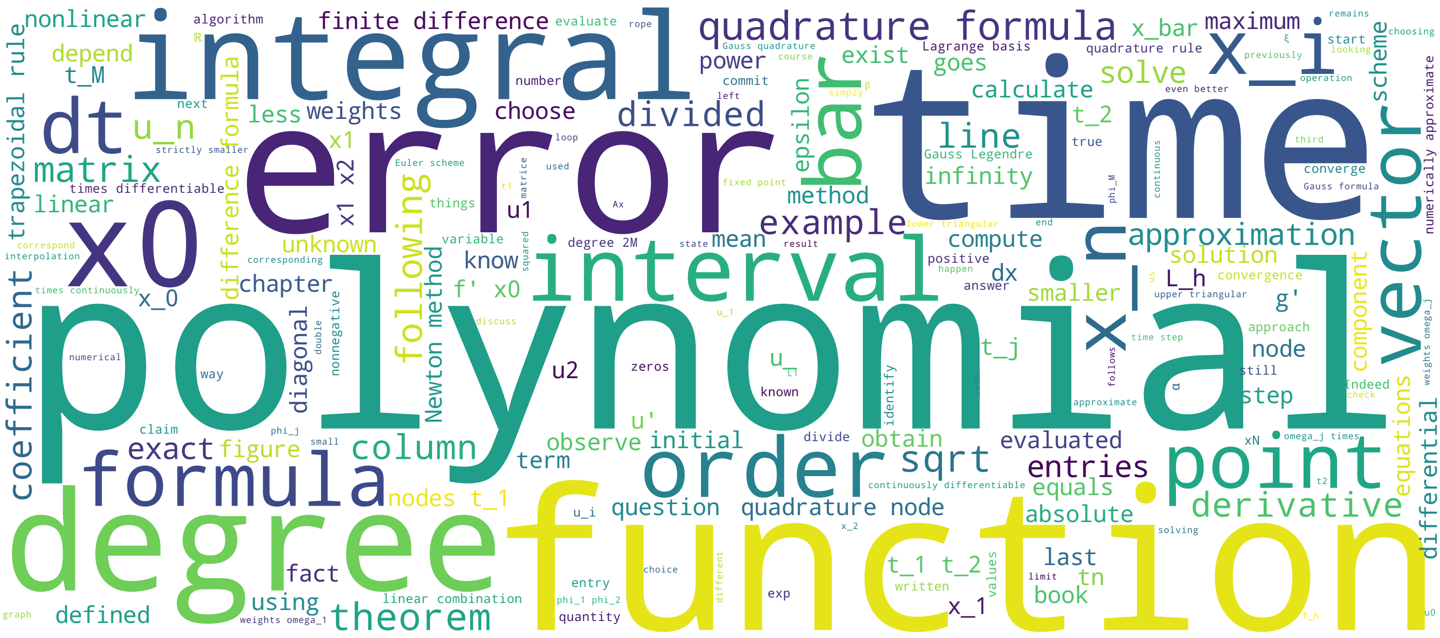

Chap. 3 : points d'intégration - formules de Gauss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Video

Chap. 3 : points d'intégration - formules de Gauss

- Soit une formule de quadr. $J(g) = \sum_{j=1}^{M} w_j \, g(t_j)$ appr $\int_{-1}^{1} g(t)\,dt$
- $w_j = \int_{-1}^{1} \varphi_j(t)\,dt \quad j = 1,\dots,M$
- Thm 3.2 $\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_{M-1}$
- Existe-t-il un choix judicieux de $t_1 t_2 \dots t_M$?
- $M = 2 \quad t_1 = -\frac{1}{\sqrt{3}} \quad t_2 = \frac{1}{\sqrt{\ }}$

It now remains to discuss about the choice of the quadrature nodes. This will bring me to the definition of the Gauss quadrature formula. Consider a quadrature formula, which I write as J(g), so the sum for i from 1 to M of the weights omega_j times g evaluated in t_j, this to numerically approximate the integral over -1 to 1 of g(t) dt. Once the nodes are chosen, the weights are given by theorem 3.2, omega_j is the integral from -1 to 1 of phi_j(t) dt, the functions phi_1,phi_2 up to phi_M being the Lagrange basis of P_M-1 associated to the nodes t_1,t_2,...,t_M. donc *j* allant de 1 jusqu'à *M*. From theorem 3.2, I know that the quadrature rule built this way is exact for polynomials of degree less or equal M-1. Therefore the integral from -1 to 1 of p(t) dt is equal to J(p) for all polynomial p of order M-1. The question is: Is there a good choice of the quadrature nodes t_1 up to t_M, a choice better than others? The answer is yes! This choice corresponds to the Gauss quadrature formula. So, in the case M equal to 2, I could choose as quadrature nodes those of the trapezoidal formula, t_1 equal to -1 and t_2 = 1. Now I will choose t_1 equal to -1 / sqrt(3) and t_2 equal to 1 / sqrt(3).

Notes

Summary

0m 00s

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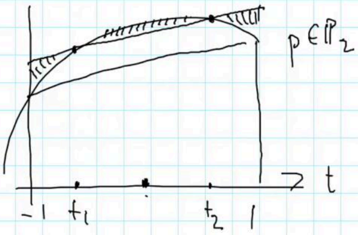
- Existe-t-il un choix judicieux de $t_1 t_2 \dots t_M$?
- $M=2 \quad t_1 = -\frac{1}{\sqrt{3}} \quad t_2 = \frac{1}{\sqrt{3}} \quad w_1 = w_2 = 1 \quad J(g) = g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) \quad \int_{-1}^{1} p(t)\,dt = J(p) \ \forall p \in \mathbb{P}_1$ (thm 3.2)



$\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}$

I calculate the weights omega_1 and omega_2 using theorem 3.2, and I get omega_1 equal to 1 and omega_2 equal to 1 as well. Now the quadrature formula is J(g) equal to g(-1/sqrt(3)) plus g(1/sqrt(3)). By construction, this formula is exact for polynomials of degree 1. Therefore the integral from -1 to 1 of p(t) dt is equal to J(p) for all polynomial of degree 1, this being a consequence of theorem 3.2. in fact, it is even better then this. Indeed, this can be seen from this figure, where t varies from -1 to 1. Let us choose a polynomial of degree 2. I immediately see that by using the trapezoidal rule I commit a certain error. If I consider a polynomial of degree 2, I do not have J(p) equal to the integral over -1 to 1 of p(t) dt for the trapezoidal rule. On the other hand, by using this formula here things are much better, why? Since my quadrature nodes are -1 / sqrt(3) and 1 / sqrt(3), more or less -0.577 and 0.577, and reporting on the figure, I observe that the error committed here is compensated by this error here. In fact, in this case the integral between -1 and 1 of p(t) dt is equal to J(p) for all polynomials of degree 2.

Notes

Summary

2m 05s

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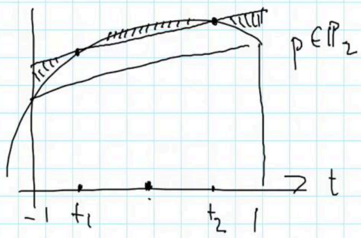
- thm 3.2 $\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_{M-1}$
- Existe-t-il un choix judicieux de $t_1 t_2 .. t_M$?
- $M = 2$ $t_1 = -\frac{1}{\sqrt{3}}$ $t_2 = \frac{1}{\sqrt{3}}$ $w_1 = w_2 = 1$ $J(g) = g(-\frac{1}{\sqrt{3}}) + g(\frac{1}{\sqrt{3}})$ $\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_1$ (thm 3.2)



$\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_2$

$\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_3$

$\left| \int_a^b f(x)\,dx - L_h(f) \right| = O(h^4) \quad f \in \mathcal{C}^4[a,b]$

$L_h(f) = \frac{h}{2} \sum_{i=0}^{N-1} \left(f\left(x_i + h \frac{-\frac{1}{\sqrt{3}}+1}{2}\right) + f\left(x_i + h \frac{\frac{1}{\sqrt{3}}+1}{2}\right) \right)$

- M qque $t_1 t_2 .. t_M$ zéros du polyn. Gauss-Legendre L_M $\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_{2M-1}$

It happens that things are even better, this is also true for polynomials of degree 3 by symmetry reasons. So the integral over -1 and 1 of p(t) dt is equal to J(p) for all polynomial p of degree 3. And now, by considering the difference between the integral between a and b of f(x) dx and the corresponding L_h(f) formula, I don't have an error of order h to the power 2 as in the trapezoidal rule, but I obtain an error of order h to the power 4. Of course assuming that f is 4 times differentiable on [a,b]. Now what is the L_h(f) formula? I have not explicitly given the formula for L_h(f), here is the corresponding formula: it is h / 2 times the sum over each sub-interval, over i from 0 to N-1, and one must evaluate the function f in x_i + h times (t_1 + 1) over 2, with t_1 equal to -1 / sqrt(3); next f evaluated in x_i + h times (t_2 + 1) over 2 with t_2 equal to 1 / sqrt (3). All this can be generalized for any M. For t_1, t_2 up to t_M being the zeros of the polynomials of Gauss - Legendre, there is exactly M zeros on the interval [-1,1] of the Gauss - Legendre polynomials of degree M to be defined, I get the general result, that is the quadrature formula is exact for polynomials of degree 2M-1, well the integral from -1 to 1 of p(t) dt equal to J(p) for any polynomial p of degree 2M-1.

Notes

Summary

4m 02s

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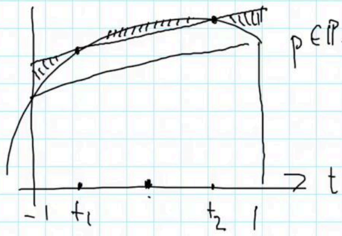
- $M=2$ $\quad t_1 = -\frac{1}{\sqrt{3}}$ $\quad t_2 = \frac{1}{\sqrt{3}}$ $\quad w_1 = w_2 = 1$ $\quad J(g) = g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$ $\quad \int_{-1}^{1} p(t)\,dt = J(p) \ \forall p \in \mathbb{P}_1$ (thm 3.2)



$\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_2$

$\int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_3$

$\left| \int_a^b f(x)\,dx - L_h(f) \right| = O(h^4) \quad f \in \mathcal{C}^4[a,b]$

$L_h(f) = \frac{h}{2} \sum_{i=0}^{N-1} \left(f\left(x_i + h\,\frac{-\frac{1}{\sqrt{3}}+1}{2}\right) + f\left(x_i + h\,\frac{\frac{1}{\sqrt{3}}+1}{2}\right) \right)$

- M qque $t_1 t_2 \dots t_M$ zéros du polyn. Gauss-Legendre L_M $\quad \int_{-1}^{1} p(t)\,dt = J(p) \quad \forall p \in \mathbb{P}_{2M-1}$

  $\left| \int_a^b f(x)\,dx - L_h(f) \right| = O(h^{2M}) \quad f \in \mathcal{C}^{2M}[a,b]$

For example for M = 2, the quadrature formula is exact for a polynomial of degree 3. Now, the error I commit between the integral over a and b of f(x) dx minus L_h(f), the corresponding approximation formula, this error is of order h t the power 2M if the function f is 2M times differentiable on the interval [a,b]. In fact, compared to a conventional quadrature formula we double the order of convergence, so for example for the trapezoidal rule we have order 2 but here for the Gauss formula with 2 nodes we have order 4 only by choosing these magical nodes t_1, t_2 equal to -1 / sqrt(3) and 1 / sqrt(3).

Notes

Summary

6m 22s