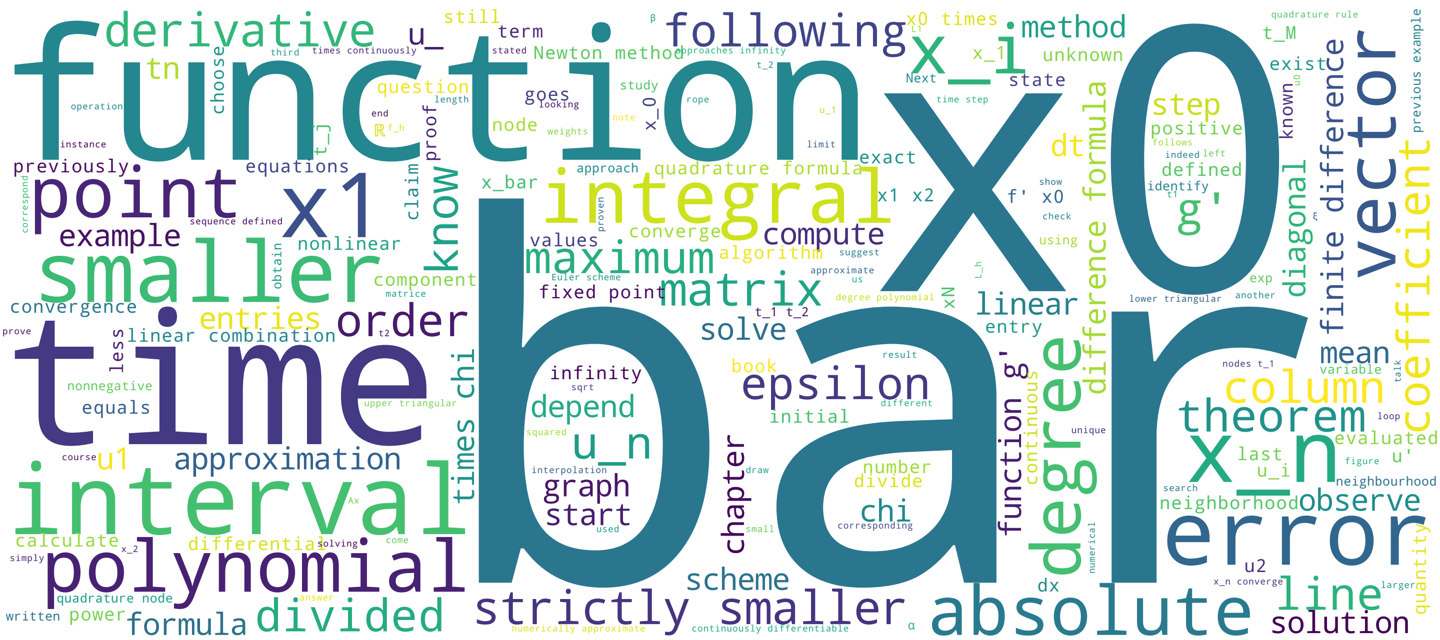
# Chapitre 8 : Méthode de point fixe (fin)

## Introduction à l'analyse numérique

Prof. Marco Picasso



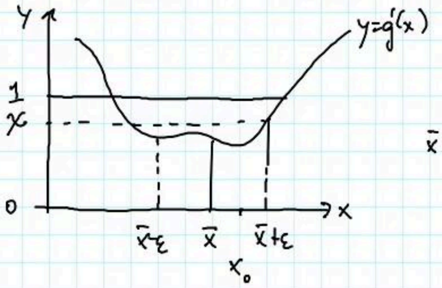

Video

EPFL

# Chap 8 - Méthode de point fixe (suite)

**Thm 8.3:** Soit $g : \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^1$, soit $\bar{x}$ tq $g(\bar{x}) = \bar{x}$, supposons $|g'(\bar{x})| < 1$.
Alors $\exists \varepsilon > 0$ $\forall$ $\bar{x} - \varepsilon \leq x_0 \leq \bar{x} + \varepsilon$, la suite déf. par $x_{n+1} = g(x_n)$ converge vers $\bar{x}$. De plus la convergence est linéaire :
$\exists\, 0 < C < 1$ $\forall n$ $|\bar{x} - x_{n+1}| \leq C |\bar{x} - x_n|$.

**Dem:**




$\exists \varepsilon > 0$ $\exists\, 0 < \chi < 1$ $\forall \bar{x} - \varepsilon \leq x \leq \bar{x} + \varepsilon$ $|g'(x)| \leq \chi < 1$

Soit $\bar{x} - \varepsilon \leq x_0 \leq \bar{x} + \varepsilon$ on a

$\bar{x} - x_1$

The proof of this theorem is very instructive, and we will study it. " Proof " Here is the function g', the graph of g', I know that g'(x bar) is smaller than 1 in absolute value, the graph of the function g', I know that in x bar, g'(x bar) is strictly smaller than 1. Since the function g' is continuous, I claim that this function g' remains smaller than one in a neighbourhood of x bar, that is, there exists a positive epsilon and chi positive but strictly smaller than 1, such that for all x between x bar minus epsilon and x bar plus epsilon, g'(x) in absolute value is smaller or equal to chi, which is strictly smaller than 1. I can draw the corresponding figure, here is the neighbourhood, here x bar plus epsilon, and this is x bar minus epsilon, in this neighborhood, the function is smaller or equal to chi, which is strictly smaller than 1. Choose x0 be in this neighborhood, x bar minus epsilon to x bar plus epsilon, here for instance. We can do the following calculation. I want to calculate x bar minus x1, why x bar minus x1, because I want to prove that x bar minus x_n+1 is smaller or equal than something strictly smaller than 1 times x bar minus x_n, therefore I start with x1.

Notes

Summary

0m 03s

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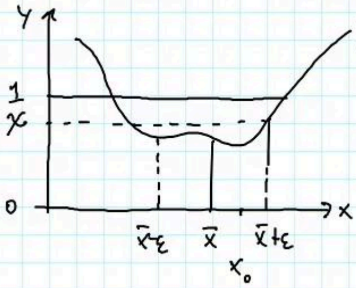
$\exists\ 0<C<1\ \forall n\quad |\bar{x}-x_{n+1}|\le C|\bar{x}-x_n|$.

**Dem:**




$\exists \varepsilon>0\ \exists\ 0<\chi<1\ \forall\ \bar{x}-\varepsilon\le x\le\bar{x}+\varepsilon\quad |g'(x)|\le\chi<1$

Soit $\bar{x}-\varepsilon\le x_0\le\bar{x}+\varepsilon$ on a

$$|\bar{x}-x_1| = |g(\bar{x})-g(x_0)| = \left|\int_{x_0}^{\bar{x}} g'(s)\,ds\right| \le |\bar{x}-x_0| \max_{\bar{x}\le s\le x_0}|g'(s)| \le |\bar{x}-x_0|\ \chi$$

$$|\bar{x}-x_2| = |g(\bar{x})-g(x_1)| = \left|\int_{x_1}^{\bar{x}} g'(s)\,ds\right| \le |\bar{x}-x_1|$$

Take x bar minus x1, in absolute value, x bar is by definition equal to g(x bar), since it is a fixed point of g, and x1 is equal to g(x0). Now I use the fundamental theorem of calculus; well this is equal to the integral between x0 and x bar of g'(s) ds. This is smaller or equal to the length of the interval, in absolute value, x bar minus x0, times the maximum of g'(s), which is the integrand. First I stated that the absolute value of the integral is smaller than the integral of the absolute value. I take the integrand g'(s) and search for the maximum over the interval [x bar,x0]. In the figure, here isthe maximum over the interval [x bar,x0]. But on this interval, I know that the function is smaller or equal to chi; therefore I can state that x bar minus x1 is smaller or equal than x bar minus x0 times chi, which is strictly smaller than 1. Lets carry on, this time for x bar minus x2, I start over: x bar is equal to g(x bar), this comes from the definition of a fixed point, x2 is equal to g(x1), which is the integral between x1 and x bar of g'(s) ds taken in absolute value. As previously, I bound this quantity, here I have x bar minus x1, and I must take the maximum on the interval [x bar, x1] or [x1, x bar].

Notes

Summary

2m 14s

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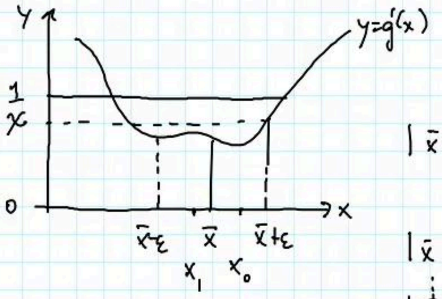


$\exists \varepsilon>0\ \exists\ 0<\chi<1\ \forall\ \bar{x}-\varepsilon\le x\le \bar{x}+\varepsilon \quad |g'(x)|\le \chi<1$

Soit $\bar{x}-\varepsilon \le x_0 \le \bar{x}+\varepsilon$ on a

$$|\bar{x}-x_1| = |g(\bar{x})-g(x_0)| = \left|\int_{x_0}^{\bar{x}} g'(s)\,ds\right| \le |\bar{x}-x_0| \max_{\bar{x}\le s\le x_0} |g'(s)| \le |\bar{x}-x_0|\ \chi$$

$$|\bar{x}-x_2| = |g(\bar{x})-g(x_1)| = \left|\int_{x_1}^{\bar{x}} g'(s)\,ds\right| \le |\bar{x}-x_1|\ \chi$$

$$\vdots$$

$$|\bar{x}-x_{n+1}| \le |\bar{x}-x_n|\ \chi \le |\bar{x}-x_0|\ \chi^{n+1} \xrightarrow{n\to\infty} 0$$

**Exemple:** cf ex page pré

Previously, I said that the error x bar minus x1 was strictly smaller than x bar minus x0, this means I can put x1 here for example in the graph. I repeat, x bar minus x0 is larger than x bar minus x1. The maximum, on this interval, of g'(s) on the interval [x bar, x1] is again smaller or equal, so x1 is in this interval and the derivative is smaller than chi, and hence this will be smaller or equal than x bar minus x1 times chi. And so on, you can observe that this methodology can be recursively repeated, and I get that x bar minus x_n+1 is smaller or equal to x bar minus x_n times chi. Et donc, comme cette quantité-là, Well by induction I get that, x bar minus x_n+1 is smaller or equal to x bar minus x0 times chi to the power n+1, since chi is included in (0,1), as n approaches infinity, this quantity approaches 0. I have therefore proven that the sequence defined by x_n+1 equal to g(x_n) converges, and that this convergence is linear. This means that the error at step n+1 is strictly smaller than the error at step n. Now I suggest to come back to, the previous example. Lets illustrate this theorem using the previous example. the example from the previous page.

Notes

Summary

4m 21s