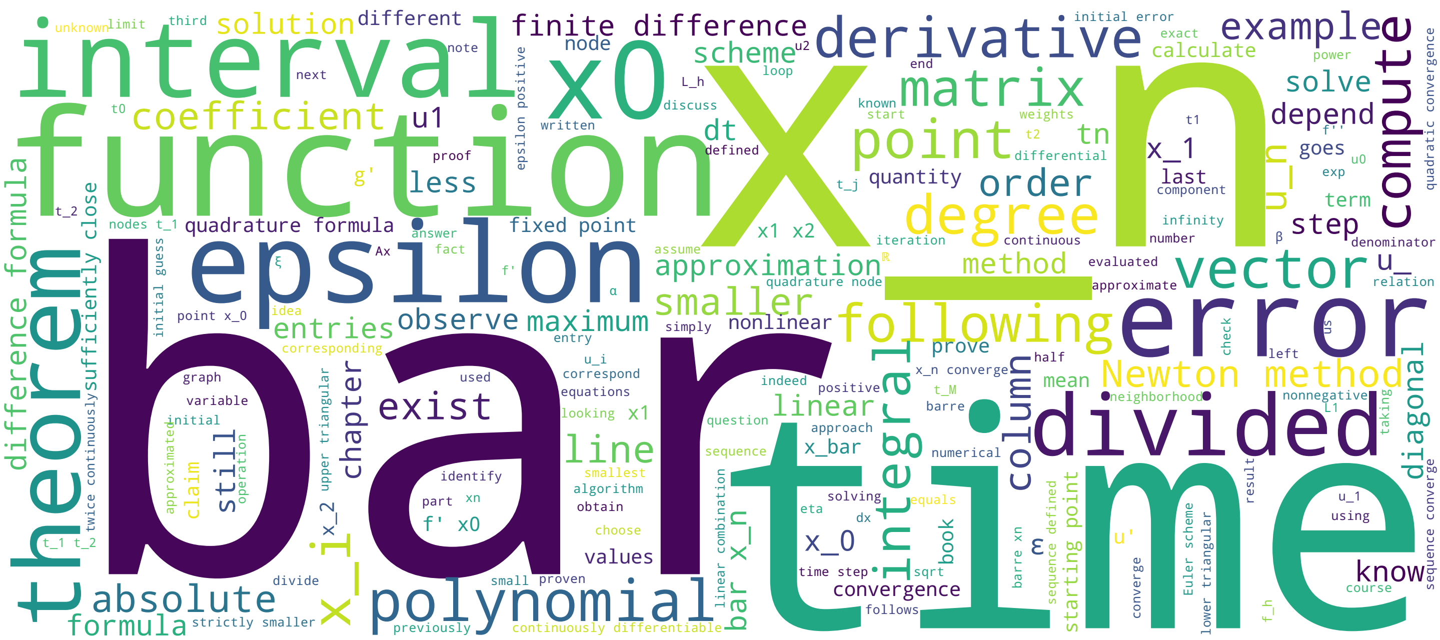
Chapitre 8 : Méthode de Newton (suite)

Introduction à l'analyse numérique

Prof. Marco Picasso



Video

EPFL

Chap 8 - Méthode de Newton (suite)

Thm 8.4. Soit $f: \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^2$, soit $\bar{x}$ tq $f(\bar{x}) = 0$, supp $f'(\bar{x}) \neq 0$. Alors $\exists \varepsilon > 0$ $\forall$ $\bar{x} - \varepsilon \leq x_0 \leq \bar{x} + \varepsilon$ la suite déf. par $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ converge vers $\bar{x}$. De plus la convergence est quadratique : $\exists C > 0$ $\forall n$

$$|\bar{x} - x_{n+1}| \leq C\, |\bar{x} - x_n|^2.$$

Int ay

Let me now state theorem 8.4 from the book. this theorem allows to better understand Newton's method. Let f from R to R be of class C2, twice continuously differentiable. Let x bar be a zero of f, x bar such that f(x bar) is equal to 0, I assume that such a zero exists, and I also assume that f ' (x bar) is different from 0, c'est une quantité qui intervient au dénominateur. The claim is that there exists an epsilon positive, such that for all x_0, the starting point for Newton's method, between x bar minus epsilon and x bar plus epsilon, in other words x_0 is sufficiently close to x bar, well in this case, the sequence defined by x_n+1 equal to x_n minus f (x_n) / f ' (x_n), converges towards x bar; this is in fact a consequence of theorem 8.3, the fixed point theorem. There is some extra information, furthermore the convergence is quadratic, which means very fast, more precisely, there exists C such that for all n, at step n+1, x bar minus x_n+1 is smaller or equal than C times the error at step n, (x bar minus x_n) squared. voilà la fin du théorème. Before doing the proof of the second part of the theorem, we shall discuss about the quadratic convergence.

Notes

Summary

0m 03s

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$$|\bar{x} - x_{n+1}| \leq C\,|\bar{x} - x_n|^2.$$

Interprét : Si x_0 suff proche de $\bar{x}$, converg. rapide $C = 1$

$|\bar{x} - x_0| = 0,1$
$|\bar{x} - x_1| \leq 10^{-2}$
$|\bar{x} - x_2| \leq 10^{-4}$
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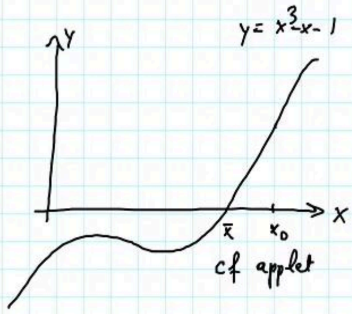
Il existe des itérations où la méth. de Newton ne conv. pas



The theorem says that if x_0 sufficiently close to x bar, donc ça c'est la condition : il existe un *ε* tel que pour tout *x0* minoré par *x* barre - *ε* et majoré par *x* barre + *ε*, then the convergence is fast. An example situation, to settle some ideas say that C=1, and x bar minus x_0 the initial error, donc l'erreur, je choisis un *x0*, the initial error is 0.1. Now I compute x bar minus x_1, which is smaller than (x bar minus x_0) squared, thus 10^-2, the error at step 2, x bar minus x_2, is smaller than C times the error at the previous step, (x bar minus x_1) squared, thus less than 10^-4, the error at the third step, x bar minus x_3, is smaller than 10^-8, therefore in three iterations I have approximated the zero up to 10^-8. The convergence is therefore very fast. Now there exists cases where Newton's method does not converge. For example, consider the following case. I consider the function f given by x^3 minus x minus 1, a java applet is available, to illustrate the calculations. From an initial guess x_0, here, you arrive very rapidly, by taking the tangent, here, you reach x bar very quickly.

Notes

Summary

2m 22s

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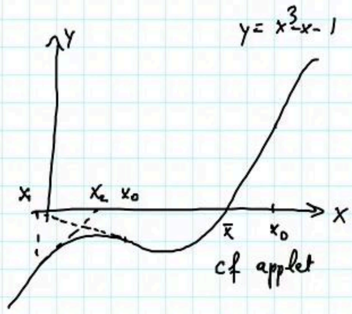

Il existe des situations où la méth. de Newton ne conv. pas

cf applet

Dem : $x_{n+1} = g(x_n)$ $\quad g(x) = x - \frac{f(x)}{f'(x)}$ $\quad g'(x) = 1 - \frac{(f'(x))^2 - f f''(x)}{(f'(x))^2}$ $\quad |g'(\bar{x})| = 0 < 1$

thm 8.3 $\exists \varepsilon > 0$ $\forall$ $\bar{x} - \varepsilon \leq x_0 \leq \bar{x} + \varepsilon$ $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ conv.

On the other hand, from an x_0 here, by applying Newton's method, you take this tangent, x_1 will be here, next x_2 will be here, so you will either oscillate between two values, or even diverge. Therfore, we observe that the condition on x_0 being sufficiently close to x bar, qui est ici, this condition cannot be removed. Now we can prove the result. Proof: Donc, j'ai tout à l'heure calculé, I already said that Newton's method was a fixed point method, x_n+1 equals g(x_n), with g(x) equal to x minus f(x) / f ' (x). J'ai calculé-- Let's note that is f' (x bar) is different from 0, it remains different from 0 in a neighborhood of x bar. I already compute the derivative g ' (x) = 1 - (f ' (x)^2 - f " (x)) / (f ' (x)^2) divisé par *(f'(x))²*, and I observed that g ' (x bar) was equal to 0, which is strictly smaller than 1. Therefore, from theorem 8.3, there exists an epsilon positive such that, if my starting point x_0 is between x bar minus epsilon and x bar plus epsilon, then the sequence given by x_n+1 equal to g(x_n), wheer g(x_n) = x_n - f(x_n) / f ' (x_n), this sequence converges. We still need to prove the quadratic convergence, the second part of the result; how can we do?

Notes

Summary

4m 28s

Chap 8 - Méthode de Newton (suite)

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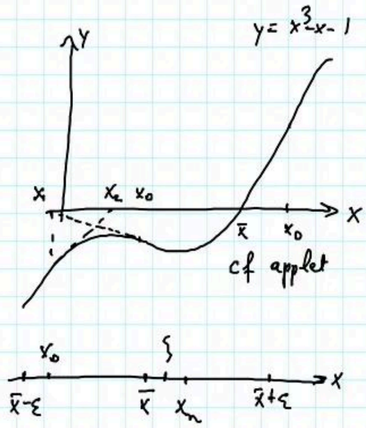

Dem: $x_{n+1}=g(x_n)$ $\quad g(x)=x-\dfrac{f(x)}{f'(x)}$ $\quad g'(x)=1-\dfrac{f'(x)^2-ff''(x)}{f'(x)^2}$ $\quad |g'(\bar{x})|=0<1$

thm 8.3 $\exists\,\varepsilon>0\ \forall\ \bar{x}-\varepsilon\le x_0\le\bar{x}+\varepsilon$ $\quad x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$ conv.

$$f(\bar{x})=f(x_n)+(\bar{x}-x_n)f'(x_n)+\frac{(\bar{x}-x_n)^2}{2}f''(\xi)$$

$$|\bar{x}-x_{n+1}|=\frac{|\bar{x}-x_n|^2}{2}\,\frac{|f''(\xi)|}{|f'(x_n)|}$$

I compute a Taylor expansion, I compute f(x bar) equal to f(x_n) plus (x bar - x_n) f ' (x_n), plus (x bar - x_n)^2 * f '' (eta) divided by 2 factorial. fois *f''* en un point intermédiaire ξ. Here is x bar the zero of f, here is x_n, here we have the interval x bar minus epsilon to x bar plus epsilon in which I placed the initial starting point x_0, et je sais que l'erreur diminue à chaque itération. eta is somewhere between x bar and x_n, but I also know that it is between x bar minus epsilon and x bar plus epsilon. Now I can calculate x bar minus x_n+1 in absolute value. *x barre - xn x barre - xn+1* en valeur absolue, pardon. It is equal to (x bar - x_n)^2 divided by 2, I have divided this relation, here, by f ' (x_n), and I still have f '' (eta) in absolute value, divided by f ' (x_n) in absolute value. Now, I must prove that there exists C such that for all n, thus C does not depend on n. The candidate for C is half f '' (eta) times f ' (x_n), where eta is between x bar and x_n, but this quantity depends on x_n, thus n.

Notes

Summary

6m 44s

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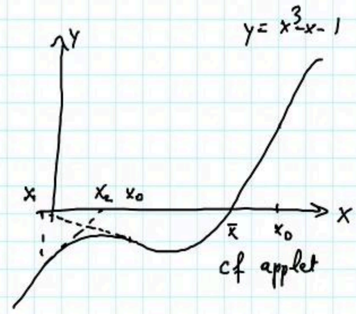

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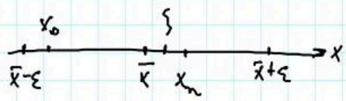
Dem : $x_{n+1} = g(x_n)$ $g(x) = x - \frac{f(x)}{f'(x)}$ $g'(x) = 1 - \frac{(f'(x))^2 - ff''(x)}{(f'(x))^2}$ $|g'(\bar{x})| = 0 < 1$

thm 8.3 $\exists\, \varepsilon > 0$ $\forall\, \bar{x} - \varepsilon \leq x_0 \leq \bar{x} + \varepsilon$ $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ conv.

$$f(\bar{x}) = f(x_n) + (\bar{x} - x_n) f'(x_n) + \frac{(\bar{x} - x_n)^2}{2} f''(\xi)$$

$$|\bar{x} - x_{n+1}| = \frac{|\bar{x} - x_n|^2}{2} \frac{|f''(\xi)|}{|f'(x_n)|} \leq |\bar{x} - x_n|^2 \frac{1}{2} \frac{\max_{\bar{x} - \varepsilon \leq x \leq \bar{x} + \varepsilon} |f''(x)|}{\min_{\bar{x} - \varepsilon \leq x \leq \bar{x} + \varepsilon} |f'(x)|}$$



But I can bound this quantity by (x bar - x_n)^2 times 1/2, times the largest value on the interval x bar minus epsilon to x bar plus epsilon, and the denominator by the smallest value on the interval x bar minus epsilon to x bar plus epsilon, thus maximum of f " (x) in absolute value, over the interval x bar minus epsilon to x bar plus epsilon, divided by the smallest value, hence the min of f ' (x) in absolute value, for x in [x bar minus epsilon, x bar plus epsilon], and this the C in the theorem, C is equal to 1/2 times the maximum of the second derivatives divided by the minimum of the first derivatives on the interval x bar minus epsilon to x bar plus epsilon. I have proven the theorem.

Notes

Summary

8m 48s