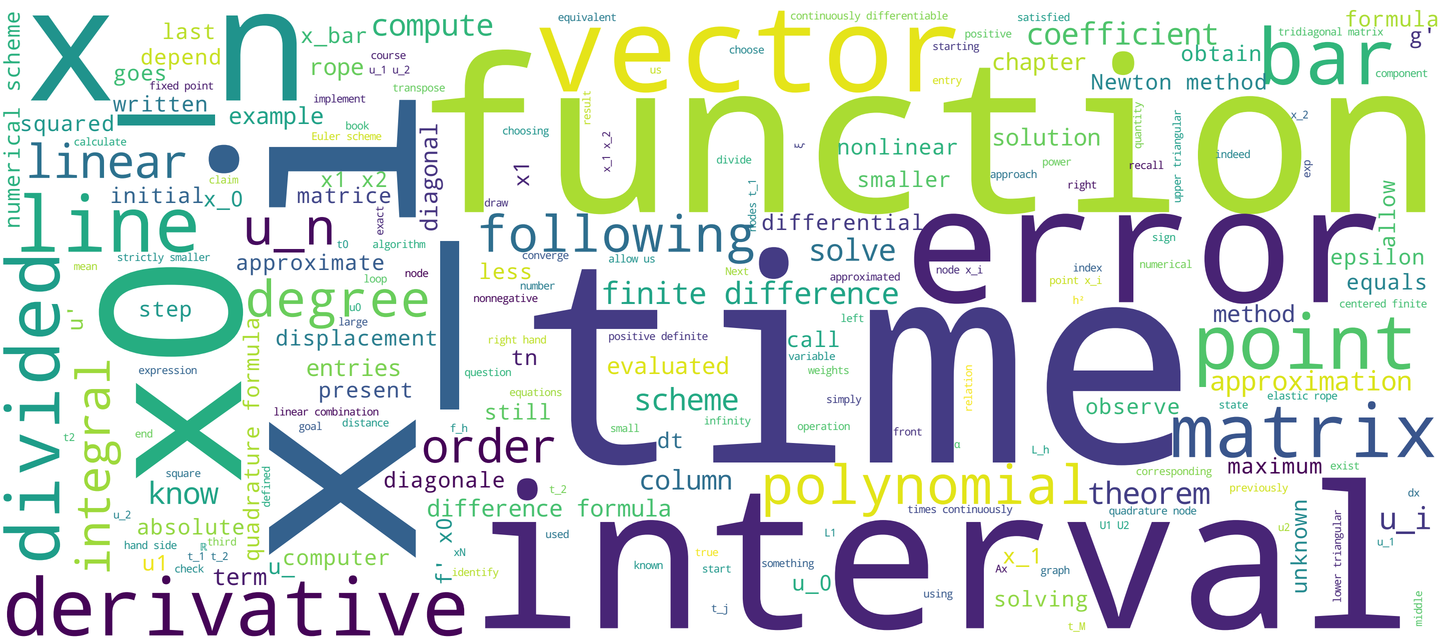
Chapitre 10 : Méthode de différences finies

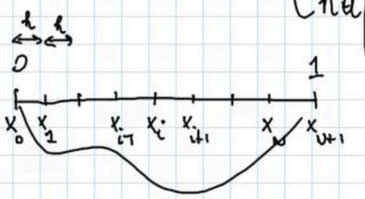
Introduction à l'analyse numérique

Prof. Marco Picasso



Video

Chap 10 : Méthode de différences finies



N entier pos (grand) $h = \frac{1}{N+1}$ pas d'espace (petit)

$x_i = ih \quad i = 0, 1, \dots, N, N+1.$

But : calculer des valeurs u_i approx de $u(x_i)$ $i = 1, 2, \dots, N$

I will now present a finite difference method which allow us to compute an approximation of the solution u of our toy problem, the elastic rope problem. Firstly, I will subdivide the interval [0,1] which models the rope into sub-intervals. So I cut the rope into 2, then 4, then 8. I can write x_1 the first point and x_N the last point inside the rope. Here N is a positive integer which should be large, and I write h = 1 over N plus 1 the step size, which should be small if N is large. N will approach infinity, and h will approach 0. Therefore, in the middle I have a point x_i, to its left x_i minus h, that is x_i-1 and to its right x_i plus h, that is x_i+1. x_0 coincides with 0 and x_N+1 with 1, hence I have x_i equal to h times i for i starting at 0 up to N+1. The goal of the method is the following: the goal of the numerical method is to present a scheme which will allow us to compute values u_i, approximations of u at point x_i, which I do not know. I recall that u(x) is the displacement of the elastic rope I don't know u(x), I will approximate u(x_i) by u_i, for all i from 1 up to N. Here I can draw a picture, when the x_i are equidistant separated by h, the distance between two consecutive points.

Notes

Summary

0m 04s

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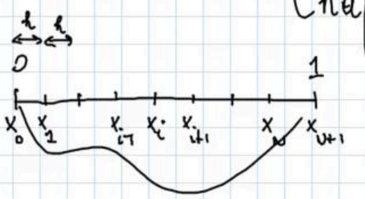

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$$-u''(x_i) = f(x_i) \qquad i = 1, 2, \dots, N$$

$$-\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} = f(x_i) + O(h^2)$$

Formule de diff. finie centrée (chap 2)

I now want to present the numerical scheme. The differential equation is the following: minus the second derivative of u with respect to x equal to f (x): here I write it down: -u''(x) = f(x). Which is satisfied for all x in the interval [0,1]. I can write it at points x_i -u''(x_i) = f(x_i) and this is true for all interior nodes with i from 1 up to N. I use a centered finite difference formula to approximate the second derivative. We saw it in chapter 2. The second derivative in node x_i can be approximated by minus 2 times the value in x_i of u, with u evaluated in x_i minus h, but x_i minus h is x_i-1; and the value if u in x_i plus h that is x_i+1. All this divided by h squared. There is a minus sign in front of the second derivative and this term is equal to f evaluated in x_i. Of course, as I approximated the second derivative with this finite difference formula I have a remainder which is a term of order h^2, this remainder term will be divided by 4 each time h is divided by 2, assuming that u is 4 times continuously differentiable. We know that this O(h^2) term involves the fourth derivatives of u in the interval [0,1]. This was discussed in chapter 2.

Notes

Summary

2m 10s

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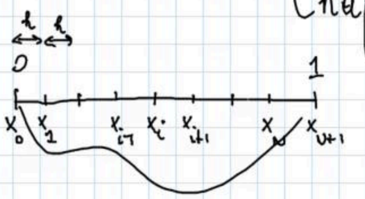

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Formule de diff. finie centrée (chap 2)

schéma: $$\begin{cases} \dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i) & i = 1, \dots, N \quad \text{syst} \\ u_0 = 0 \\ u_{N+1} = 0 \end{cases}$$

Now I can write a scheme: A numerical scheme is something I will be able to implement on a computer, which will allow me to compute the approximations u_i. So to obtain this scheme, I replace u at node x_i, which is unknown, by its approximation u_i, which will be computed thanks to a computer. The same thing for u evaluated in x_i-1 by u_i-1 and u in x_i+1 by u_i+1. Therefore I have the following scheme: I have two times u_i, on the left minus u_i-1 and u_i+1 on the right, all this divided by the square of h, this must be equal to f evaluated in x_i, and I discard the O(h^2) term, since it don't know how to implement it. these relations must be satisfied for all i from 1 to N, for i=1 to N. Et vous voyez ici dans cette expression When choosing the index i = 1 I have u_1, here u_2 and here u_0, so what is u_0 worth? u_0 is the displacement of the rope in x_0, which is zero. Hence I write that u_0 = 0. The same on the right end, the displacement of the rope in x_n+1 is zero which allows me to write u_n+1 = 0. Now I have my numerical scheme which allows me to approximate u in x_i with u_i, for all i from 1 to N. This scheme is equivalent to solving a linear system.

Notes

Summary

3m 57s

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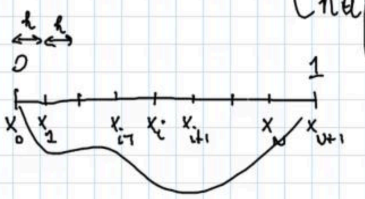

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schéma : $\begin{cases} \dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f(x_i) & i = 1, \dots, N \\ u_0 = 0 \\ u_{N+1} = 0 \end{cases}$

système linéaire :

$$\underbrace{\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}}_{\vec{u}} = \underbrace{\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{pmatrix}}_{\vec{f}}$$

This system can be written the following way: a matrix which I call A and the vector of unknowns named u. So the vector of unknowns is simply u_1, u_2, up to u_N. Donc ici U1 U2... jusqu'à U N and on the right hand side, the vector, which is known, contains the forces applied in x_1, x_2 up to x_N. I call this vector f. I state that this scheme corresponds to solving a linear system. So, I still need to define what this matrix A contains. You see here that on each line, there is 1 / h^2, so this can be a factor in front, I look at the first line for the index i = 1: u_0 = 0, I have 2 u_1 minus u_2 divided by h squared, which equals f (x_1), so, the first line is 2 and -1. The second line is (- u_1 + 2u_2 - u_3)/h^2 = f(x_2). Therefore on the second line I will write -1, 2 and -1, and so on up to the line one before last: -1, 2 and -1 again and then -1 and 2 on the last line. I obtain a tridiagonal matrix, on the diagonal we have 2s, and on the off diagonals we have -1s. et sur la sur-diagonale vous avez aussi un -1 So now I can compute approximations of u_1, u_2 up to u_N of u at the nodes x_1, x_2, up to x_N, provided I solve this linear system.

Notes

Summary

5m 53s

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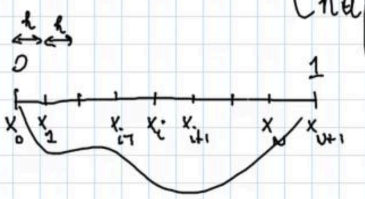

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système linéaire :

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & (0) \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ (0) & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{pmatrix}$$

$$A \qquad \vec{u} \quad = \quad \vec{f}$$

A est sym. déf. pos. $\quad A = LL^T$

et je prétends que la matrice A cette matrice qui contient 1 sur h² 2 sur la diagonale et - 1 sur la sur et la sous-diagonales The matrix A is tridiagonal, symmetric and positive definite, so I can use a cholesky factorization A = L times the transpose of L to solve this system A u = f.

Notes

Summary

7m 51s