Chapitre 9 : Schéma d'Euler rétrograde

Introduction à l'analyse numérique

Prof. Marco Picasso

Video

Chap 9 - Schéma d'Euler rétrograde

Euler progr : $\frac{u^{n+1}-u^n}{h} = f(u^n, t_n)$

Euler rétr : $\frac{u^{n+1}-u^n}{h} = f(u^{n+1}, t_{n+1})$ origine ? $\dot{u}(t_{n+1}) = f(u(t_{n+1}), t_{n+1})$ form. diff. finie rétr.

$$\frac{u(t_{n+1})-u(t_n)}{h} = f(u(t_{n+1}), t_{n+1}) + O(h)$$

sch Euler rétr. implicite : $\underbrace{u^{n+1}-u^n - h\, f(u^{n+1}, t_{n+1})}_{g(u^{n+1})} = 0$

I just introduced the forward Euler's method. (u_(n+1)- u_n)/h = f(u_n, tn). Now I will present Euler's backward method, which is written : (u_(n+1) - u_n)/h, as in the forward method and instead of f(u_n,tn) we write f(u_(n+1),t(n+1)). How do we get this method ? Well, we write the differential equation at time t(n+1,) so u'(t(n+1))= f(u(t(n+1)),t(n+1)) this time we use a backward finite difference formula, in other words we'll approach u'(t(n+1)) by (u(t(n+1))-u(t(n+1)-h))/h this must be equal to f(u(t(n+1)),t(n+1)) plus obviously a remainder of order 1 in h, since I used a backward finite difference formula. That was in chapter 2 of the course. Now, to get the scheme, I simply replace u(t(n+1)), the exact solution at the time t(n+1), which I do not know, by u_(n+1). I replace u(tn) which I do not know, by u_n, and likewise here, f(u_(n+1),t(n+1)), I give up on the term in O(h) and get Euler's backward method. This scheme is an implicit scheme. What does this mean? Well, there is an implicit relation between u_(n+1) and u_n, so I cannot write u_(n+1) =... since u_(n+1) appears here and there, But I can write the method in the following way: I can write (u_(n+1)-u_n)-h*f(u_(n+1),t(n+1)), this must be equal to 0, so if I denote this as g(u_(n+1)), it boils down to finding the 0 of the function g.

Notes

Summary

0m 04s

Chap 9 - Schéma d'Euler rétrograde

Euler progr : $\frac{u^{n+1}-u^n}{h} = f(u^n, t_n)$

Euler rétr : $\frac{u^{n+1}-u^n}{h} = f(u^{n+1}, t_{n+1})$ origine ? $\dot{u}(t_{n+1}) = f(u(t_{n+1}), t_{n+1})$ form. diff. finie rétr.

$$\frac{u(t_{n+1})-u(t_n)}{h} = f(u(t_{n+1}), t_{n+1}) + O(h)$$

sch Euler rétr. implicite : $\underbrace{u^{n+1}-u^n - h\, f(u^{n+1}, t_{n+1})}_{g(u^{n+1})} = 0$ u^{n+1} zéro de g

$$g(x) = x - u^n - f(x, t_{n+1})$$
$$g'(x) = 1 - \frac{\partial f}{\partial x}(x, t_{n+1})$$

Newton : $x_0 = u^n$

$m = 0, 1, 2, \dots$

$$x_{m+1} = x_m - \frac{g(x_m)}{g'(x_m)} = x_m - \frac{x_m - u^n - f(x_m, t_{n+1})}{1 - \frac{\partial f}{\partial x}(x_m, t_{n+1})}$$

Si $|x_{m+1} - x_m| < 10^{-8}$ $u^{n+1} = x_{m+1}$

So u_(n+1) is the zero of the function g. The function g which is defined for all x in ℝ by g(x)=x, so I replace u_(n+1) by x, x-u_n, which is known, minus f(x,t(n+1)). And to find the zero of the function I will use Newton's method, so I will need g': g'(x)=1, u_n is know, minus here df*dx at (x,t(n+1)). So Newton's method is written in the following way: The Newton method that let us from u_n which we know, to find u_(n+1), such that g(u_(n+1))=0. I start from u_n, which is the last value I computed. Then I do a loop m=0,1,2, etc... It is the Newton's loop, so I let x_(m+1)=x_m-g(x_m), (I am looking for the zero of g) divided by g'(xm). So g(x_m) = x_m-u_n (which we know) - f(x_m,t(n+1))/g'(x_m), which is 1-df*dx at the point x_m and time t(n+1). To finish, I exit the loop when x(m+1)-x(m) is very small for example when less than 10^-8. In this case, I let u_(n+1) be the last computed value, that is, x_(m+1). So, see that to go from u_n to u_(n+1) I must do a loop, m=0,1,2,... but in practice, you know that Newton's method, if it converges, that is to say its starting point is close enough from the solution, converges quickly, at least when g'(u_(u+1)) is nonzero, il est dérivé au dénominateur, donc converge rapidement, in practice after a few iterations, we will get the solution to our non linear system, in other words, u_(n+1).

Notes

Summary

2m 32s

Chap 9 - Schéma d'Euler rétrograde

Euler progr : $\frac{u^{n+1}-u^n}{h} = f(u^n, t_n)$

Euler rétr : $\frac{u^{n+1}-u^n}{h} = f(u^{n+1}, t_{n+1})$ origine ? $\dot{u}(t_{n+1}) = f(u(t_{n+1}), t_{n+1})$ form. diff. finie rétr.

$$\frac{u(t_{n+1})-u(t_n)}{h} = f(u(t_{n+1}), t_{n+1}) + O(h)$$

sch Euler rétr. implicite : $\underbrace{u^{n+1}-u^n - h\, f(u^{n+1}, t_{n+1})}_{g(u^{n+1})} = 0$ u^{n+1} zéro de g

$g(x) = x - u^n - f(x, t_{n+1})$

$g'(x) = 1 - \frac{\partial f}{\partial x}(x, t_{n+1})$

Newton : $x_0 = u^n$

$m = 0, 1, 2, \dots$

$$x_{m+1} = x_m - \frac{g(x_m)}{g'(x_m)} = x_m - \frac{x_m - u^n - f(x_m, t_{n+1})}{1 - \frac{\partial f}{\partial x}(x_m, t_{n+1})}$$

Si $|x_{m+1} - x_m| < 10^{-8}$ $u^{n+1} = x_{m+1}$

inconvénient : impl, plus diff à progr.

avantage : stabilité ?

"Drawbacks of this method" We see that this method is implicit, so more difficult to implement. It is also more costly since I have to do some iteration, but not too many, in practice less than 10. The advantage of this method, compared to a forward Euler's method, which is explicit, is that this method will be stable, no matter what the time step h is. We now have to talk about stability of Euler's schemes.

Notes

Summary

5m 08s