Chapitre 9 : Equations différentielles du 1er ordre – Position du problème

Introduction à l'analyse numérique

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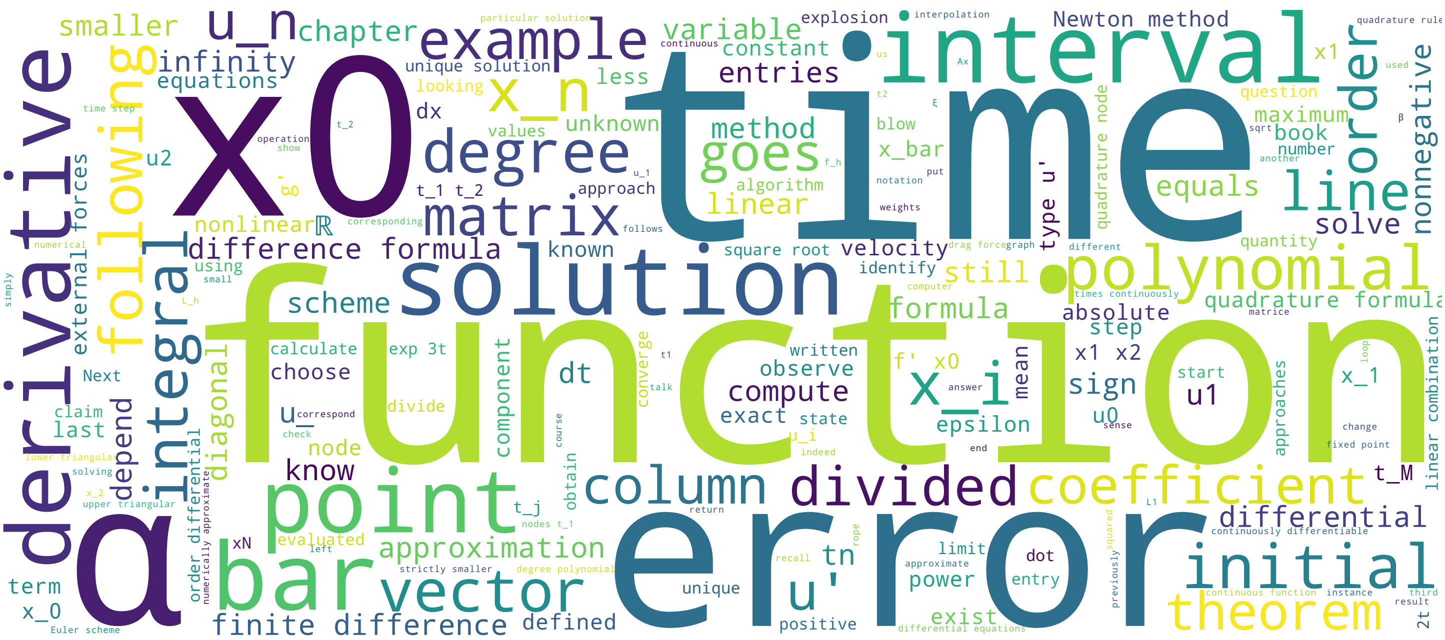
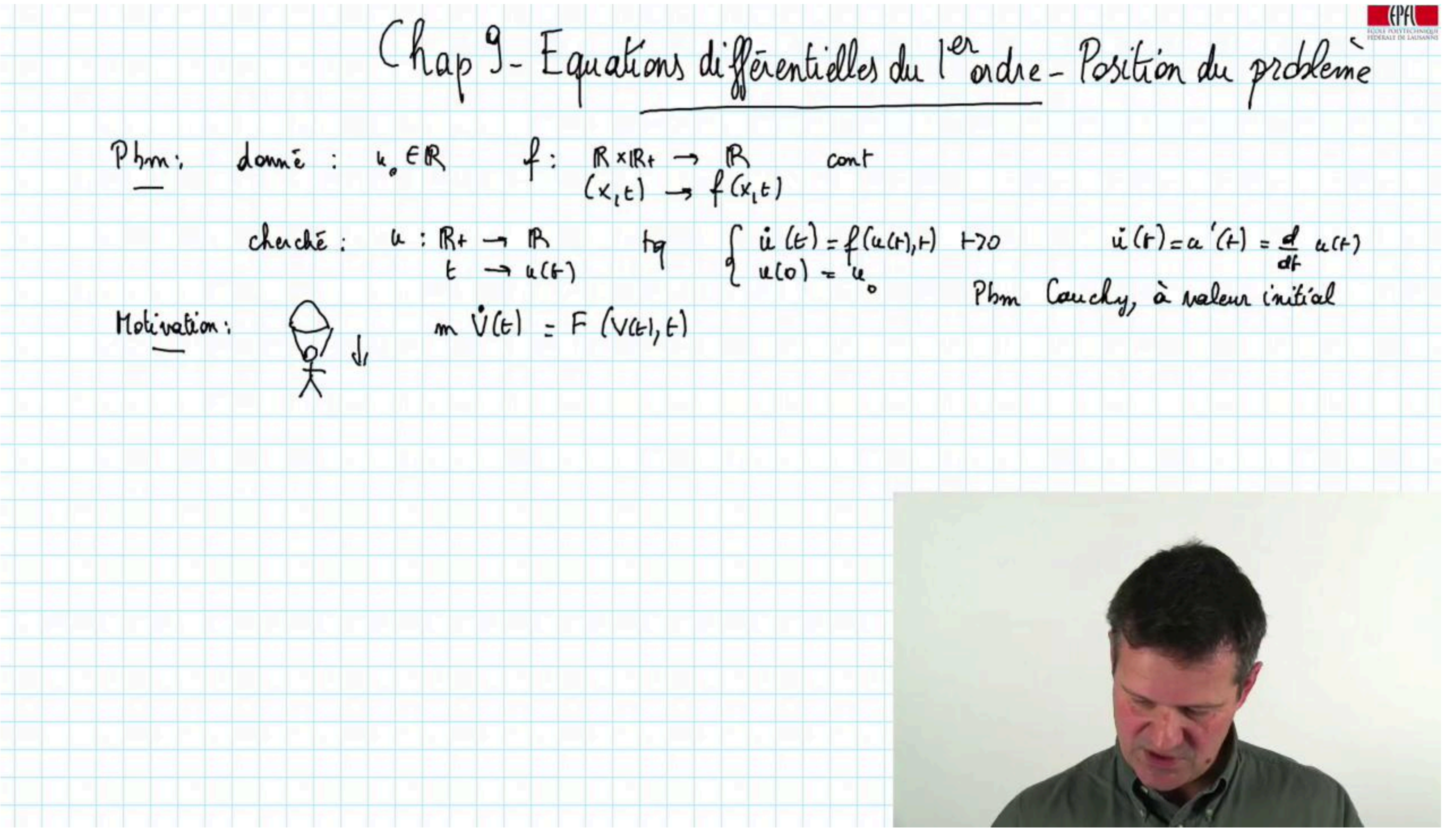


Video



First order differential equations. Let's consider the following problem: given an initial value u0 in ℝ, given a function f or two variables (x,t) which maps (x,t) x in ℝ, t is the time, so nonnegative, and f(x,t) is in ℝ, f is a continuous function. We're looking for a function u, of one variable t, that returns u(t), so t is nonnegative, u(t) is in ℝ, that satisfies the following first order differential equation: u'(t) (u dot t), dot is the notation in physics books for the derivative u'(t), we have a function u of one variable, and we consider the derivative d/dt u(t), there is no ambiguity there about the derivative, this function satisfies u'(t) =f(u (t),t) for nonnegative t, with initial value u(0)=u0, which is given. The problem is called Cauchy's problem, or initial value problem. comme traduction de l'anglais : initial value problem. What is the physical motivation behind this differential equation? Consider the case of a parachutist, falling at a velocity v, Newton's equations write: mass * acceleration, which is the derivative of velocity, is equal to the external forces, so the gravity force, and the drag force due to the parachute, so the external forces depend on the velocity and eventually on time.

Notes

Summary

0m 02s

Chap 9 - Equations différentielles du 1er ordre - Position du problème

Phm: donné : $u_0 \in \mathbb{R}$ $f: \begin{array}{l} \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R} \\ (x,t) \to f(x,t) \end{array}$ cont

cherché : $u: \begin{array}{l} \mathbb{R}_+ \to \mathbb{R} \\ t \to u(t) \end{array}$ tq $\begin{cases} \dot{u}(t) = f(u(t),t) & t>0 \\ u(0) = u_0 \end{cases}$ $\dot{u}(t) = u'(t) = \frac{d}{dt}u(t)$

Phm Cauchy, à valeur initial

Motivation:  $\downarrow$ $\begin{cases} m\,\dot{V}(t) = F(V(t),t) \\ V(0) \end{cases}$ gravité traînée

Ex 9.1: $\begin{cases} \dot{u}(t) = 3u(t) - 3t \\ u(0) = \alpha \end{cases}$ $f(x,t) = 3x - 3t$ $u(t) = \left(\alpha - \frac{1}{3}\right)e^{3t} + t + \frac{1}{3}$ une sol. unique

Ex 9.2: $\begin{cases} \dot{u}(t) = (u(t))^{1/3} \\ u(0) = 0 \end{cases}$ $\dot{u}\,u^{\alpha} = 1$ $\alpha = -1/3$ $\frac{1}{\alpha+1}u^{\alpha+1}(t) = t + c$ $u(t) = \pm\left(\frac{2}{3}t\right)^{3/2}$ $u(t) = 0$

We have the external forces : Gravity, that pushes towards the botton, and the drag force slowing the fall of the parachutist, the given initial value v(0) being known. Let's take a few examples from the book, example 9.1 for instance, the differential equation is: u'(t)=3u(t)-3t, with initial condition u(0)= α, a given real number. Here the function is f(x,t)=3x-3t, I replaced u(t) by x, you have learned in your first year calculus course how to solve this differential equation, it is a linear differential equation, the solution of the homogeneous problem is proportional to exp(3t), and t+1/3 is a particular solution. Thus u is a constant times exp(3t) plus a particular solution. The constant is such that u(0) equals the initial condition a, here we find α -1/3. So this is the unique solution. Let's now consider example 9.2 of the book. Solve u'(t)=u(t)^(1/3) with initial condition u(0)=0. I can solve this differential equation, it is of the type u'*u^α=1, donc ce terme-là, je le passe de l'autre côté, with α=-1/3. The integral of u'.u^(α)=1 is 1/(1+α)*u(t)^(α+1), =t + a constant. Finally I get u(t)=(2/3*t)^(3/2), we have this solution with a + sign and the same with a - sign and also we have the solution u(t)=0 since 0 is also a solution.

Notes

Summary

2m 09s

Chap 9 - Equations différentielles du 1er ordre - Position du problème

Phm : donné : $u_0 \in \mathbb{R}$ $\quad f: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$, $(x,t) \to f(x,t)$ cont

cherché : $u : \mathbb{R}_+ \to \mathbb{R}$, $t \to u(t)$ tq $\begin{cases} \dot u(t) = f(u(t),t) & t>0 \\ u(0) = u_0 \end{cases}$ $\quad \dot u(t) = u'(t) = \frac{d}{dt} u(t)$

Phm Cauchy, à valeur initial

Motivation :  $\downarrow$ $\begin{cases} m\,\dot V(t) = F(V(t),t) \\ V(0) \end{cases}$ gravité traînée

Ex 9.1 : $\begin{cases} \dot u(t) = 3u(t) - 3t \\ u(0) = \alpha \end{cases}$ $\quad f(x,t) = 3x - 3t$ $\quad u(t) = \left(\alpha - \frac{1}{3}\right) e^{3t} + t + \frac{1}{3}$ une sol. unique

Ex 9.2 : $\begin{cases} \dot u(t) = (u(t))^{1/3} \\ u(0) = 0 \end{cases}$ $\quad \dot u\, u^\alpha = 1$ $\quad \alpha = -1/3$ $\quad \frac{1}{\alpha+1} u^{\alpha+1}(t) = t + c$ $\quad u(t) = \pm\left(\frac{2}{3}t\right)^{3/2}$ $\quad u(t) = 0$

3 solutions

Ex 9.3 : $\begin{cases} \dot u(t) = (u(t))^3 \\ u(0) = 1 \end{cases}$ $\quad \dot u\, u^\alpha = 1$ $\quad \alpha = -3$

Therefore, there are 3 solutions which are: the first, (2/3*t)^(3/2), then minus this solution, and finally the 0 constant function, donc trois solutions à ce problème. Finally, example 9.3, which is u'(t)=u(t)^3 with initial value u(0)=1 this time. Still an equation of the type u'*u^α=1. This times α is -3, the integral is 1/(1+α)*u(t)^(α+1)=t + c I get the solution u(t)=1/square root of(1-2t). At the time 0, u(0)= 1 and when t approaches 1/2, there is a blow-up in the sense that the limit when t goes to 1/2 with lower values, then u(t) goes to + infinity. J'ai explosion, on dit qu'il y a explosion de la solution. Note that if I just change the sign of the differential equation, u'(t) = -u(t)^3, with the same initial value u(0)=1, I still have an equation of the type u'.u^(α) = -1, α is still -3, and I find that the solution is given by 1/square root of (1+2t) so the - sign that was here became + sign, à la place d'un signe moins avant, so I have no more blow-up, I have a unique solution for all time t, this solution goes from 1 at time 0, to 0 when t goes to the infinity. Ici il y a une solution unique. Now I will state a theorem taken from the book, this theorem states that there exists a unique condition under a certain assumption on the function f(x,t).

Notes

Summary

5m 07s