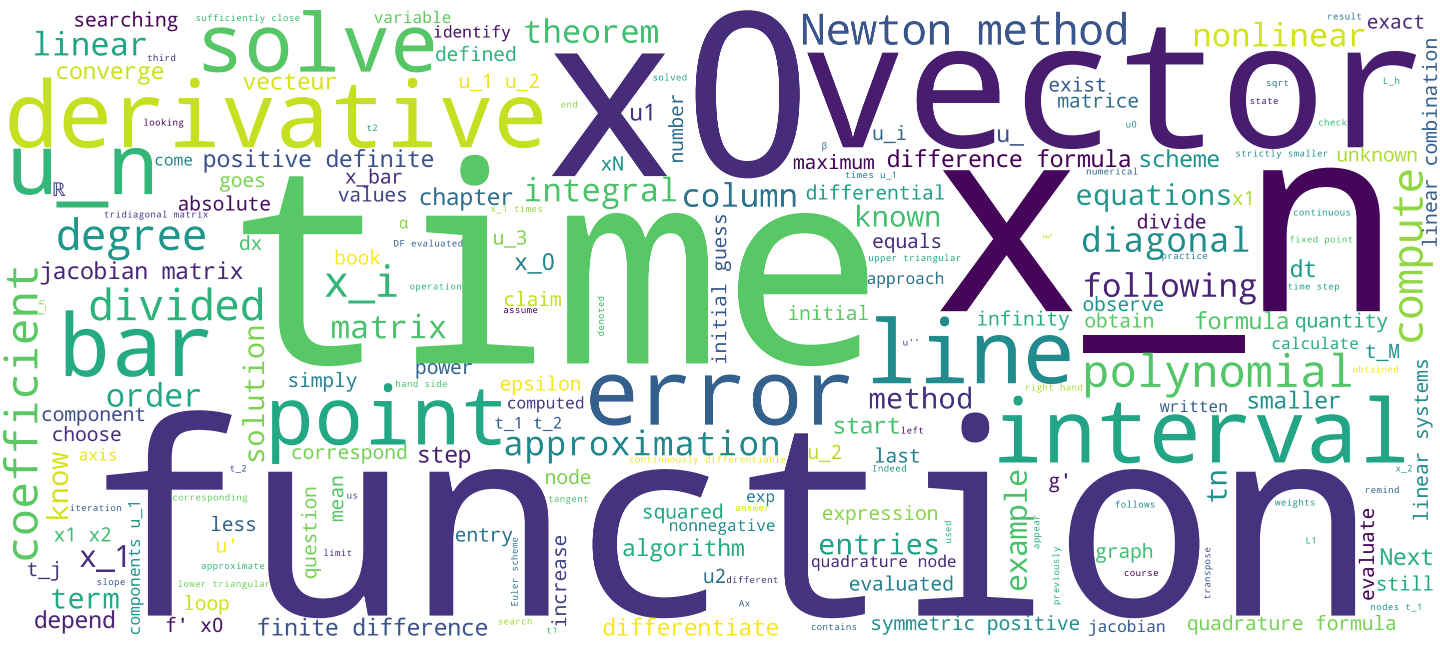
Chapitre 10 : Un problème non linéaire (suite)

Introduction à l'analyse numérique

Prof. Marco Picasso



Video

Chap 10 : un problème non linéaire

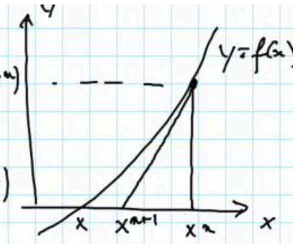
Cherche $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}$ tq $\vec{F}(\vec{u}) = \vec{0}$: $\begin{pmatrix} \frac{2u_1 - u_2}{h^2} + x_1(u_1)^3 - f(x_1) \\ \vdots \\ \vdots \end{pmatrix}$

$\vec{u}^n$ connu $\vec{u}^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_N^n \end{pmatrix}$

x tq $f(x)$

$f'(x^n) = \frac{f(x^n)}{x^n - x^{n+1}}$

$f'(x^n)(x^n - x^{n+1}) = f(x^n)$



I will now solve the nonlinear system of N equations and N unknowns using Newton's method. I remind you that we are searching for the vector u, with components of u_1, u_2 up to u_N, the approximations of u in x_1, u in x_2 and so on to u in x_N such that the vector F(u) = 0. And F(u) contains these equation, the first equation is 2u_1 - u_2 all divided by h^2, the approximation of u" in x_1, plus x_1 times u_1^3 minus f(x_1), and so on for all equations. I shall now use Newton's method to find a zero of this function. A short recall on Newton's method. I am searching x such that f(x) = 0. Here the graph of the function f, and here the zero. I start from x_n and I want to calculate x_n+1. So I take the tangent and search for the intersection with the x axis. I have f(x_n), this is the graph of the function f, I have the slope f '(x_n) which is equal to f(x_n), the increase in y, f(x_n) - 0 divided by the increase in x which is x_n minus x_n+1. Beware of the sign: x_n minus x_n+1 ! Well f ' (x_n) times x_n minus x_n+1 must be equal to f(x_n). This is for a function from R to R. Now for a nonlinear system, let's assume u^n is known, well u^n is the vector of components u_1^nn, u_2^n and so on up to u _N^n with N lower index and n the upper index.

Notes

Summary

0m 03s

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$\vec{u}^n$ connu $\vec{u}^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_N^n \end{pmatrix}$ $\quad \vec{u}^{n+1}$ tq $DF(\vec{u}^n)\,(\vec{u}^n - \vec{u}^{n+1}) = \vec{F}(\vec{u}^n)$

$\forall \vec{u} \in \mathbb{R}^N \qquad DF(\vec{u}) = \begin{pmatrix} \frac{2}{h^2} + 3x_1(u_1)^2 & -\frac{1}{h^2} \\ & \end{pmatrix}$

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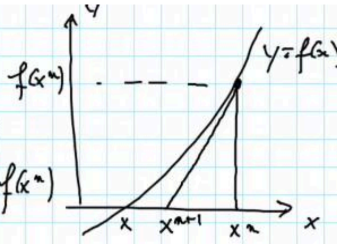

u^n is an approximation of the vector u such that F(u) = 0, and u_n+1 is such that f '(u_n) becomes the jacobian matrix DF evaluated in u^n, which I know, times the vector u^n minus u^n+1 and all this equal to F evaluated in u^n, which is also known since u^n is known. Now I must specify what the jacobian matrix is. Here is the expression of F(u), for all u in R^N, with components u_1, u_2 up to u_N. I will compute DF evaluated in u. So DF(u), what must I do ? I must compute all the derivatives, so I take the first line, I must differentiate this expression with respect to x_1. This will give me the coefficient " 1 - 1 " of this jacobian. So the derivative of this expression with respect to u_1, I get 2 / h^2 and here 3 * x_1 * u_1^2. I have 2 over h squared plus 3 times x_1 times u_1 squared. This is the derivative of this first line with respect to u_1. Then I must differentiate the first line with respect to u_2, So here I simply have -1 over h. This is the coefficient on the first line second column ("1 - 2") of the jacobian matrix. Next I must differentiate with respect to u_3, but u_3 does not appear in this equation so the derivative is 0. Ans do on up to u_N where the derivatives are o.

Notes

Summary

2m 34s

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$\forall \vec{u} \in \mathbb{R}^N \quad DF(\vec{u}) = \begin{pmatrix} \frac{2}{h^2} + 3x_1(u_1)^2 & -\frac{1}{h^2} & & (0) \\ -\frac{1}{h^2} & \frac{2}{h^2} + 3x_2(u_2)^2 & -\frac{1}{h^2} & \\ & \ddots & \ddots & -1/h^2 \\ (0) & & -1/h^2 & \frac{2}{h^2} + 3x_N(u_N)^2 \end{pmatrix}$

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$f'(x^n) = \frac{f(x^n)}{x^n - x^{n+1}}$

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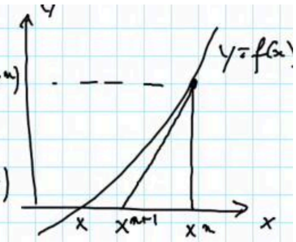


Algorithme : $\vec{u}^0$ donné

$n = 0, 1, 2, \dots$

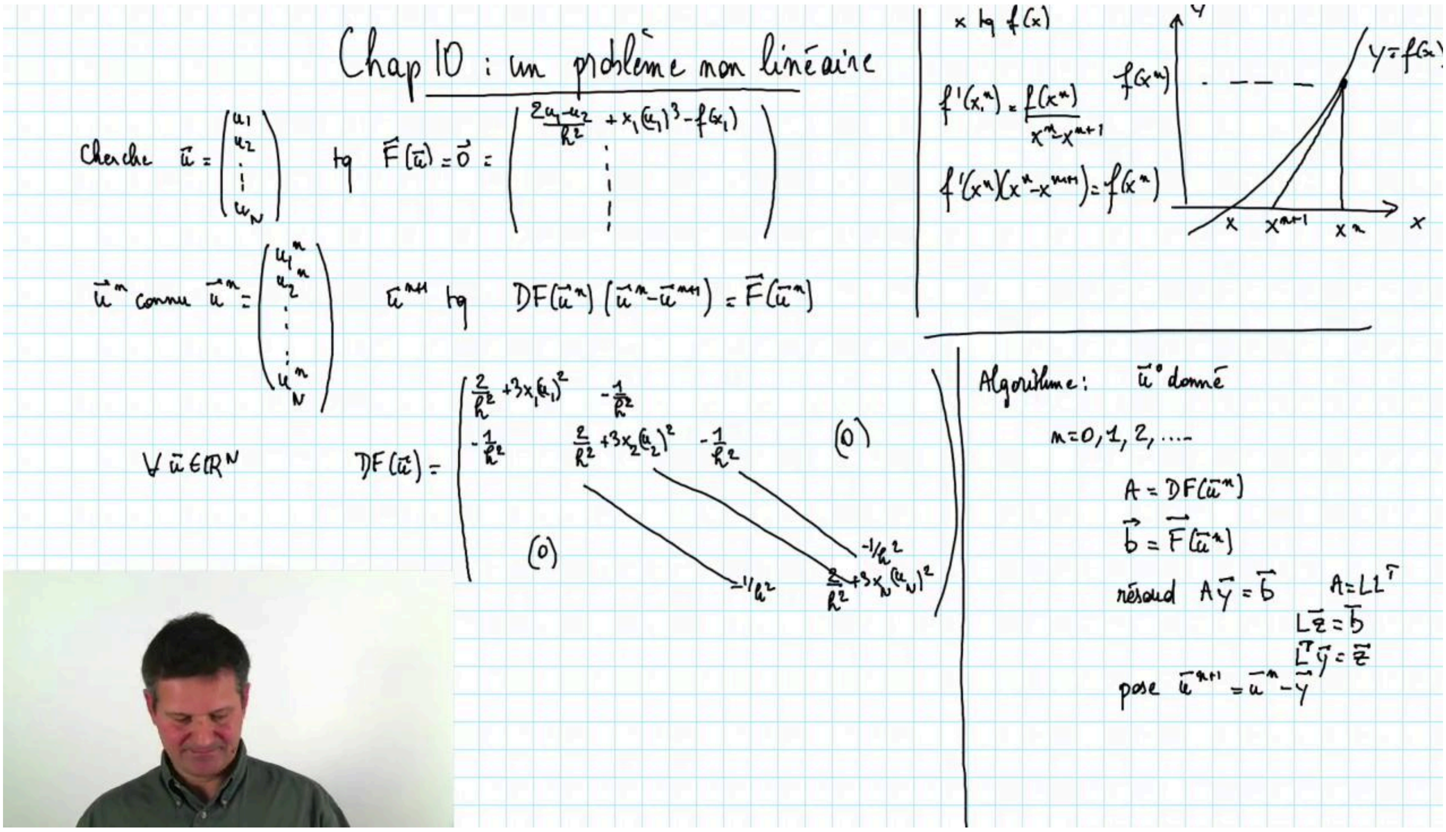
$A = DF(\vec{u}^n)$

So on the first line of this matrix I only have the derivative w.r.t. u_1, and here the derivative w.r.t. u_2. By doing the same thing on the second line, I get -1 / h^2, the term here below the diagonal- The term on the diagonal is 2 / h^2 plus 3 times x_2 times u_2 squared. And next I have I will differentiate w.r.t. u_3, and will get -1 / h^2 for the term above the diagonal. I will obtain a tridiagonal matrix. The coefficient above and below the diagonal are -1/ h^2. Whereas the coefficients on the diagonal are for example 2 over h squared plus 3 times x_N times u_N squared. All the other elements are zero. There we have it, I am able to compute the jacobian matrix. Now I can write the algorithm which corresponds to Newton's method. I start from an initial guess u^0 given, all the components u_0^0, u_1^0 up to u_N^0 are known. Next I loop: from n=0,1, 2 and so on, I must solve a linear system, this one here, a matrix times unknown vector equal to this known vector. I set A equal to DF(u^n). This is the jacobian matrix I have computed I did the computation for any vector u I have to evaluate it at u^n which is known. So at the first step of the algorithm, n=0, I must evaluate the jacobian at u^0.

Notes

Summary

4m 28s



Next I must compute the right hand side of the linear system, denoted as b. Here b is equal to F(u^n). C'est un vecteur. F vecteur de uN vecteur. We solve A y = b. Donc il se trouve, ici, que la matrice, I claim that DF (u) is a symmetric positive definite matrix. Indeed, it is a matrix with 2s and -1s here, which is symmetric positive definite, plus a diagonal matrix with positive entries therefore the sum is symmetric positive definite. I can compute the cholesky factorization L times L-transpose, A = L L^T. Next I must solve both of these linear systems. I must solve L z = b, then L-transpose times y = z. So I have solved this linear system A y = b. Here, y is u^n - u^n+1, so I define: u^n+1 equal to u^n-y. And when F(u^n) is small enough, I exit this loop. I know that Newton's method, if it converges, it will converge if the initial guess is sufficiently close to the solution u, then Newton's method will converge quickly. In practice, I will only do a few iterations of Newton's method. You can notice that we have a nonlinear system to solve, a nonlinear system with N equations and N unknowns, which comes from a nonlinear differential equation, and this nonlinear system is obtained by solving several linear systems.

Notes

Summary

6m 27s