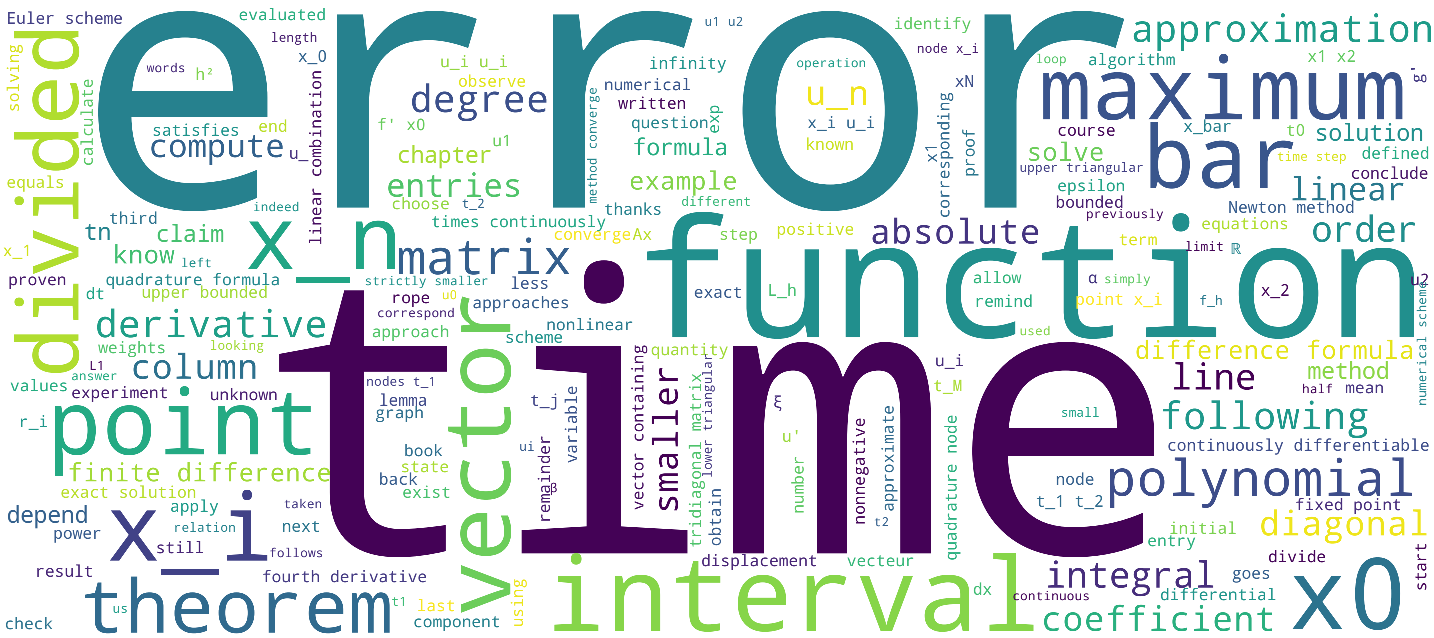
Chapitre 10 : Méthode de différences finies (suite)

Introduction à l'analyse numérique

Prof. Marco Picasso



Video

EPFL

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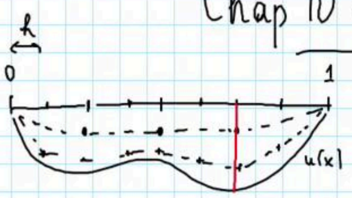

$u_i \xrightarrow{h \to 0} u(x_i)$?

l'erreur est divisée par $2^2 = 4$ si h est divisé par 2. $O(h^2)$

Thm: $u \in \mathcal{C}^4[0,1]$ $\exists$

I remind you that I subdivided this elastic rope on the interval [0,1] into sub-intervals of equal length h, here. This is the exact displacement of the rope which is unknown, here is the experiment I will do. I will first choose three points, I will compute the approximation of the displacement in 3 points, here is the approximate deformed rope, next I will divide h by 2, therefore I compute everything in 7 points instead of 3, I cut h in half and get the first node, second up to seventh point. Here is the approximation of the displacement with seven points. The question is the following. Does u_i, the approximation of u in point x_i, converge to the exact solution u in point x_i, which is unknown, as h approaches 0? I claim this is the case, if you look here, you see the error for 3 points this quantity was the error, and by dividing h by 2 the error has been divided by 4. I claim that the error is divided by 2^2 = 4 each time h is divided by 2. This method converges at rate O(h^2). I give now the precise mathematical statement of this result. Theorem: let u be 4 times continuously differentiable on the interval [0,1], in other words the data must be such that this function u is 4 times continuously differentiable.

Notes

Summary

0m 04s

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l'erreur est divisée par $2^2 = 4$ si h est divisé par 2. $O(h^2)$

Thm: $u \in \mathcal{C}^4[0,1]$ $\exists C > 0$ $\forall\, 0 < h < 1$ $\max\limits_{1 \le i \le N} |u_i - u(x_i)| \le C h^2$.

Dem: schéma: $\dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i)$ $\qquad A\vec{u} = \vec{f}$

$$\frac{-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})}{h^2} = f(x_i) + O(h^2) \qquad A\vec{w} = \vec{f} + \vec{r} \qquad \vec{w} = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{pmatrix} \quad \vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix}$$

Then there exists a C positive such that for all h, positive and smaller than one, the maximum over i=1,...N, of the error u(x_i)-u_i in absolute value I claim that this error is smaller than C times h^2. Proof of this theorem. Here is the numerical scheme, le schéma c'était schéma numérique, je vous rappelle le schéma, -u_i-1 + 2 u_i - u_i+1 all divided by h^2, moins *ui + 1*, la valeur à droite divisée par *h²*, equal to f(x_i) for all i from 1 up to N. It can be written in a more compact form A u = f where A is the tridiagonal matrix with 2 on the diagonal and -1 off diagonal, f the vector containing the values f(x_i) for i from 1 to N. Now the exact solution, so u(x) satisfies the following equation, do not mix up u(x_i) with u_i, u_i is the approximation and u(x_i) is the exact value at node x_i which I do not know. So u satisfies 2 u(x_i) minus u(x_i-1) minus u(x_i+1) divided by h^2, all this equal to f(x_i) plus a term of order O(h^2), I will write these relations for i from 1 to N, Je vais écrire ces lignes sous la forme suivante : I write A w = f +, where w is the vector containing the exact solution at nodes x_i, so u(x_1), u(x_2) up to u(x_N), with the remainder is a vector with components r_1, r_2 and so on up to r_N.

Notes

Summary

2m 19s

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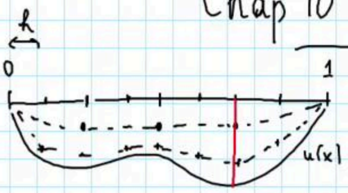

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$$\frac{-u(x_{i-1}) + 2u(x_i) - u(x_{i+1})}{h^2} = f(x_i) + O(h^2) \qquad A\vec{w} = \vec{f} + \vec{r}$$

$$\vec{w} = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{pmatrix} \qquad \vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix}$$

Chap 2

$|r_i| \le C h^2$

$C = \frac{1}{12} \max_{0 \le x \le 1} |u^{(4)}(x)|$

$$A(\vec{w} - \vec{u}) = \vec{r}$$

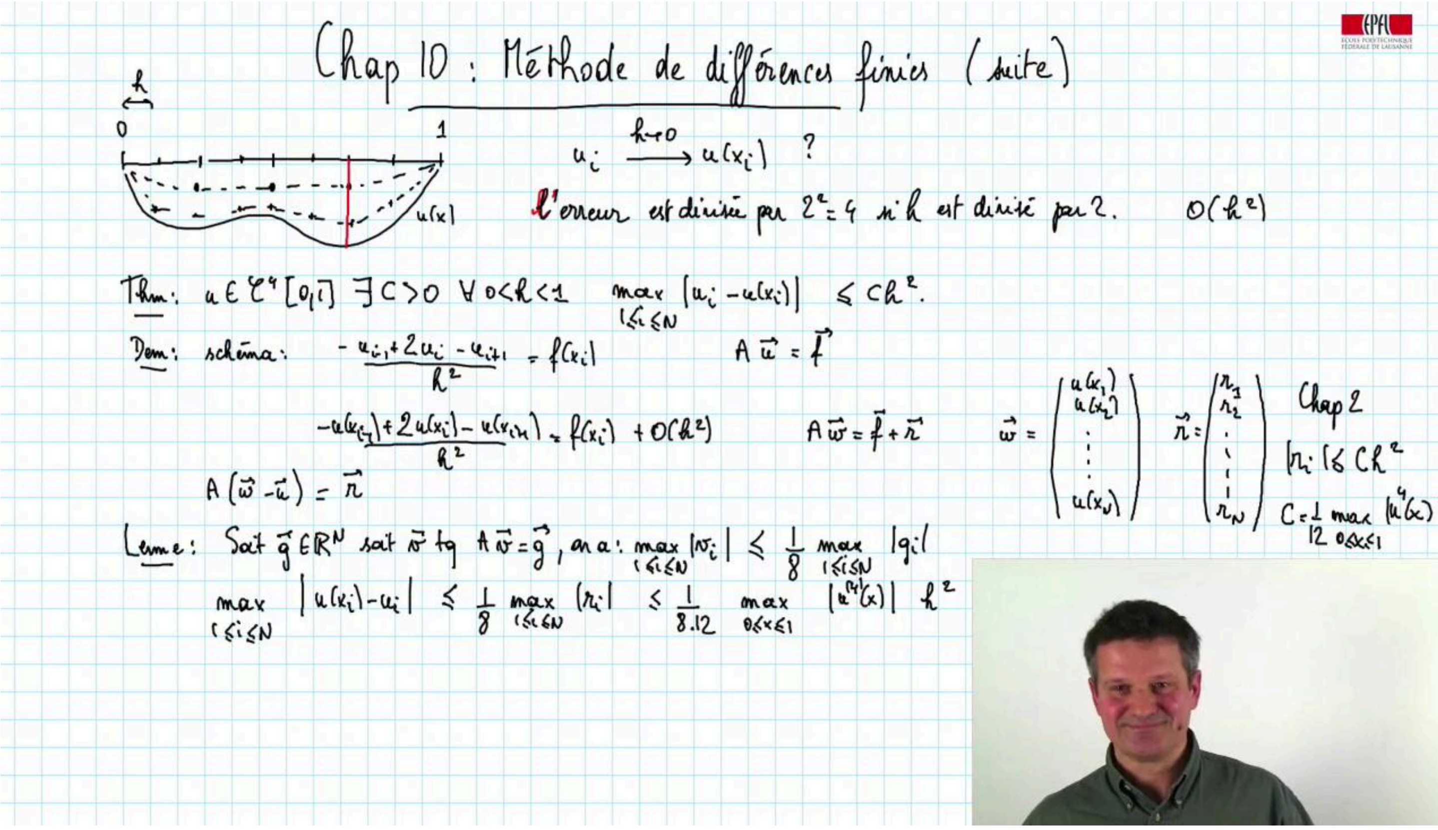
Lemme: Soit $\vec{g} \in \mathbb{R}^N$ soit $\vec{v}$ tq $A\vec{v} = \vec{g}$, on a : $\max_{1 \le i \le N} |v_i| \le \frac{1}{8} \max_{1 \le i \le N} |g_i|$

From chapter 2, I have proven that r_i was bounded by a constant time h^2, if you go back to the proof of the finite difference formula for the second derivative, you can see that this remainder is one twelfth of the maximum of the fourth derivative of x, taken in absolute value over the interval [0,1]. So A u = f, and A w = f +r with w containing the exact values at nodes x_1, x_2 up to x_N, and r the errors which are bounded by h^2. By substracting these two lines, I get A(w-u) = f+r-f, that is r. This is the equation for the vector containing the errors, that is u (x_1) - u_1 and u(x_2) - u_2 and so on. Now I state a lemma which will allow me to conclude, a linear algebra lemma given without proof. Let g be any vector of R^N, and let v such that A v = g, the matrix A being 1/h^2 times the matrix with 2 on the diagonal and -1 above and below the diagonal. So let g be a vector and v such that A v = g. Then the maximum of the absolute value of the component v_i is less than the maximum of the absolute value of the components g_i times one over eight, g_i being the components of vector g. Thanks to this lemma, I can now conclude.

Notes

Summary

5m 03s



I remind you that I have A(w-u) = r, so I apply this Lemma with g = r, where r is the remainder coming from the Taylor expansion and v is w-u. *x1xN - u* qui est le vecteur qui contient l'approximation *u1 u2 uN* I apply this result here, and I have that the maximum for all i from 1 to N, here v_i = u(x_i) - u_i, u(x_i) is the exact deformation of the rope which I do not know, and u_i its approximation obtained by solving the linear system A u = f. The maximum of this error is, thanks to this inequality, upper bounded by one over eight times the maximum of the components of the vector r. Since the r_i are smaller than C times h^2 here C is one twelfth times the maximum of the fourth derivatives of u, I finally obtain that the error is bounded by one over 8 times 12, times the maximum of the fourth derivative of u (in abs. val.) evaluated on the interval [0,1] times h^2. I have thus completed the proof of the theorem. The maximum of the error is upper bounded by C times h^2, here is C, it depends on the fourth derivative of u but not on h, as stated in the theorem. This theorem says that if u is four times continuously differentiable, then there exists a C such that for all h, the error is smaller than C times h^2. I have therefore proven this theorem.

Notes

Summary

7m 29s