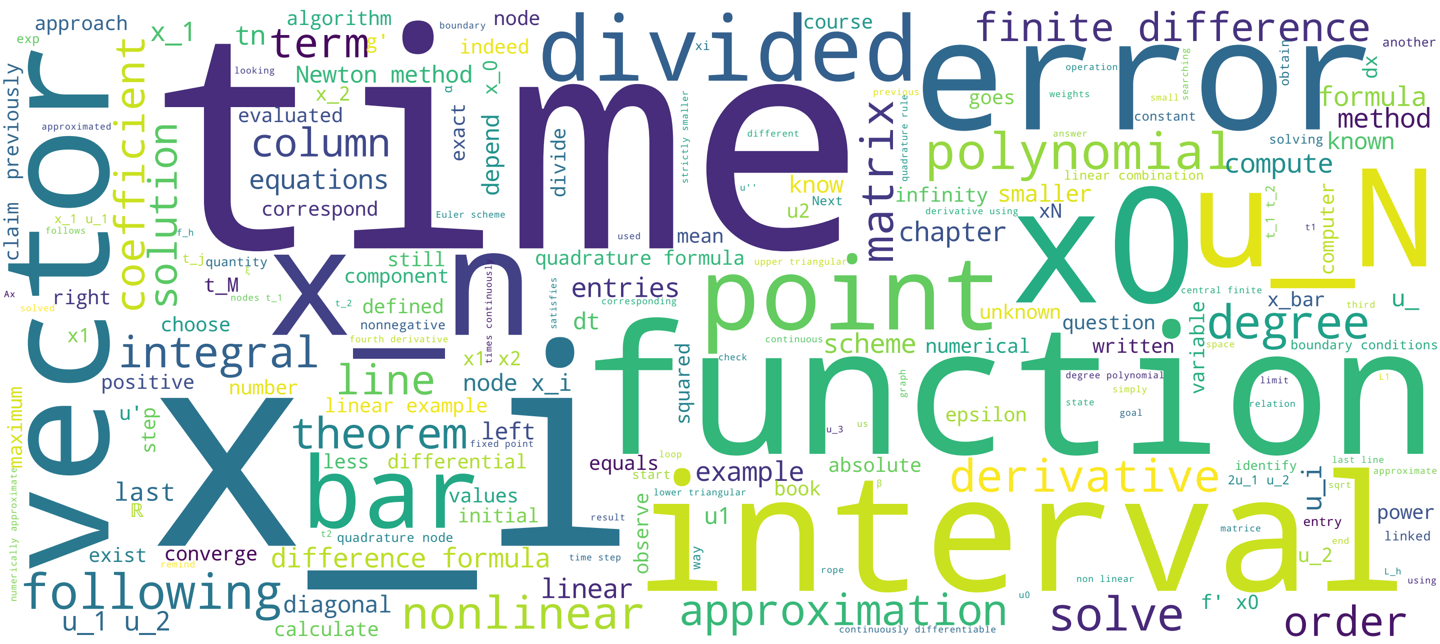
Chapitre 10 : Un problème non linéaire

Introduction à l'analyse numérique

Prof. Marco Picasso

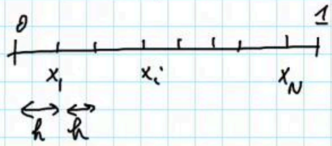


EPFL

Video

Chap 10 : un problème non linéaire

$$\begin{cases} -u''(x) + x\,(u(x))^3 = f(x) & 0<x<1 \\ u(0)=0 \\ u(1)=0 \end{cases}$$



$h = \frac{1}{N+1}$ $\quad x_i = i h \quad i=0,1,\dots,N+1$

u_i approx de $u(x_i)$

Now I want to solve a nonlinear boundary value problem. This nonlinear problem is the following. u must satisfy the following equation: -u''(x) plus a non linear term which I write x times u(x)^3, equal to f(x) with x in the interval [0,1] with the boundary conditions u (0) = 0 and u(1) = 0. I remind you that the function f is defined on the 0,1 interval and I am searching for u which satisfies this equation. In the previous linear example dans le cas linéaire, c'est-à-dire si cet terme n'existe pas, when I apply 2 times f, well the solution to the problem is 2 times u. This is not the case in the nonlinear problem. As before, I subdivide the interval [0,1] into sub-intervals. Here in 2, 4 and 8 and so on. And again like previously, I name the first node x_1, and x_N the last. So h, the space between two consecutive nodes is constant; it is 1 over N + 1. I set x_i = i times h, for i from 0 to N + 1. Here I have a node x_i, to its left the node x_i-1 and to its right the node x_i+1. And still like before, the goal is to calculate u_i, an approximation of u at point x_i. la solution du problème au point *xi*. So what do we do? Like the linear example, we write the differential equation at node x_i.

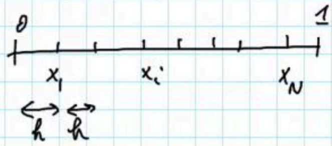
Notes

Summary

0m 03s

Chap 10 : un problème non linéaire

$$\begin{cases} -u''(x) + x(u(x))^3 = f(x) & 0<x<1 \\ u(0)=0 \\ u(1)=0 \end{cases}$$



$h = \frac{1}{N+1}$ $\quad x_i = ih \quad i=0,1,\dots,N+1$

u_i approx de $u(x_i)$

$$-u''(x_i) + x_i(u(x_i))^3 = f(x_i) \qquad i=1,\dots,N$$

$$\frac{-u(x_{i-1})+2u(x_i)+u(x_{i+1})}{h^2} + x_i(u(x_i))^3 = f(x_i) + O(h^2)$$

FDF centrée

schéma: $\frac{-u_{i-1}+2u_i-u_{i+1}}{h^2} + x_i(u_i)^3$

So I have - u"(x_i) + x_i * (u(x_i)^3), where x_i is i times h, all this equal to f(x_i) which is known. I can write these equations for all i from 1 to N. Now I can approach the second derivative using a central finite difference formula, *FDF*, centrée, pour l'approximation de la dérivée seconde, the same formula as before. Hence I have 2u(x_i), to the left minus u(x_i - h), the same as u(x_i-1), and to the right u(x_i+1), divided by h^2. comme tout à l'heure. There is an extrra term which is x_i times u(x_i)^3, this must be equal to f(x_i). And of course, I have approximated the second derivative using a finite difference formula, so we have a remainder term O(h^2), this error term depends on the fourth derivative of u. Now I can write a scheme, as before, I replace u(x_i) by its approximation u_i, which I can compute using a computer. So the scheme is the following. Scheme: something which can be implemented on a computer. I have 2 u_i, minus u_i-1 to the left and u_i+1 to the right, all divided by h squared. This term is the same as the linear example since it corresponds to the second derivative. There is another term which is x_i times u_i^3, which corresponds to x * u(x)^3, with x_i equal to i times h.

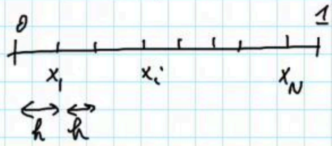
Notes

Summary

1m 55s

Chap 10 : un problème non linéaire

$$\begin{cases} -u''(x) + x\,(u(x))^3 = f(x) & 0<x<1 \\ u(0)=0 \\ u(1)=0 \end{cases}$$



$h = \frac{1}{N+1}$ $\quad x_i = ih \quad i=0,1,\dots,N+1$

u_i approx de $u(x_i)$

$$-u''(x_i) + x_i\,(u(x_i))^3 = f(x_i) \qquad i=1,\dots,N$$

$$\frac{-u(x_{i-1}) + 2u(x_i) + u(x_{i+1})}{h^2} + x_i\,(u(x_i))^3 = f(x_i) + O(h^2) \qquad \text{FDF centrée}$$

schéma : $\begin{cases} \dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + x_i\,(u_i)^3 = f(x_i) & i=1,\dots,N \\ u_0 = 0 \\ u_{N+1} = 0 \end{cases}$

Système non linéaire $\vec{F}(\vec{u}) = \vec{0}$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \qquad \vec{F}(\vec{u}) = \begin{pmatrix} F_1(u_1, u_2, \dots, u_N) \\ F_2(u_1\ u_2 \cdots u_N) \\ \vdots \\ F_N(u_1, u_2 \cdots u_N) \end{pmatrix} =$$

These two terms are equal to f(x_i), and I insist that f(x_i) is known. Like the previous case I drop the term in O(h^2). So this must be satisfied for all i from 1 to N, with the same convention where for i = 1 the term u_0 which is involved is given by the boundary conditions: u_0 = 0. The same for i = N, I have u_N+1 also equal to 0, u_N+1 = 0 from the other boundary condition. Here we now have scheme which can be implemented. Solving the linear boundary value problem led to solving a linear system, and now a nonlinear system must be solved. I started from a nonlinear differential equation and I have a nonlinear system to solve, which I will re-write in a more compact form, the vector F(u) = 0. Here u is the vector of unknowns, u_1, u_2 up to u_N, which are approximations of u(x_1), u(x_2) up to u(x_N), and then F(u), here I make a reference to chapter 8, about nonlinear systems, so F(u) can be written as F_1(u) where u is a vector, but the vector u has the components u_1, u_2 up to u_N, this is the first line, F_2 depends on u_1, u_2 up to u_N, and so on up to the last equation F_N depending on u_1, u_2,... u_N. In this particular case, the first equation is (2u_1 - u_2)/h^2 + x_1 * u_1^3 - f(x_1).

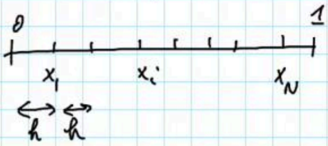
Notes

Summary

3m 55s

Chap 10 : un problème non linéaire

$$\begin{cases} -u''(x) + x\,(u(x))^3 = f(x) & 0<x<1 \\ u(0)=0 \\ u(1)=0 \end{cases}$$



$h = \frac{1}{N+1}$ $\quad x_i = ih \quad i=0,1,\dots,N+1$

u_i approx de $u(x_i)$

$$-u''(x_i) + x_i\,(u(x_i))^3 = f(x_i) \qquad i=1,\dots,N$$

$$\frac{-u(x_{i-1})+2u(x_i)+u(x_{i+1})}{h^2} + x_i\,(u(x_i))^3 = f(x_i) + O(h^2) \qquad \text{FDF centrée}$$

$$\text{schéma :}\begin{cases} \frac{-u_{i-1}+2u_i-u_{i+1}}{h^2} + x_i\,(u_i)^3 = f(x_i) & i=1,\dots,N \\ u_0=0 \\ u_{N+1}=0 \end{cases}$$

Système non linéaire $\vec{F}(\vec{u}) = \vec{0}$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \quad \vec{F}(\vec{u}) = \begin{pmatrix} F_1(u_1,u_2,\dots,u_N) \\ F_2(u_1\, u_2 \cdots u_N) \\ \vdots \\ F_N(u_1, u_2 \cdots u_N) \end{pmatrix} = \begin{pmatrix} \frac{2u_1-u_2}{h^2} + x_1(u_1)^3 - f(x_1) \\ \frac{-u_1+2u_2-u_3}{h^2} + x_2(u_2)^3 - f(x_2) \\ \vdots \\ \frac{-u_{N-1}+2u_N}{h^2} + x_N(u_N)^3 - f(x_N) \end{pmatrix}$$

Méthode de Newton Chap 8.

Here I have (2u_1 - u_2)/h^2, plus x_1 * u_1^3 minus f(x_1). This must be equal to 0, and is my first equation. In the same way I can write the second equation. It will be - u_1 + 2_u2 - u_3 all divided by h^2, now this term for i=2, x_2 times u_2^3 minus f(x_2). And so on until the last line which I will write. So for i = N, I have -u_N + 2u_N+1 all divided by h^2 plus x_N times u_N^3 minus f(x_N) and this equation must be equal to 0. This is indeed a nonlinear system of N equations, here the N equations. And I also have N unknowns, these are u_1, u_2 up to u_N. These equations can't be solved independently one from another, they are linked, u_1 is linked to u_2, and u_2 is linked to u_3 and so on. These relations between the variables are nonlinear, so we have a nonlinear system to solve. To solve this system of nonlinear equations, we shall use Newton's method from chapter 8.

Notes

Summary

6m 20s