Chapitre 9 : Théorème 9.1 (existence)

Introduction à l'analyse numérique

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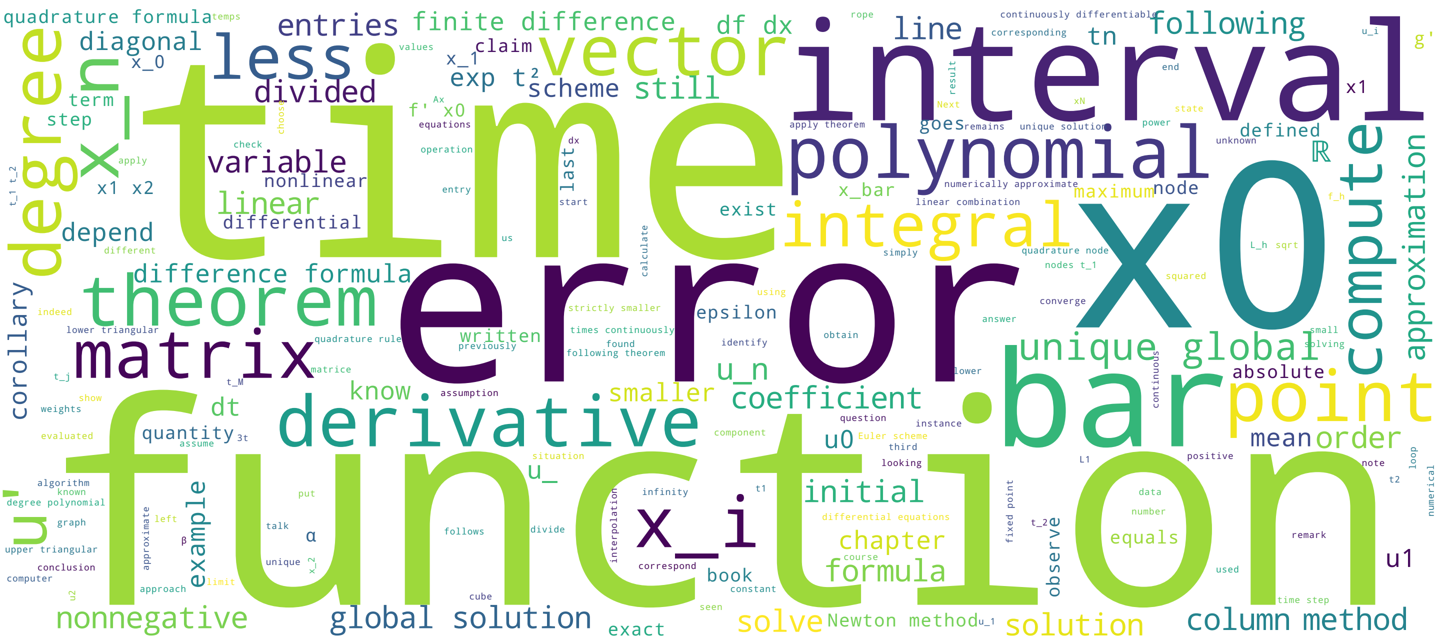


Video

Chap 9 - Théorème 9.1 (existence)

Thm 9.1 : données : $u_0 \in \mathbb{R}$ $f(x,t)$

pbm : trouver $u(t)$ tq $\dot{u}(t) = f(u(t),t)$ $t \geq 0$ $u(0) = u_0$

hyp : $f(x,t)$ continue et $\exists\, \ell : \mathbb{R}_+ \to \mathbb{R}$, $t \to \ell(t)$ tq $\forall x,y \in \mathbb{R}$ $\forall t \geq 0$ $(f(x,t)-f(y,t))(x-y) \leq \ell(t)\,(x-y)^2$.

Concl : ℓ

We have seen that there are are situations where the differential equation has a unique solution at all time t, and there are situations where there are several solutions, or no solution at all after a certain time. il n'y a plus de solution. The following theorem states that, under a certain assumptions, there is one and only one solution. Here is the theorem without proof, theorem 9.1 of the book. Recall that the data of the problem are the initial value u0 and the function f that depends on the two variables x, t, x in ℝ, and t nonnegative. The problem to solve is to find u, a function of t, such that u'(t)=f(u(t),t) for t nonnegative with initial value u(0)=u0. The assumption on the function f is: f is continuous, moreover I have to find another function l(t) qui à t retourne l de t t here is the time, it is nonnegative, l(t) is in ℝ, and such that if I take any x,y in ℝ and any t nonnegative, I compute f(x,t)-f(y,t) multiplied by x minus y, it must be less or equal than l(t) times (x-y)². Let's repeat, from f, I must find a function l such that for all x,y,t (f(x,t)-f(y,t))*(x-y) is less or equal to l(t)*(x-y)². In that case the conclusion is as follows.

Notes

Summary

0m 04s

Chap 9 - Théorème 9.1 (existence)

Thm 9.1 : données : $u_0 \in \mathbb{R}$ $\quad f(x,t)$

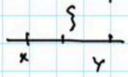
pbm : trouver $u(t)$ tq $\dot{u}(t) = f(u(t),t)$ $\quad t > 0 \quad u(0) = u_0$

hyp : $f(x,t)$ continue et $\exists\, \ell : \mathbb{R}_+ \to \mathbb{R}$, $t \to \ell(t)$ tq $\forall x,y \in \mathbb{R}\ \forall t \geq 0\ \big(f(x,t)-f(y,t)\big)(x-y) \leq \ell(t)\,(x-y)^2$.

concl : le pbm admet une solution globale unique.

Corollaire : $f(x,t)$ C^1 $\exists K \in \mathbb{R}\ \forall x \in \mathbb{R}\ \forall t \geq 0\ \frac{\partial f}{\partial x}(x,t) \leq K$. Alors le pbm admet une solution globale unique.

Dem : Soit $x,y \in \mathbb{R}$, soit $t \geq 0$, on a $\big(f(x,t)-f(y,t)\big)(x-y) = \frac{\partial f}{\partial x}(\xi,t)(x-y)^2 \leq K(x-y)^2$



Ther exists a unique u such that u'(t)=f(u(t),t), for all t, with u(0)=u0, we shall say that the problem has a unique global solution u. c'est-à-dire pour tout temps t et unique et une seule. Corollary of this theorem: assume that f(x,t) is now continuously differentiable with respect to the first variable, and assume that there exists K in ℝ such that for all x in ℝ, for all nonnegative t, the derivative of f with respect to the first variable is less than or equal to this K. I claim that in this case the problem has a unique global solution. Proof of this corollary: Take any real values x,y, take any nonnegative t, compute (f(x,t)-f(y,t))*(x-y), we want to show that quantity is less or equal some l(t) that I must find, times (x-y)^2. The mean value theorem tells me that f(x,t)-f(y,t) can be written as df/dx alors c'est bien la première variable qui change : xy at some intermediate point, denoted ξ (ξ between x and y) so (f(x,t)-f(y,t))*(x-y) is df/dx(ξ)*(x-y)^2. that is the mean value theorem and I had still x-y here donc pour finir, j'arrive à un x moins y au carré. The assumption is that df/dx is always less or equal K, So his is less than or equal to K times (x-y)².

Notes

Summary

2m 14s

Chap 9 - Théorème 9.1 (existence)

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hyp : $f(x,t)$ continue et $\exists\, \ell : \mathbb{R}_+ \to \mathbb{R},\ t \to \ell(t)$ tq $\forall x,y \in \mathbb{R}\ \forall t \geq 0\ \ (f(x,t)-f(y,t))(x-y) \leq \ell(t)(x-y)^2$.

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Corollaire : $f(x,t)$ C^1 $\exists K \in \mathbb{R}\ \forall x \in \mathbb{R}\ \forall t \geq 0\ \ \frac{\partial f}{\partial x}(x,t) \leq K$. Alors le pbm admet une solution globale unique.

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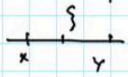

on applique le thm 9.1 $\ell(t)=K$

Ex 9.1 : $\begin{cases} \dot{u}(t) = 3u(t)-3t \\ u(0)=\alpha \end{cases}$ $\quad f(x,t)=3x-3t \quad \frac{\partial f}{\partial x}(x,t)=3=K$. le pbm admet 1 sol. glob. unique

$\begin{cases} \dot{u}(t) = -(u(t))^3 \\ u(0)=1 \end{cases}$ $\quad f(x,t) = -x^3 \quad \frac{\partial f}{\partial x}(x,t) = -3x^2 \leq 0 = K$.

Therefore, the function l(t) to be found is defined by l(t)=K. We can apply theorem 9.1 of the book with l(t)=K, the problem has a unique global solution. Let's consider the previous examples. Example 9.1 was: u'(t)=3u(t)-3*t u(0)=α the function f(x,t) here is f(x,t)=3x-3t I compute df/dx(x,t), it is equal to 3, and I can use the corollary of theorem 9.1: the problem has a unique global solution. We found explicitly the solution. une exponentielle 3t et une solution particulière. Now, if I replace 3t in the definition of f(x,t) by exp(t²) for instance, I cannot find the solution u explicitly anymore, f(x,t)=3x+exp(t²), but df/dx(x,t) is still equal to 3 so I can still use the corollary of the theorem to state that there is a unique global solution u. que je ne peux pas expliciter. Mais je sais qu'il y a une solution globale unique. Another example was u'(t)=-u(t)³ égal U de t en cube, avec un signe moins with an initial value u(0)=1. Remark that in the corollary of the theorem, the initial value u0 does not matter. So u'(t)=-u(t)³ so the function f is f(x,t)=-x³, its derivative with respect to the first variable is -3x², which is les or equal 0, so I can apply the corollary of the theorem with K=0, the problem has a unique global solution.

Notes

Summary

4m 46s

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on applique le thm 9.1 $\ell(t) = K$

Ex 9.1 : $\begin{cases} \dot{u}(t) = 3u(t) - 3t \\ u(0) = \alpha \end{cases}$ $f(x,t) = 3x - 3t$ $\frac{\partial f}{\partial x}(x,t) = 3 = K$. le pbm admet 1 sol. glob. unique

$\begin{cases} \dot{u}(t) = -(u(t))^3 \\ u(0) = 1 \end{cases}$ $f(x,t) = -x^3$ $\frac{\partial f}{\partial x}(x,t) = -3x^2 \leq 0 = K$.

I can again do the same remark : I could add $\exp(t^2)$ here, $f(x,t)$ would be $-x^3 + \exp(t^2)$, the derivative with respect to x remains the same and so the problem still has a unique global solution, even if I cannot write down a formula for it. Now we will solve numerically differential equations having unique global solutions.

Notes

Summary

7m 02s