Joo, Eitan Grinspun**
Columbia Computer Graphics Group
Columbia University

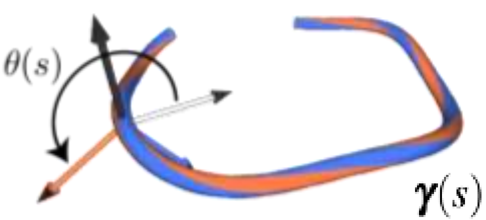
PipeLine Under The Ocean (PLUTO), 1944

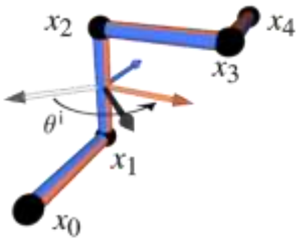
Numerics: Discrete Elastic Rods

Bergou et al, *SIGGRAPH* 2008



Twist-free frame | Material frame





$$E_{bending} = \frac{1}{2}\int EI\boldsymbol{\kappa}^2 \mathrm{ds} \qquad \boldsymbol{\kappa} = \frac{\mathrm{d}^2\boldsymbol{\gamma}}{\mathrm{ds}^2}$$

$$E_{twisting} = \frac{1}{2}\int GJ\psi^2 \mathrm{ds} \qquad \psi = \frac{\mathrm{d}\theta}{\mathrm{ds}}$$

$$E_{stretching} = \frac{1}{2}\int EA\alpha^2 \mathrm{ds} \qquad \alpha = \frac{\Delta l}{l_o}$$

↓

Discrete formulation
Discrete Differential Geometry

↓

$$\mathbf{F} = -\nabla E$$

↓

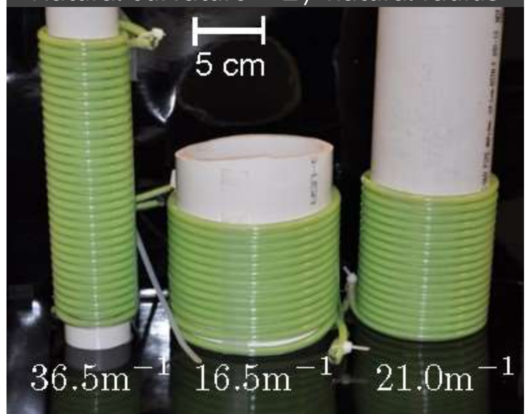
$$\mathbf{F}_{bending} + \mathbf{F}_{twisting} + \mathbf{F}_{stretching} = \mathbf{F}_{ext}$$

Experiments: rod fabrication

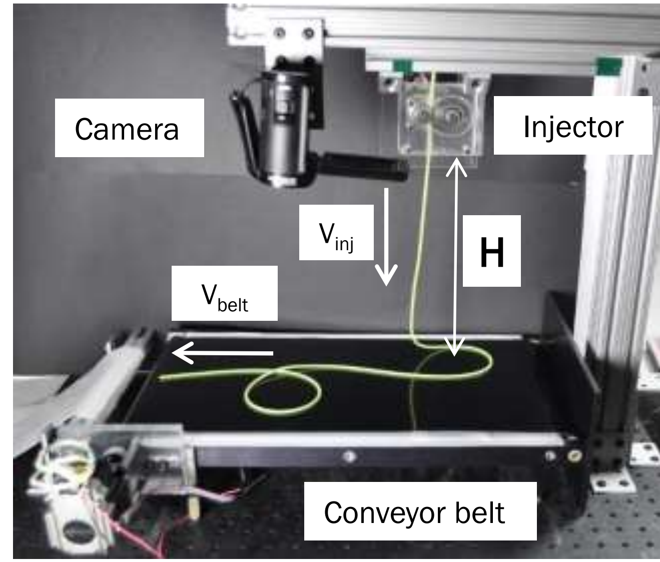
Miller et al, *Phys. Rev. Lett.* 2014
Jawed et al, *Proc. Natl. Acad. Sci.* 2014

Vinyl polysiloxane polymer

Natural curvature = 1 / natural radius



Curvature induced by wrapping around cylinders



Demolding

Curvature in engineering

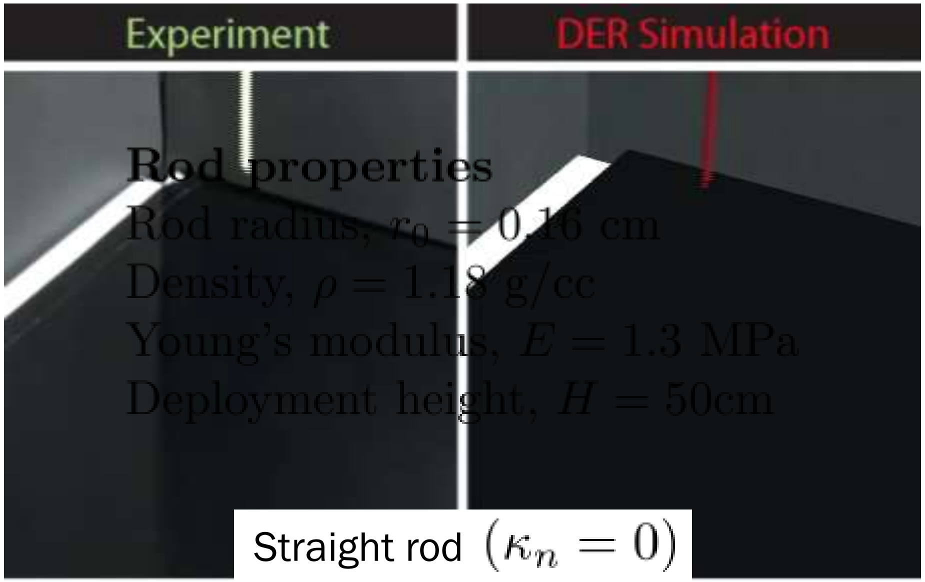
Jindaltubes.com

$$\epsilon = \frac{v_{inj} - v_{belt}}{v_{inj}}$$

Control parameter:
Dimensionless speed mismatch

Static coiling

Jawed et al, *Proc. Natl. Acad. Sci.* 2014



Rod properties

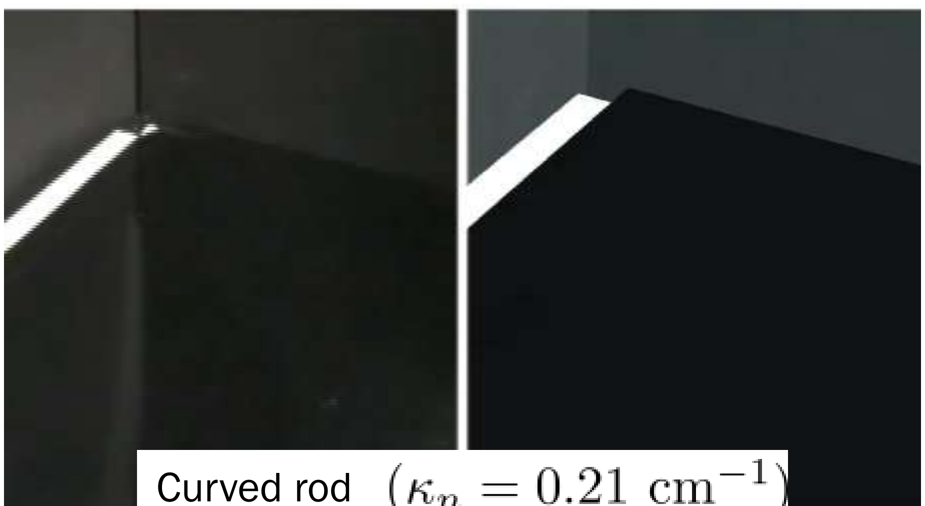
Rod radius, $r_0 = 0.16$ cm

Density, $\rho = 1.18$ g/cc

Young's modulus, $E = 1.3$ MPa

Deployment height, $H = 50$cm

Straight rod $(\kappa_n = 0)$

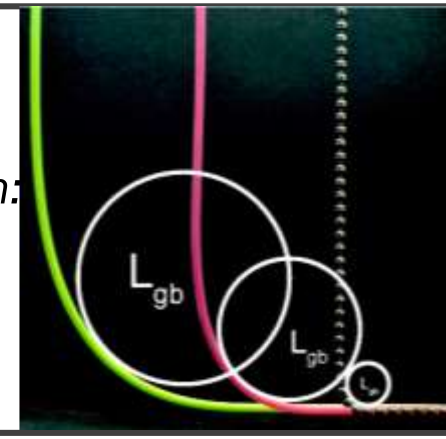


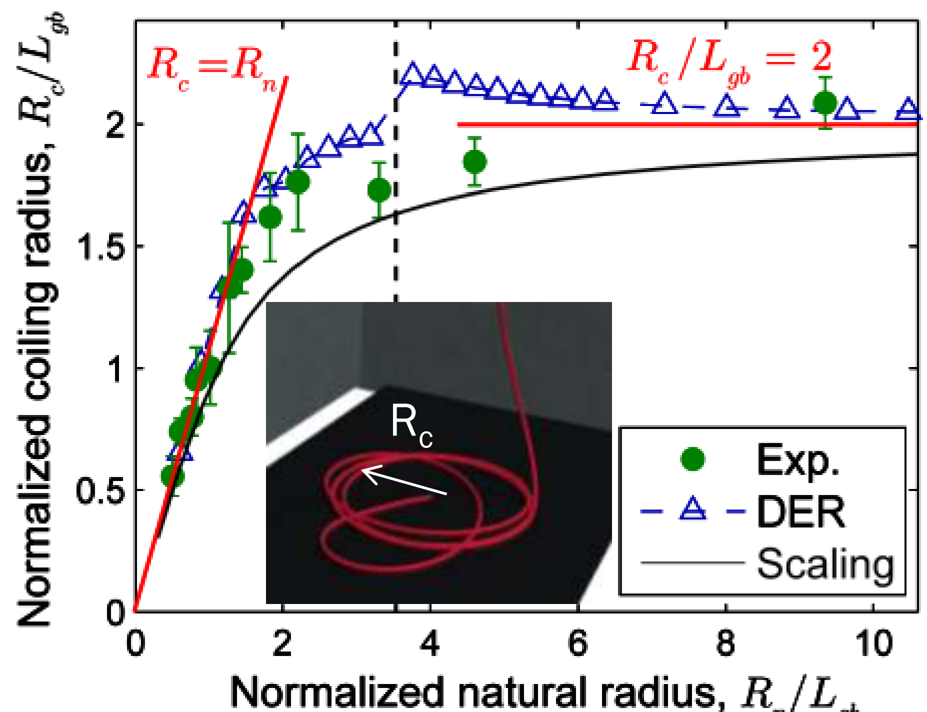
Curved rod $(\kappa_n = 0.21 \text{ cm}^{-1})$

$$E_{bending} \sim E_{gravity}$$

Gravito-bending length:

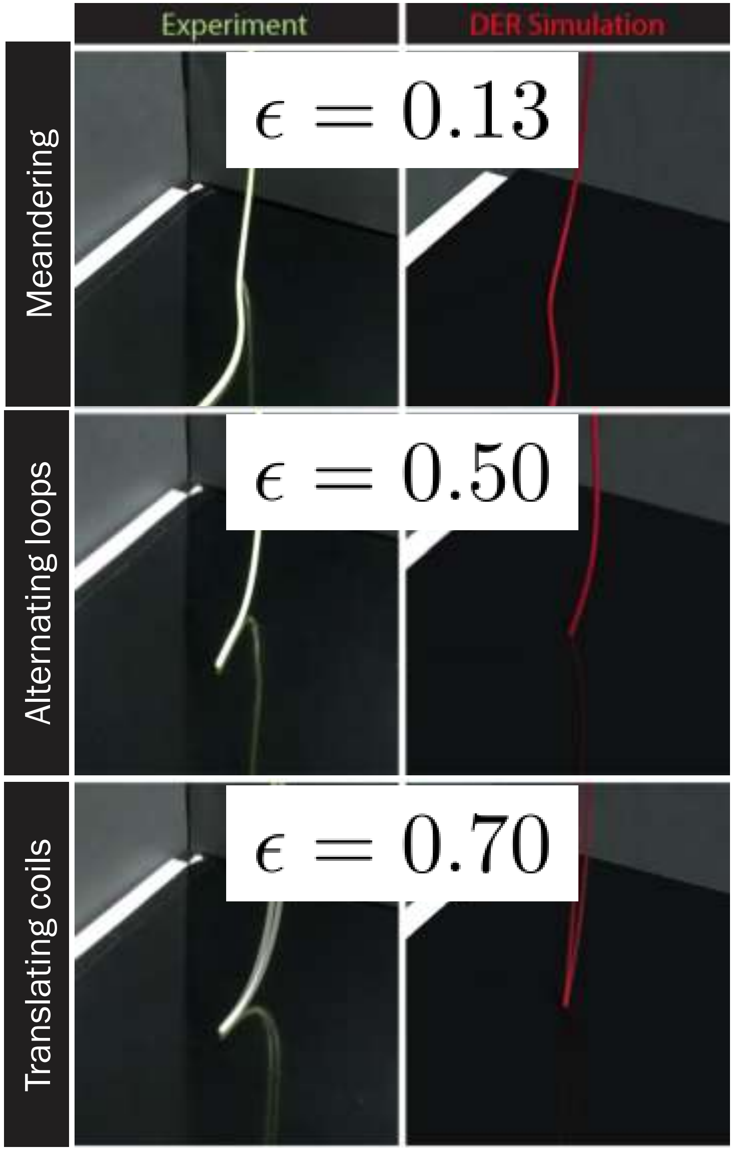
$$L_{gb} = \left(\frac{r_0^2 E}{8 \rho g} \right)^{\frac{1}{3}}$$

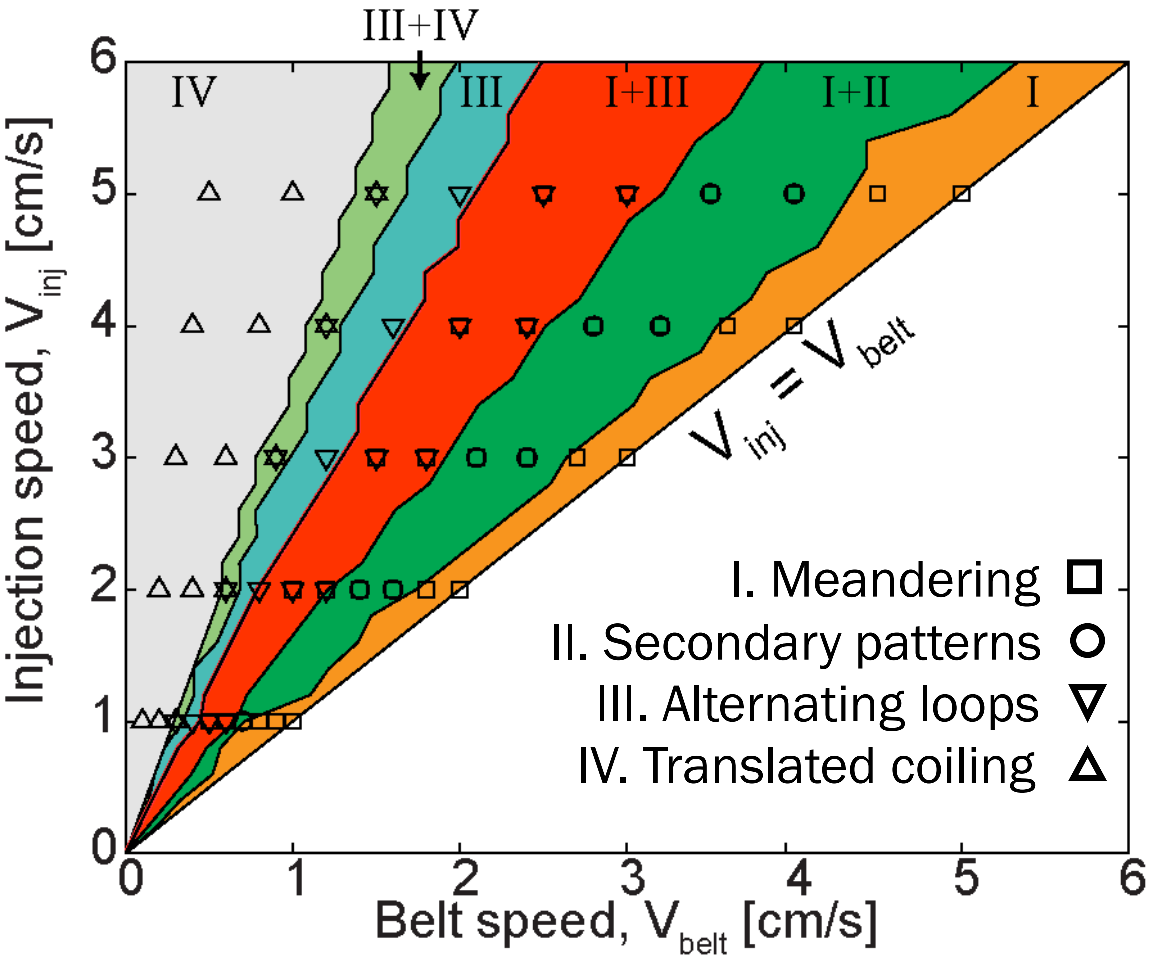




Dynamic coiling: straight rod

Jawed et al, *Proc. Natl. Acad. Sci.* 2014
Jawed et al, *Extreme Mechanics Letters, 2014*



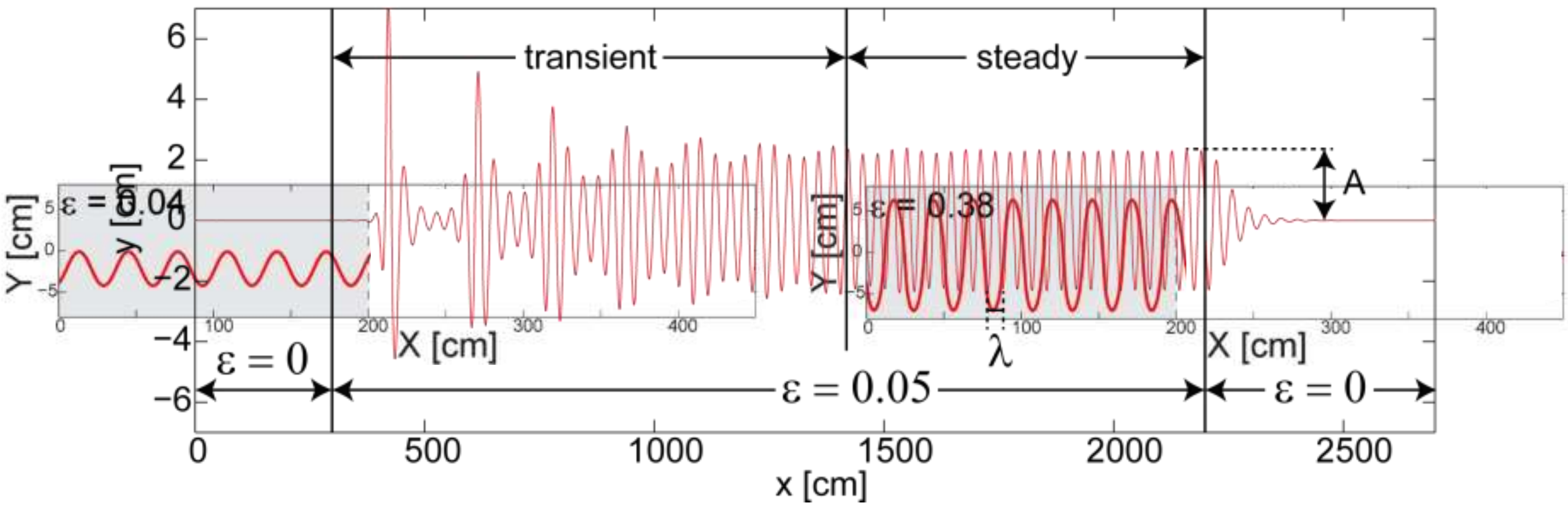


$$\epsilon = \frac{v_{inj} - v_{belt}}{v_{inj}}$$

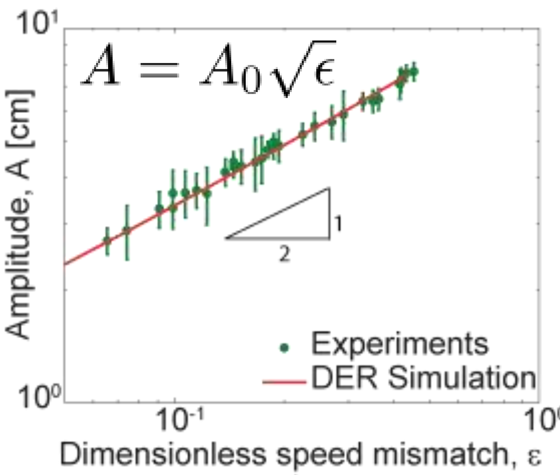
Dimensionless speed mismatch

Meandering: Hopf bifurcation?

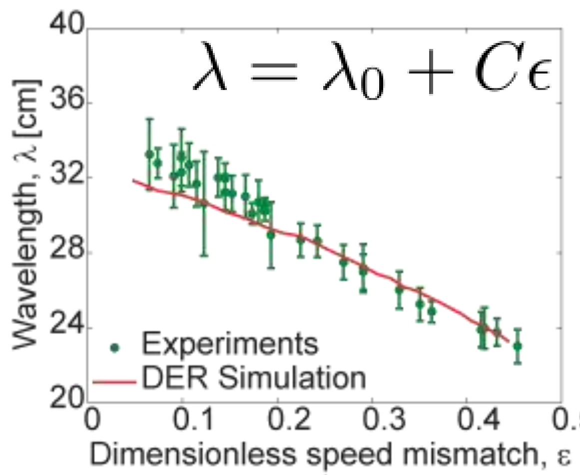
Jawed et al, *Proc. Natl. Acad. Sci.* 2014



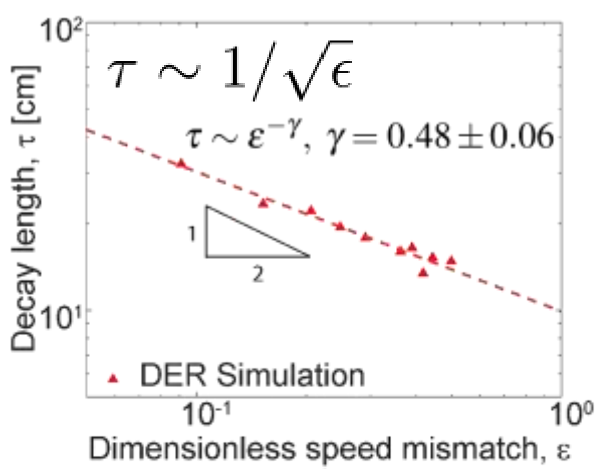
Amplitude



Wavelength

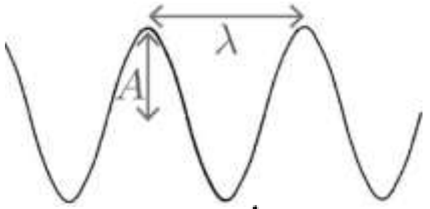


Decay length



Meandering length scale

Jawed et al, *Proc. Natl. Acad. Sci.* 2014



Geometry

$$A = A_0\sqrt{\epsilon}$$

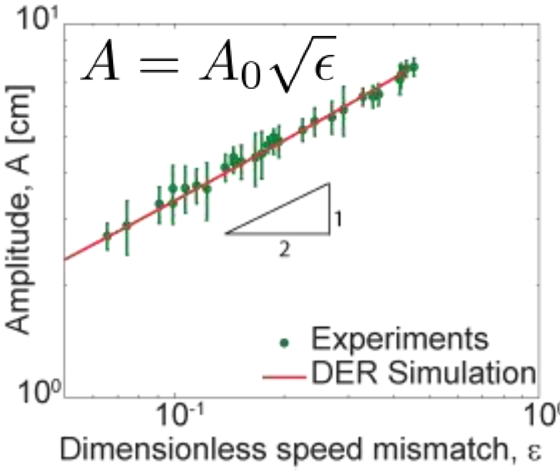
$$\lambda \approx A_0(-2.48\epsilon + 3.2)$$

Onset wavelength

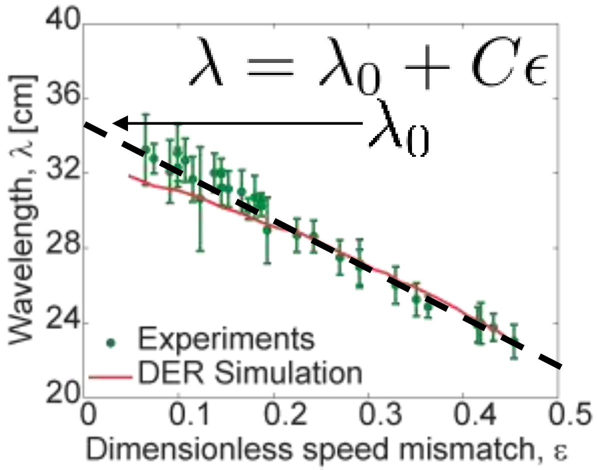
$$\lambda_0 = \lambda(\epsilon \to 0) \approx 3.2A_0$$

What sets λ_0?

Amplitude

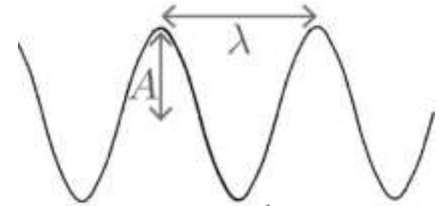


Wavelength



Meandering length scale

Jawed et al, *Proc. Natl. Acad. Sci.* 2014



Geometry

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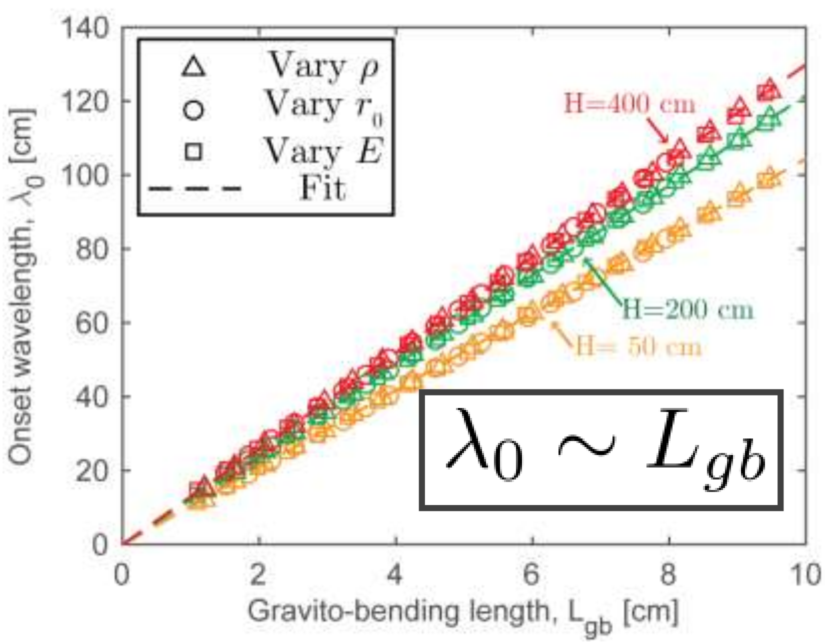
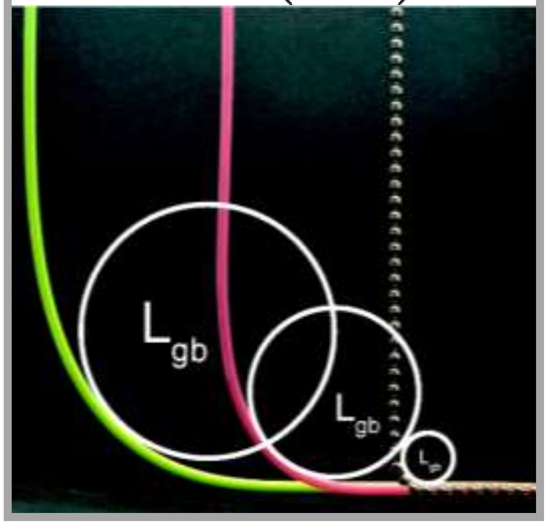


Physical parameters
Density, ρ
Young's modulus, E
Rod radius, r_0

Numerical Experiment
Vary one parameter keeping others fixed

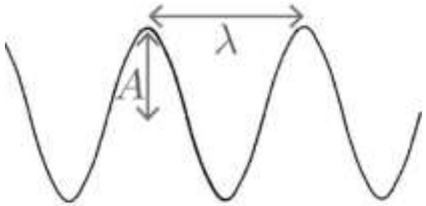
Gravito-bending length scale

$$L_{gb} = \left(\frac{r_0^2 E}{8\rho g}\right)^{\frac{1}{3}}$$



Meandering length scale

Jawed et al, *Proc. Natl. Acad. Sci.* 2014



Geometry

$$A = A_0\sqrt{\epsilon}$$

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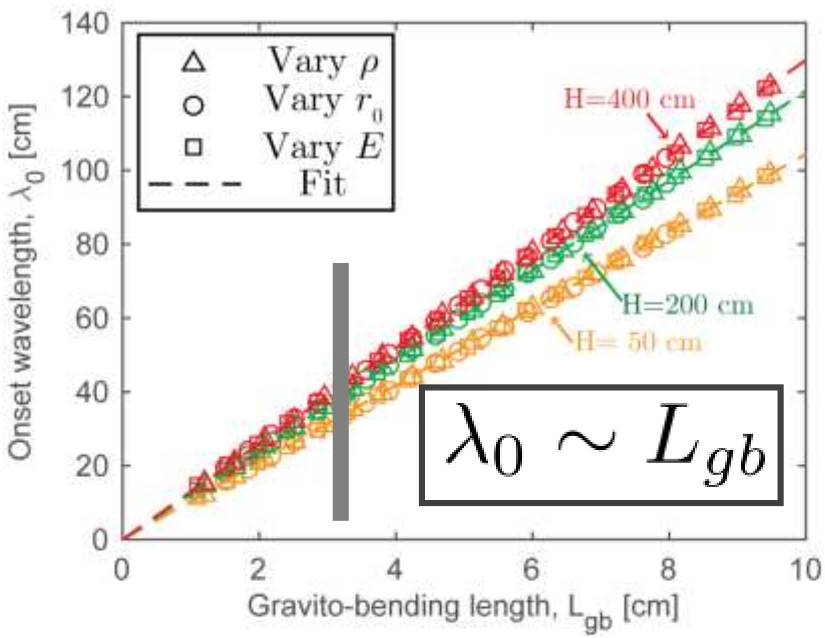
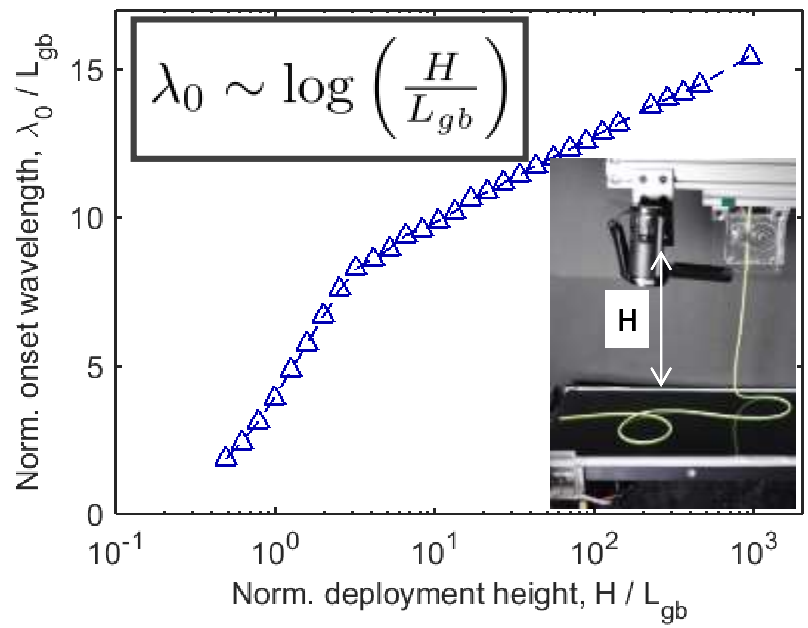
Onset wavelength

$$\lambda_0 = \lambda(\epsilon \to 0) \approx 3.2A_0$$

What sets λ_0?

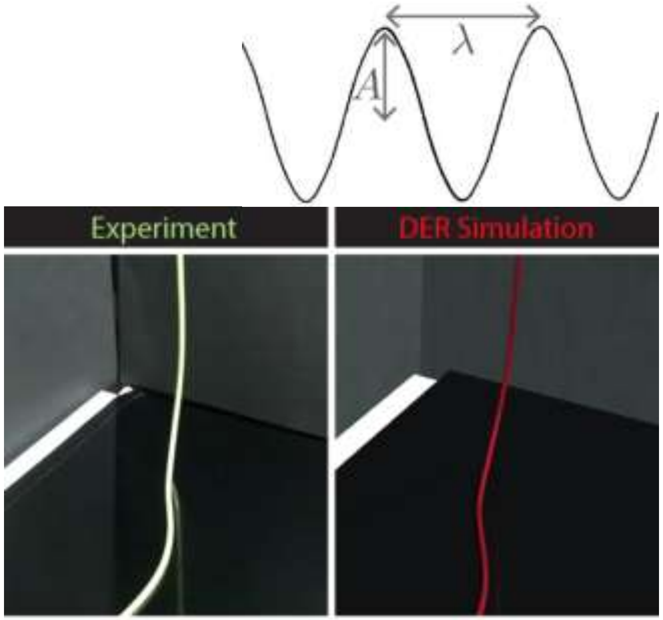
Deployment height, H





Meandering length scale

Jawed et al, *Proc. Natl. Acad. Sci.* 2014



Putting it all together:
Onset wavelength

$$\lambda_0 = L_{gb}\left(D_1 \log\left(\frac{H}{L_{gb}}\right) + \beta\right)$$

$$D_1 = 1.22 \pm 0.01$$

$$\beta = 7.14 \pm 0.02$$

Meandering wavelength

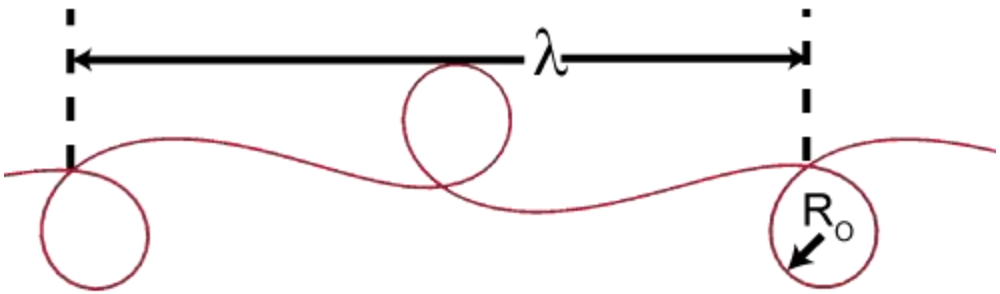
$$\lambda = \frac{-2.48\epsilon + 3.2}{3.2} L_{gb}\left(D_1 \log\left(\frac{H}{L_{gb}}\right) + \beta\right)$$

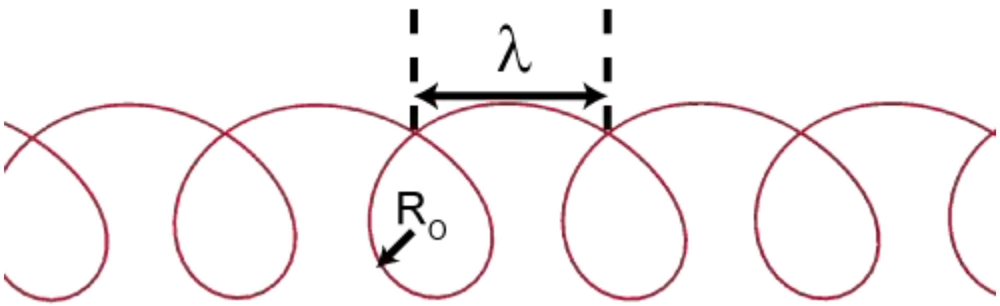
Meandering amplitude

$$A = \frac{L_{gb}}{3.2}\left(D_1 \log\left(\frac{H}{L_{gb}}\right) + \beta\right)\sqrt{\epsilon}$$

~~Meandering~~ Coiling length scale

Jawed et al, *Extreme Mechanics Letters* 2014

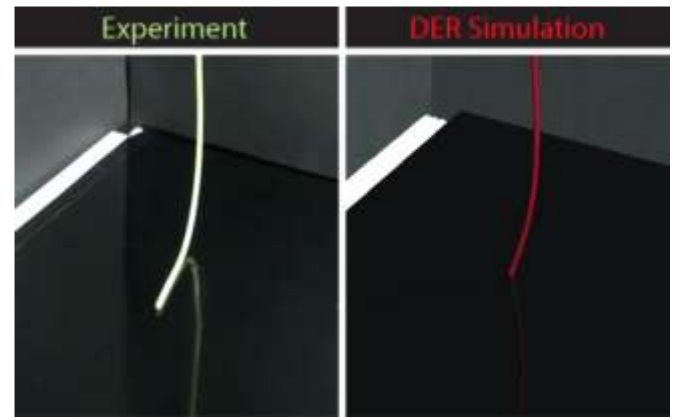




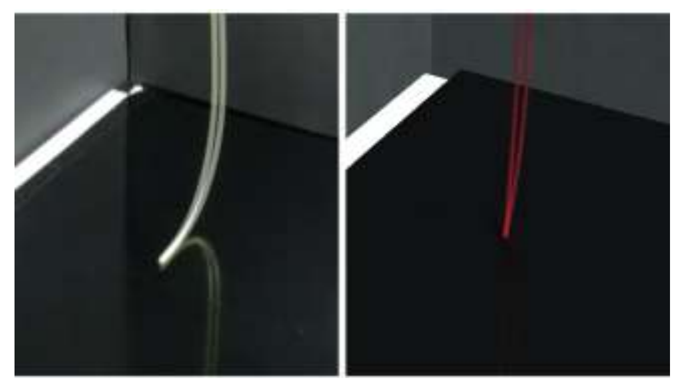
$$\lambda \sim L_{gb} \log\left(\frac{H}{L_{gb}}\right)$$

$$R_o \sim L_{gb} \log\left(\frac{H}{L_{gb}}\right)$$

Alternating loops



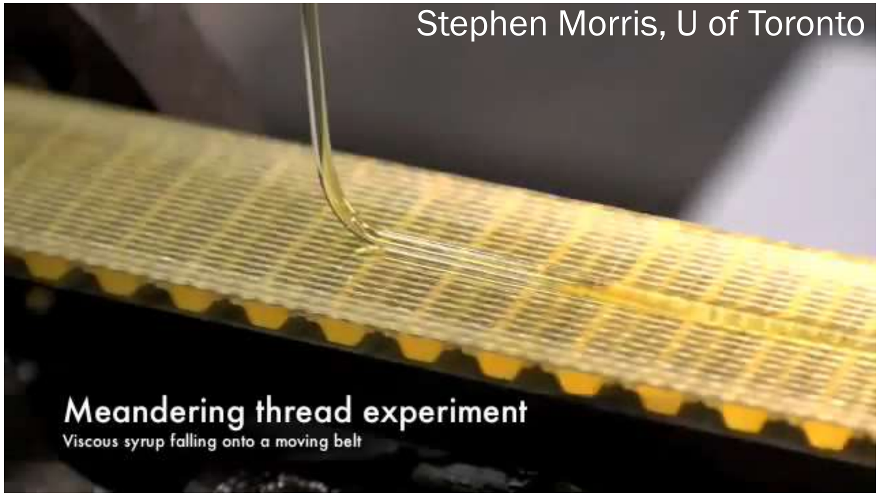
Translated coiling

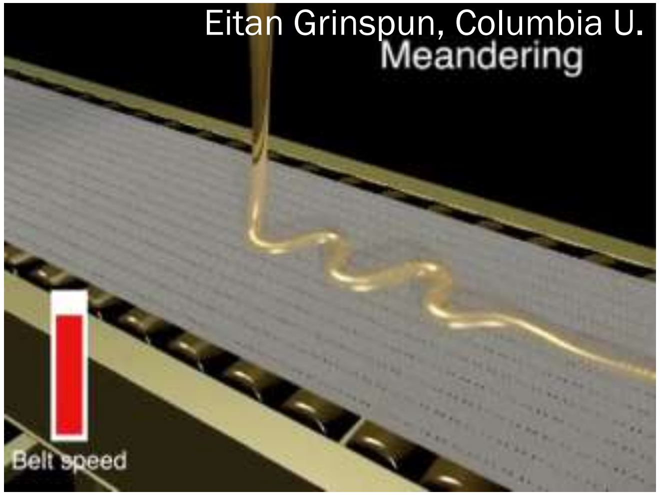


Prominence of geometry

Fluid-mechanical sewing machine

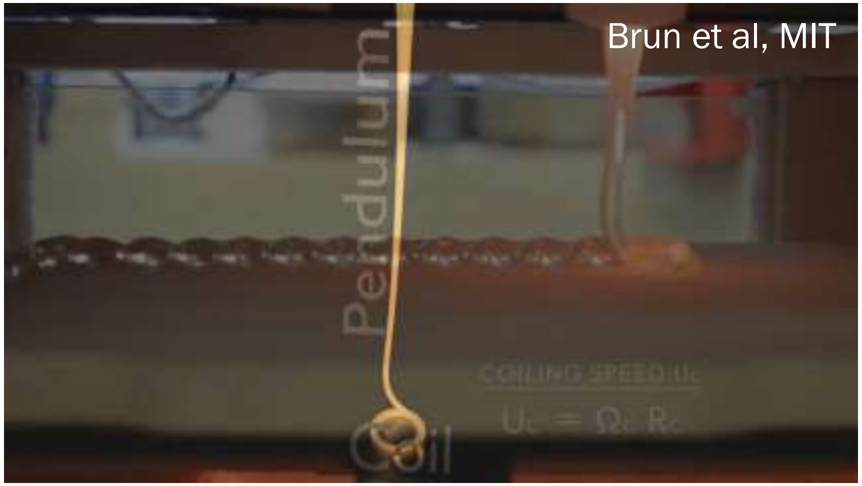
Why similarity?





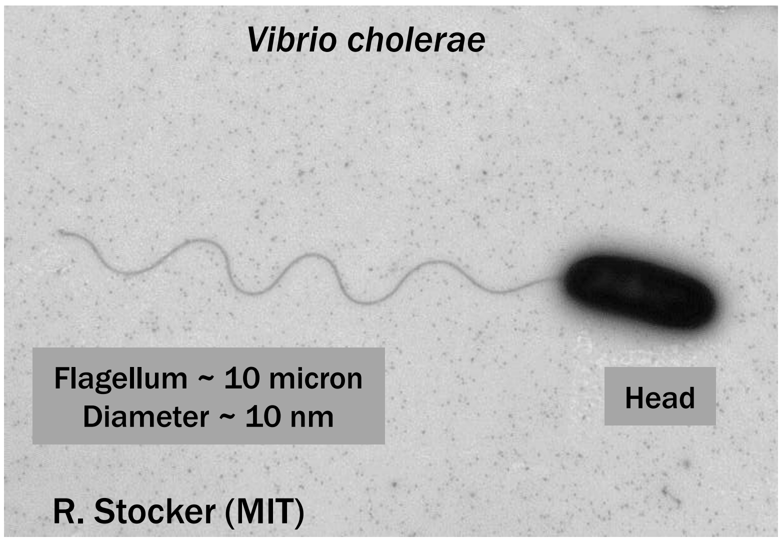
Jawed, M. K. et al
A Geometric Model for the Coiling of an Elastic Rod Deployed Onto a Moving Substrate
Journal of Applied Mechanics, 82 (12):121007, 2015

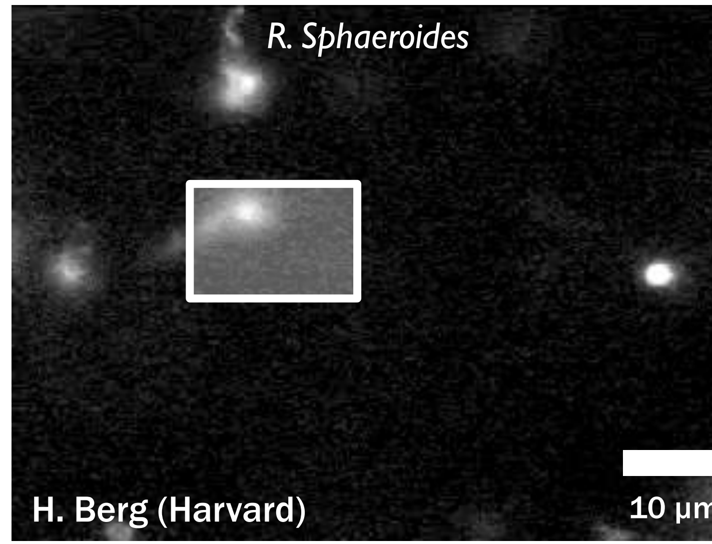
Molten glass sewing machine

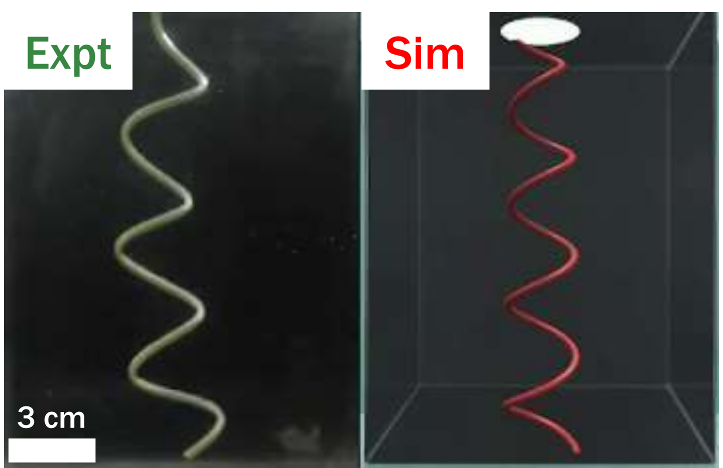


Cooking of *Jalebi*

Propulsion and instability of bacterial flagella







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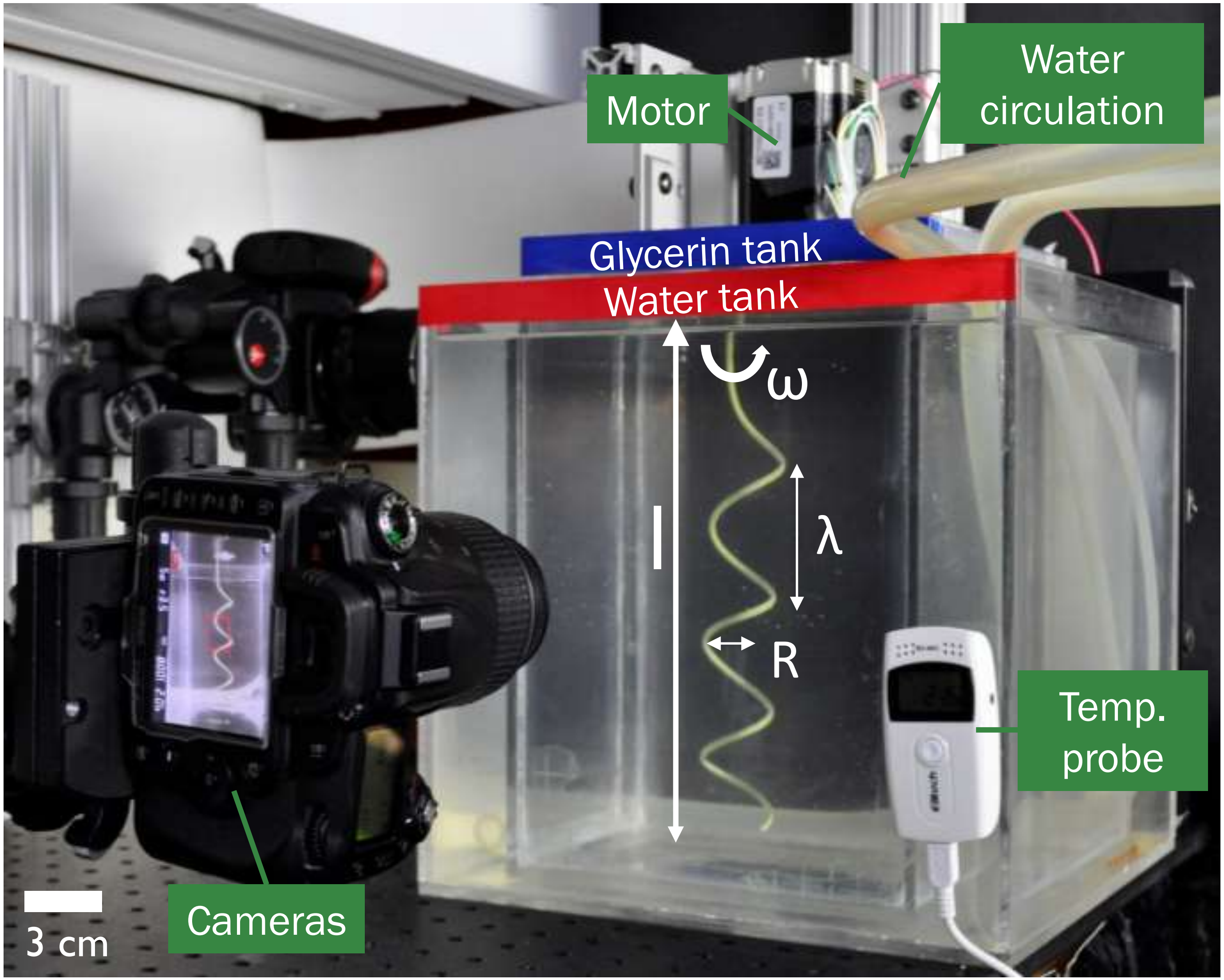
Experiments



Physical properties

Axial length
$12 \leq l[\mathrm{cm}] \leq 20$

Helix pitch
$3 \leq \lambda[\mathrm{cm}] \leq 12$

Helix radius
$0.95 \leq R[\mathrm{cm}] \leq 2.22$

Rod radius
$0.94 \leq r_0[\mathrm{mm}] \leq 2.38$

Rod density
$1.13 \leq \rho_r[\mathrm{g/cm}]^3 \leq 1.37$

Young's Modulus
$636 \leq E[\mathrm{kPa}] \leq 1255$

Temperature of glycerin
$7.55 \leq \theta[^\circ\mathrm{C}] \leq 20$

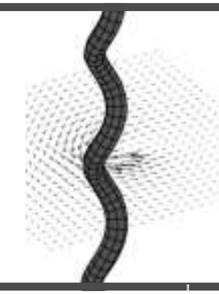
Viscosity of glycerin
$0.5 \leq \mu[\mathrm{Pa \cdot s}] \leq 4.45$

Numerics

Discrete Elastic Rod

+

Lighthill Slender Body Theory



Lighthill Slender Body Theory

Relation between velocity $\boldsymbol{u}(s)$ at each point on flagellum and the force $\boldsymbol{f}(s)$ exerted by the fluid.

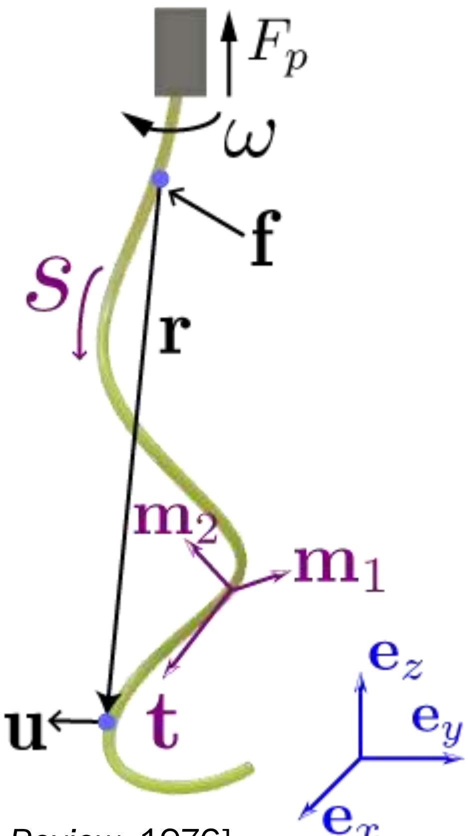
local — non-local

$$\mathbf{u}(s) = \frac{\mathbf{f}_\perp}{4\pi\mu} + \int_{r(s',s)>\delta} \mathbf{f}(s') \cdot \mathbf{J}(\mathbf{r})\, ds'$$

where $\mathbf{f}_\perp = \mathbf{f} \cdot (\mathbf{I} - \mathbf{t}\mathbf{t}^T)$

$$\delta = \frac{1}{2} r_0 \sqrt{e}$$

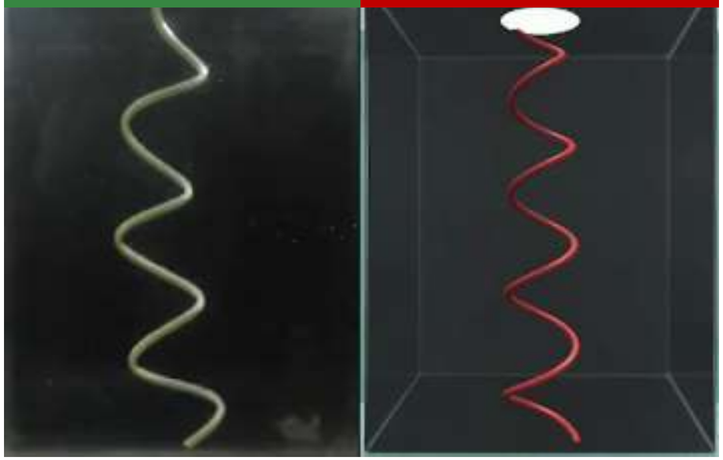
$$\mathbf{J}(\mathbf{r}) = \frac{1}{8\pi\mu}\left(\frac{\mathbf{I}}{|\mathbf{r}|} + \frac{\mathbf{r}\mathbf{r}^T}{|\mathbf{r}|^3}\right)$$



[Lighthill *SIAM Review*, 1976]

Flagellar dynamics

Experiments | **Simulations**



Physical properties

$l = 20 \pm 0.5$ cm

$\lambda = 5 \pm 0.5$ cm

$R = 1.59 \pm 0.1$ cm

$r_0 = 1.58 \pm 0.02$ mm

$\rho_r = 1.273 \pm 0.022$ g/cm^3

$Y = 1255 \pm 49$ kPa

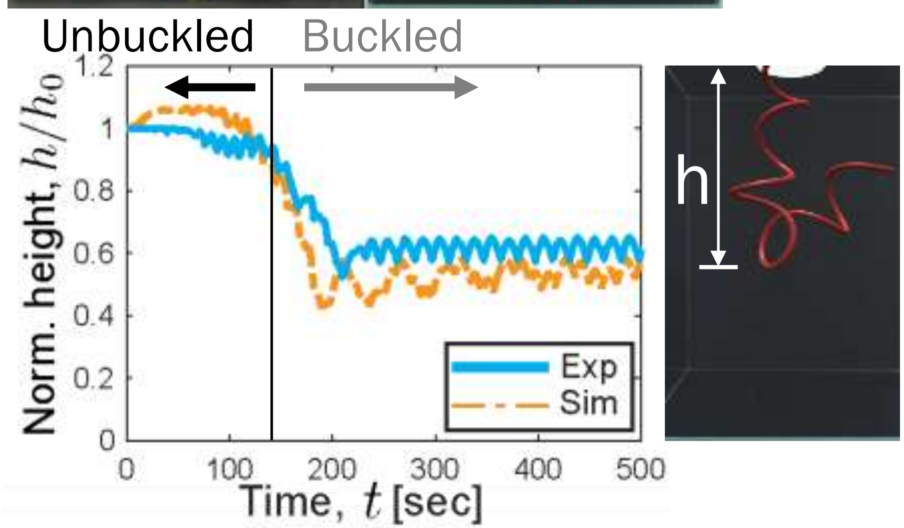
$\mu = 1.6 \pm 0.05$ Pa ·s

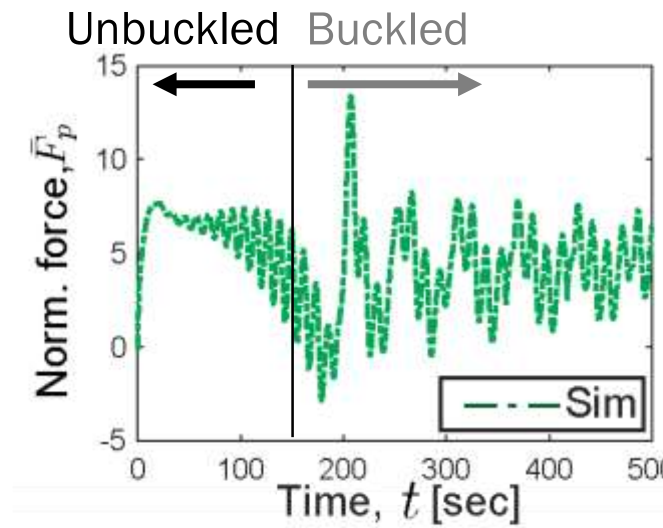
$\rho_m = 1.240$ g/cm^3

$Re \sim 10^{-2}$

Bacteria $3 \leq \lambda/R \leq 11$, $2.7 \leq l/\lambda \leq 11$

Model $\lambda/R = 3.14$, $l/\lambda = 4.0$

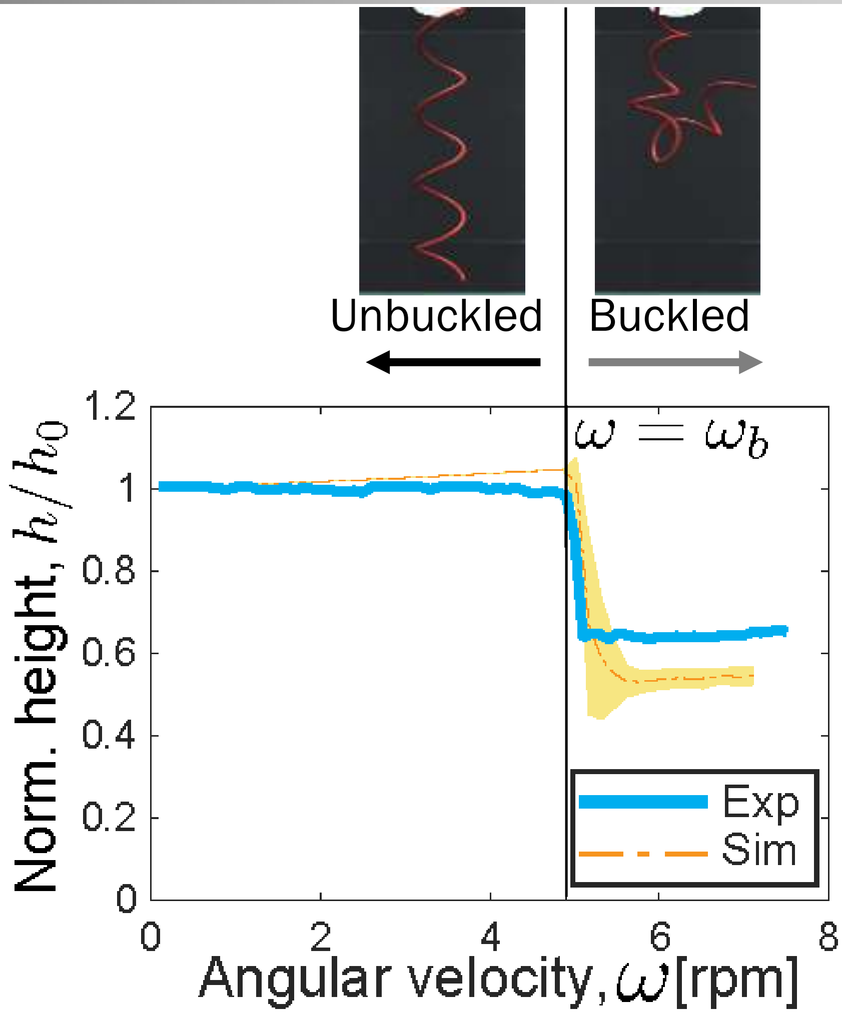


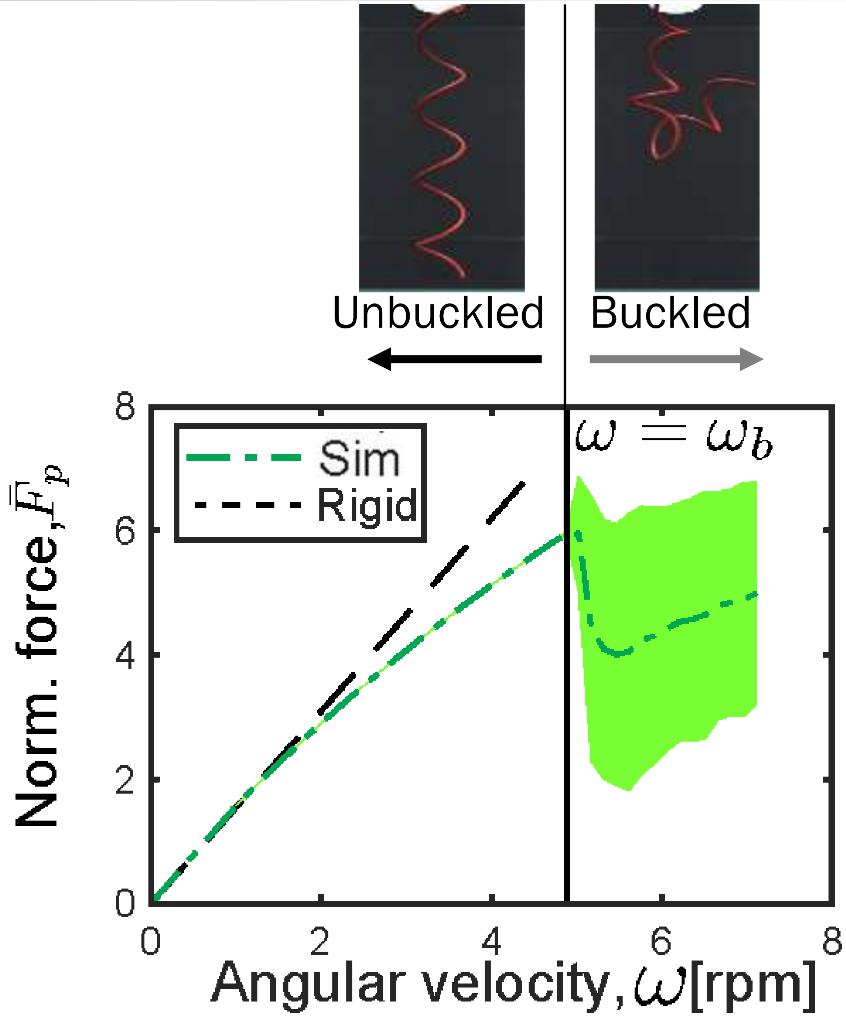


Fluid loading can cause buckling

$$\bar{F}_p = \frac{F_p}{EI/l^2}$$

Critical buckling velocity





Critical angular velocity for buckling: ω_b

Buckling characterization

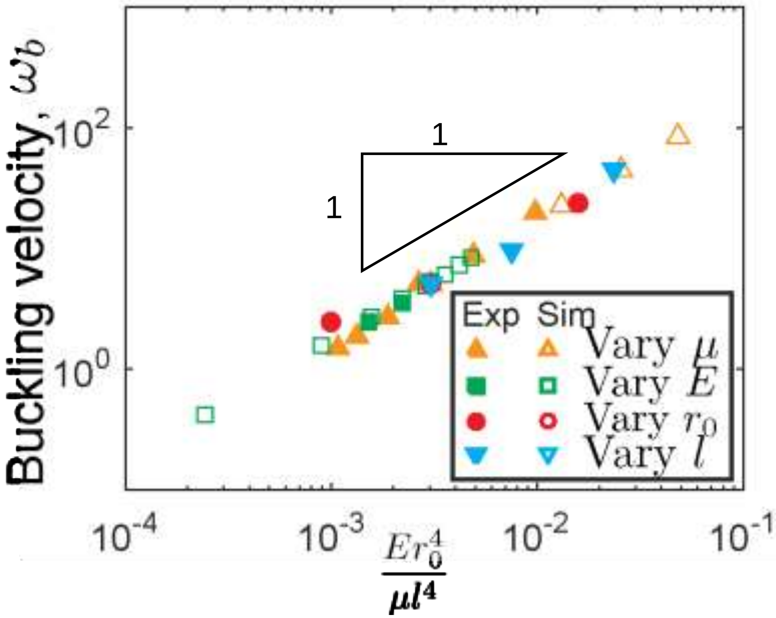
Critical force ~ Viscous force

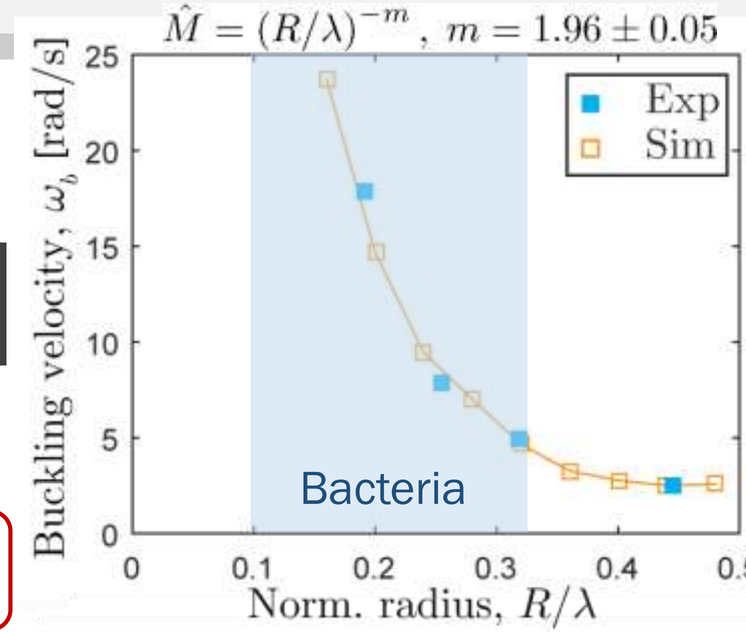
$$F_c \sim F_v$$

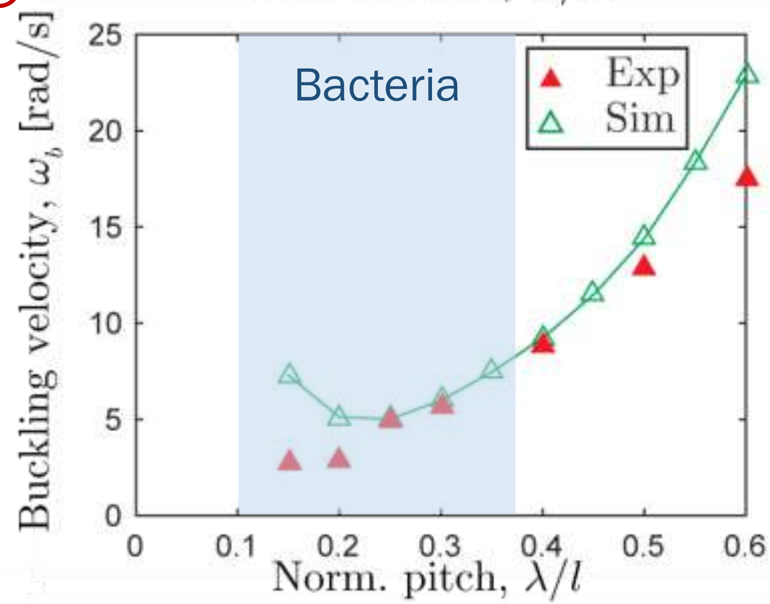
$$\omega_b = \frac{E r_0^4}{\mu l^4} \bar{\omega}_b$$

Geometry dependent coefficient

$$\bar{\omega}_b(R/\lambda, \lambda/l) = \hat{M}(R/\lambda)\hat{N}(\lambda/l)$$



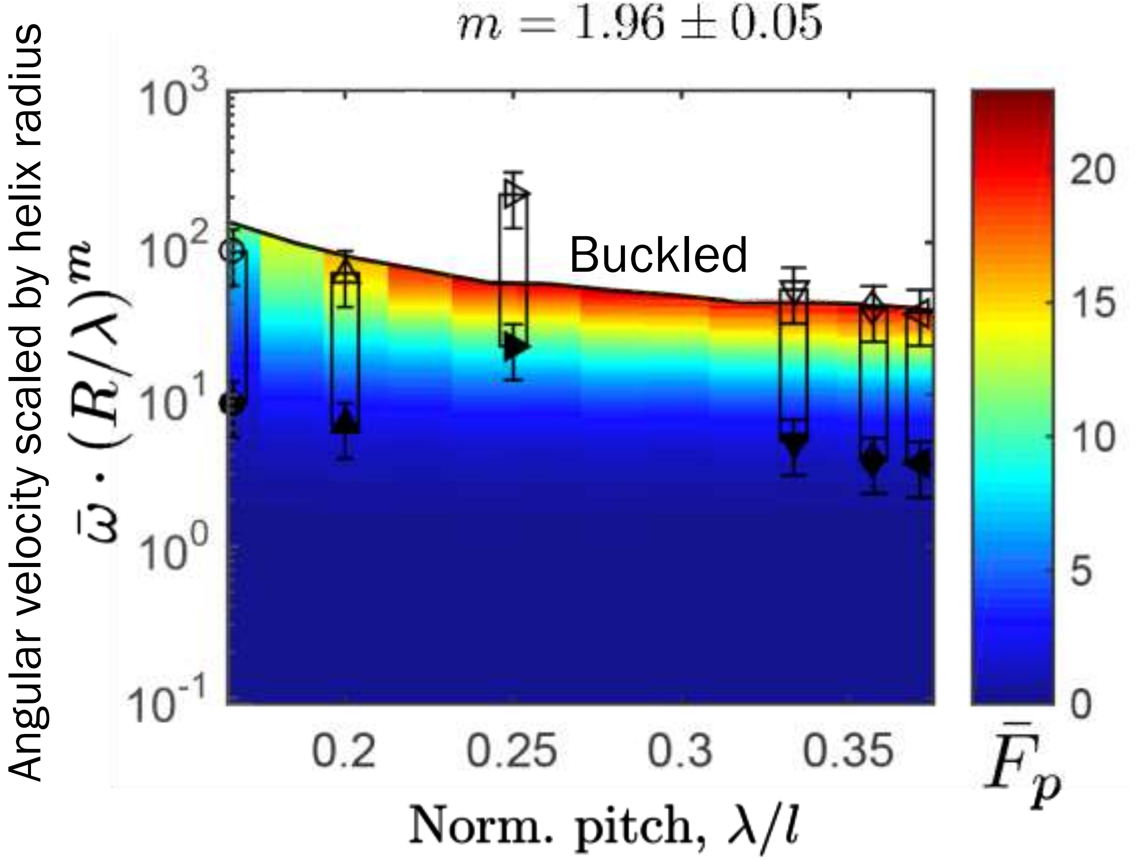




Phase diagram

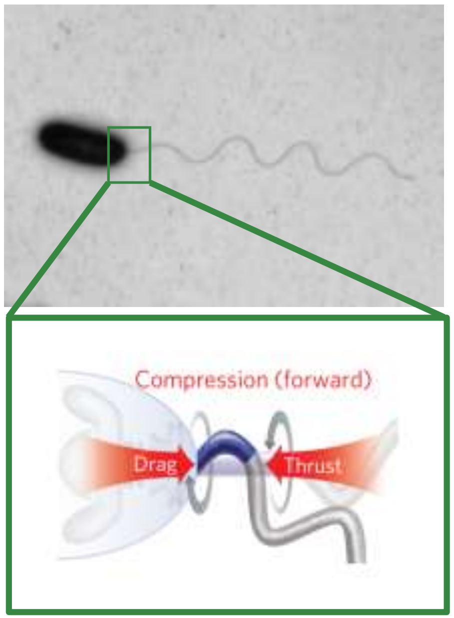
Norm. buckling velocity: $\bar{\omega}_b = (R/\lambda)^{-m}\hat{N}(\lambda/l)$

$m = 1.96 \pm 0.05$



○ Caulobacter crescentus (Wild)
◇ Escherichia coli
▽ Rhizobium lupini (Semicoiled)
△ R. lupini (Curly)
▷ Salmonella (Wild)
◁ Vibrio alginolyticus

Failure to functionality

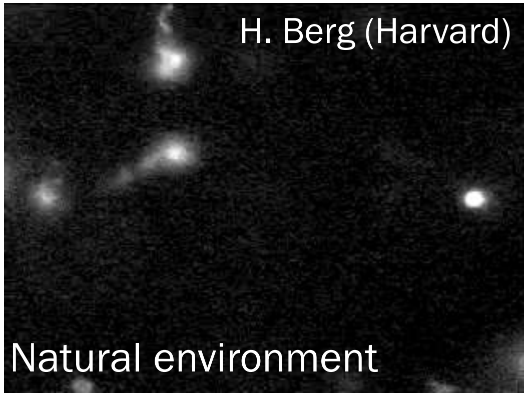


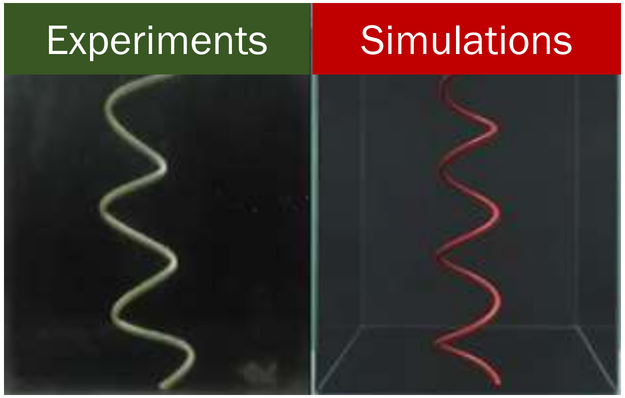
Son, K., Guasto, J., Stocker, R. (2013). Bacteria can exploit a flagellar buckling instability to change direction. *Nature physics*, *9*(8), 494-498.

Filled symbols: $EI \sim 10^{-23}$ N/m²
Y. Takano et al. JSME Int J., 2003

Open symbols: $EI \sim 10^{-24}$ N/m²
S. Fujime et al. J. Mol. Biol. (1972)

Conclusion





- Coupled effect of flexible structure + fluid
  - Structure: Discrete Elastic Rods
  - Fluid: Lighthill slender body theory
- Flagella can buckle above a threshold angular velocity
- Bacteria may exploit buckling for functionality

Jawed, M. K., Khouri, N. K., Da, F., Grinspun, E., & Reis, P. M.
Propulsion and instability of a flexible helical rod rotating in a viscous fluid
Physical Review Letters, 115(16):168101, 2015

Jawed, M. K. & Reis, P. M.
Deformation of a soft helical filament in an axial flow at low Reynolds number
Soft Matter, 2016

Email: khalidjm@mit.edu Web: khalidjawed.com

Discrete Elastic Rods

Bergou et al, *SIGGRAPH* 2008
Bergou et al, *SIGGRAPH* 2010

Simulation loop

Letting $\mathbf{q} = (\mathbf{x}_0, \theta^0, \cdots, \mathbf{x}_n, \theta^n, \mathbf{x}_{n+1})^T$ and $h_k = t_{k+1} - t_k$, we apply Euler integration (implicit on internal and explicit on external forces), using Newton's method to solve

$$\mathrm{M}\Delta\dot{\mathbf{q}} - h_k \mathbf{F}_{\mathrm{int}}(\mathbf{q}(t_k) + \Delta\mathbf{q}) = h_k \mathbf{F}_{\mathrm{ext}}(t_k, \mathbf{q}(t_k), \dot{\mathbf{q}}(t_k))$$
$$\Delta\mathbf{q} - h_k \Delta\dot{\mathbf{q}} = h_k \dot{\mathbf{q}}(t_k)$$

Explicit energy gradient and Hessian

$$E_s = \frac{1}{2}\sum_{j=0}^{n} k_s^j \left(\varepsilon^j\right)^2 \left|\bar{\mathbf{e}}^j\right|,$$

$$E_t = \frac{1}{2}\sum_{i=1}^{n} \beta_i \frac{(m_i - \overline{m}_i)^2}{\bar{l}_i},$$

$$E_b = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\bar{l}_i} (\boldsymbol{\kappa}_i - \overline{\boldsymbol{\kappa}}_i)^T \mathrm{B}_i (\boldsymbol{\kappa}_i - \overline{\boldsymbol{\kappa}}_i)$$

Time-parallel frame

Given the reference frame at time t_{k-1}, the time-parallel reference frame $\underline{\mathbf{d}}_\alpha^j(t_k)$ only depends on $\mathbf{t}^j(t_k)$, and the only nonzero terms in (3) are $i \in \{j, j+1\}$. *Thus, the force stencil is local and the energy Hessian is banded.*